Noisy Uncoupled Chaotic Map Ensembles Violate the Law of Large Numbers

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An ensemble of *uncoupled* chaotic maps with spatially synchronized parametric fluctuations violates the law of large numbers. This is clearly evident in the nonstatistical features of the mean field, whose mean-square deviation (MSD) does not fall as $1/N$ with increasing N, where N is the number of elements in the system. In fact the MSD saturates after a critical value of $N=N_c$. This amazing phenomenon is reminiscent of the nonstatistical behavior in globally coupled chaos. Interestingly though, there is no coupling in this system, and so the emergence of a subtle coherence in the global dynamics, as suggested by the existence of size-independent fluctuations, is very intriguing.

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Complex systems, ranging from economic markets and ecosystems to earthquakes and fluid mechanics, have generated a lot of research interest in recent years. The most striking feature of many composite systems containing large numbers of elements is that fascinating global phenomena arise out of seemingly simple local dynamics. It is then of considerable importance to investigate the physics generic to such spatially extended systems.

Here we study a dynamical system which is a conglomeration of many individually chaotic elements. The elements are *uncoupled*, unlike in other models of composite systems where local or global coupling is introduced [1-8]. Our system, in most general form, is then given as an ensemble of N local maps:

$$x_{n+1}(i)=f(x_n(i);a_n(i)) , \quad (1)$$

where n is discrete time, i is the site index ($i=1, 2, \ldots, N$), and the $a_n(i)$'s are the nonlinearity parameters which may fluctuate in both space and time, i.e., be a function of both n and i. In particular, the function f can be the logistic map which is used extensively to model a very wide variety of nonlinear dissipative phenomena, i.e., $f(x)=1-ax^2$ [9].

There are four distinct cases to be considered, determined by the nature of the fluctuations in the nonlinearity parameter a, of the local maps. (i) a is constant: $a_n(i)\equiv a$. (ii) $a_n(i)=a(1+\sigma\eta_n)\equiv a_n$, where σ is the strength of fluctuations in the nonlinearity parameter and η_n is a random number uniformly distributed in the interval $[-0.5,0.5]$. Note that this fluctuation is a function of time but is site independent; i.e., the noise in the parameters is synchronous for all the elements and can thus be considered spatially uniform, though random in time. (iii) $a_n(i)=a(1+\sigma\eta^i)\equiv a(i)$. Here the parameters are a random function of site but remain frozen in time, i.e., the parameters are spatially fluctuating but temporally invariant. (iv) $a_n(i)=a(1+\sigma\eta_n^i)$. Here the fluctuations are a function of both time and space.

We have simulated Eq. (1), for all the above cases, with local logistic maps in the chaotic regime. The initial conditions of the individual elements are randomly chosen in the interval $[-1,1]$, and transients are allowed to die. (For values of the nonlinearity parameter $a<2$ the iterates of the map $f(x)$ are bounded in the interval $[-1,1]$.) All results reported here are in the "turbulent" phase, i.e., a phase where the chaotic local dynamics displays no apparent coherence among the elements. [Since there is no coupling, it is not surprising that there is no explicit symptom of correlation after the elements have evolved (chaotically) from random initial states.]

Let us consider the fluctuations of the mean field. If all the state variables took quasirandom values almost independently, one would expect their aggregate, the mean field h_n,

$$h_n=\frac{1}{N}\sum_{j=1}^{N} f(x_n(j)) , \quad (2)$$

to obey the law of large numbers. If this were true the

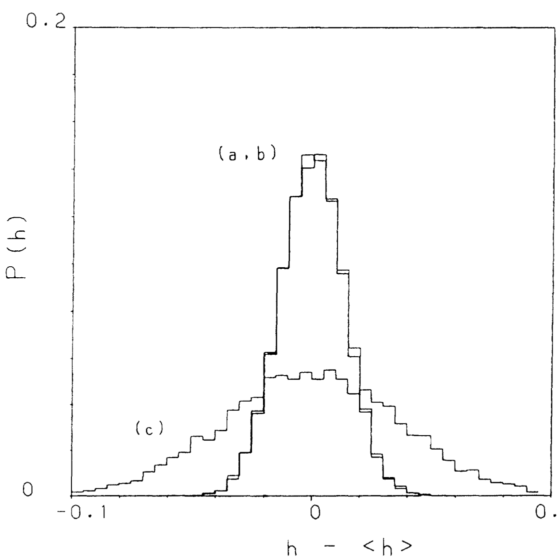

FIG. 1. The distribution of the mean field $P(h)$ for the three cases of Eq. (1): (a) case (i), (b) case (iii), and (c) case (ii), with $a=1.98$, $\sigma=0.02$, and $N=2500$. The histogram is obtained from a sampling over 10000 iterations.

mean-square deviation (MSD) $(=\langle h_n^2\rangle-\langle h_n\rangle^2)$ would decrease as $1/N$, where N is the number of elements in the system, and the mean field would converge to a fixed-point value as $N\rightarrow\infty$.

To examine the above expectation, we have numerically measured the distribution of the mean field. As shown in Fig. 1, the distribution function $P(h)$ agrees roughly with the Gaussian form, when N is large. Note, however, that the deviations of the mean field in case (ii), where the maps evolve under synchronized parametric fluctuations, are significantly larger than those in the other cases.

For the verification of the law of large numbers we calculate the MSD of the mean field with increasing number of elements N. If each element is approximated by an uncorrelated random number, it is expected that MSD$\sim 1/N$. In Fig. 2 MSD is plotted versus N, for the cases (ii), (iii), and (iv). When the nonlinearity parameter of the individual maps $a_n(i)$ are equal and constant in time [case (i)] the system is simply a set of uncoupled logistic maps and displays statistical behavior; i.e., the MSD of the mean field respects the law of large numbers, and goes as $1/N$. It is after the inclusion of noise in the system that we encounter something very interesting. When parametric fluctuations exist in space, i.e., when the nonlinearity parameter varies randomly from site to site but remains frozen in time, and when the fluctuations are both spatial and temporal [cases (iii) and (iv), respectively], we again have statistical behavior. But amazingly, when the temporal parametric fluctuations are spatially uniform, i.e., all the elements are subject to the same parametric fluctuations, which are random in time [case (ii)], we get nonstatistical behavior. It is clear that the MSD does not fall as $1/N$. In fact the MSD almost saturates after a critical value of $N=N_c$, whose value depends on the nonlinearity a and the strength of the fluctuations σ. *This observation implies that an ensemble of uncoupled chaotic maps with spatially uniform parametric fluctuations violates the law of large numbers* [9].

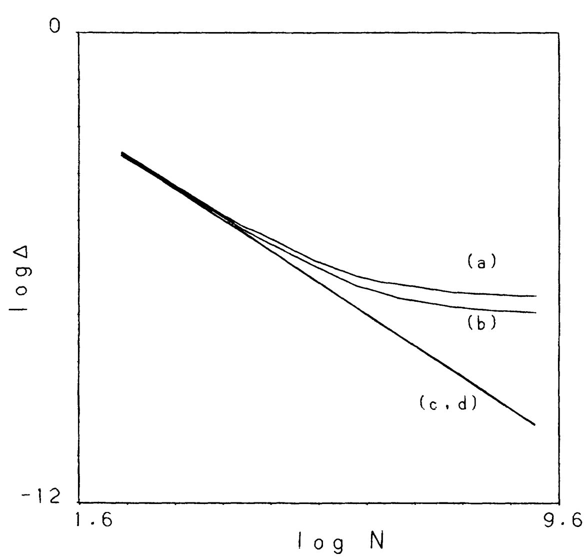

FIG. 2. Mean-square deviation Δ of the mean field vs system size N for four cases of Eq. (1) ($a=1.98$): curve (*a*) case (ii) with $\sigma=0.02$, (*b*) case (ii) with $\sigma=0.01$, (*c*) case (iii) with $\sigma=0.02$, and (*d*) case (iv) with $\sigma=0.02$. The Δ is calculated over 10000 iterations.

These features appear to hold for all parameter values of the logistic map, as long as the effective nonlinearity parameter of the local map is in the chaotic regime. Further, we have studied ensembles of different maps, such as circle maps and tent maps $[f(x)=a(0.5-|x-0.5|)]$ [10], and find similar phenomena. *This strongly suggests that the above nonstatistical properties are independent of the details of the model.*

The above set of maps can be considered the zero-coupling-parameter limit of coupled maps (with the additional important feature that the individual elements are subject to fluctuations). Coupled maps have attracted a lot of attention in recent times for two principal reasons [1–8]: First, they are important in providing models for a variety of nonlinear interactive phenomena, as diverse as Josephson junction arrays, multimode lasers, vortex dynamics in fluids, and even evolutionary dynamics, biological information processing, and neurodynamics; second, they give rise to very novel phenomena, in particular, a host of pronounced nonstatistical features.

Explicitly, a globally coupled map (GCM) is a system of N elements, consisting of a set of local mappings coupled by a "mean-field" type of interaction term, through which global information influences the individual elements. It is thus analogous to a mean-field version of coupled map lattices and can be given as [2–6]

$$x_{n+1}(i)=f(x_n(i))+\varepsilon h_n\,, \qquad (3)$$

where ε is the coupling parameter and h_n is the mean field. Remarkably, it was found that in the fully "turbulent" phase of the GCM, where coherence was completely destroyed by chaos in the individual maps and there was no explicit manifestation of correlation among the elements, a subtle collective behavior emerged [3–8]. This was evident in the marked nonstatistical behavior of the mean field, whose fluctuations saturated at large lattice size.

The anomalous dependence of the fluctuations of the mean field on system size in noisy uncoupled map ensembles is thus reminiscent of the scenario in globally coupled chaos [11]. What is amazing is the fact that we do not have any transparent way of seeing the source of coherence among the elements here, due to the complete absence of coupling interaction.

GCM had an additional interesting feature: The emergence of broad peaks in the power spectrum of the mean field. This indicated the development of a collective beating pattern in the global dynamics. Here, in contrast, the mean field is still quite aperiodic and there is no evidence of any pronounced frequencies (see Fig. 3). This marks the point of departure of this system from the GCM. Interestingly, the spectra of chaotic ensembles are distinctly less "grassy" after the addition of synchronized noise (Fig. 3). Also, the spectra of larger ensembles is somewhat smoother and flatter than those of smaller ensembles.

A rough way of seeing the correspondence of noisy maps to GCM is as follows: Expressing the mean field as a sum of an average part $\langle h \rangle$ (which is constant) and a fluctuating part (due to the deviations), we have the GCM given as

$$x_{n+1}(i) = 1 - a x_n^2(i) + \varepsilon(\text{constant} + \text{fluctuating part}) . \quad (4)$$

As a first approximation if we consider the fluctuating part to be random, we have

$$x_{n+1}(i) = 1 - a x_n^2(i) + \varepsilon\langle h \rangle + \varepsilon\sigma\eta_n \cong 1 - a_{\text{eff}}(n) x_n^2(i) , \quad (5)$$

where $\varepsilon\langle h \rangle$ is the small constant part, σ is the strength of the fluctuations, and η_n is a uniformly distributed random number. The effective nonlinearity parameter of the resulting approximate logistic map is a spatially uniform (site-independent) quantity which fluctuates randomly in time: $a_{\text{eff}}(n)$. This is analogous to case (ii) of the chaotic map ensembles considered here.

We have also studied a class of noisy systems very similar to Eq. (5) and to ensembles with parametric noise [Eq. (1)]. They are given as

$$x_{n+1}(i) = f(x_n(i);a) + \sigma\eta_n^i , \quad (6)$$

where σ is the strength of the noise and η is a uniformly distributed random number. Here too we consider different cases analogous to the ones given for Eq. (1), determined by the time and/or space dependence of the noise. Numerical studies on Eq. (6) show results very similar to that obtained on ensembles with parametric noise: When the fluctuations are spatiotemporal, i.e., the maps evolve under noise which varies randomly from site to site at every instant of time, we observe statistical behavior in the mean field, as reflected in its conformity to the law of large numbers; but, when all the elements are subject to the same noise, which is random temporally [analogous to case (ii) of Eq. (1)], the mean-field dynamics exhibits clear nonstatistical behavior (see Fig. 4). *Thus, partial coherence in global dynamics is suggested through the persisting size-independent fluctuations of the mean field, found earlier to be a result of global coupling, is now found to emerge from spatially synchronized temporal noise as well.*

In summary, we have studied the mean-field properties of an ensemble of uncoupled chaotic maps evolving under synchronized noisy dynamics (determined by the spatially uniform random fluctuations of the nonlinearity parameter in the individual elements). Such a composite system is shown to display marked nonstatistical features characterized by the violation of the law of large numbers, reminiscent of the behavior in globally coupled chaotic maps. So partial global coherence, evident through the existence of size-independent fluctuations in the mean field, can emerge from spatially synchronized stochastic influences in uncoupled maps.

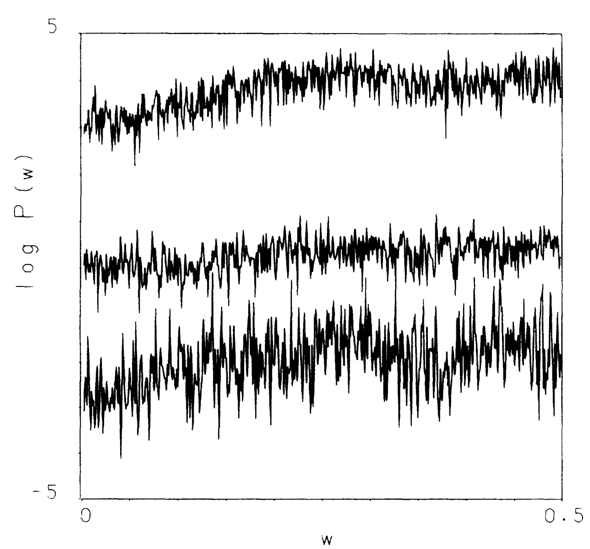

FIG. 3. Power spectra of the mean field for case (ii) of Eq. (1) with (from top to bottom) $a = 1.98$, $\sigma = 0.02$, $N = 10$; $a = 1.98$, $\sigma = 0.02$, $N = 2500$; and $a = 1.98$, $\sigma = 0.0$, $N = 2500$. Here we average over 10 runs of length 1024 each.

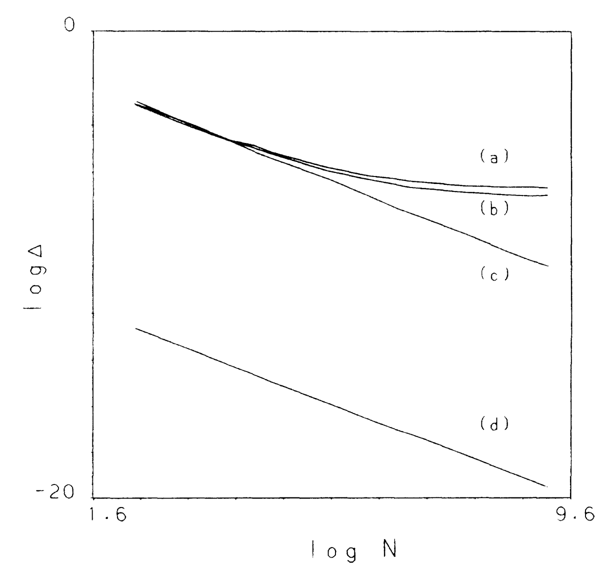

FIG. 4. Mean-square deviation Δ of the mean field vs system size N for four cases of Eq. (6) ($a = 1.98$): curve (*a*) $\sigma = 0.01$, (*b*) $\sigma = 0.005$, (*c*) $\sigma = 0.0$, and (*d*) $\sigma = 0.01$. In cases (*a*) and (*b*) the noise is spatially synchronized. In case (*d*) we have spatiotemporal noise. The Δ is calculated over 10000 iterations.

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[9] MSD of a particular ensemble of size N (N large) increases monotonically with noise strength σ [up to values of σ such that $a_n(i) < 2$]. This rise is approximately linear over a reasonably large range of noise strengths.

[10] This is in contrast to globally coupled systems, where coupled tent maps do not display the nonstatistical features found in many other coupled nonlinear maps, such as coupled logistic and circle maps. (See Ref. [4] for details.)

[11] The distribution of the mean field $P(h)$ was seen to be approximately Gaussian for the case of globally coupled logistic maps [3]. This feature too is similar to that found in our ensembles (see Fig. 1). Note, however, that in other globally coupled systems, it was observed that $P(h)$ was distinctly non-Gaussian [5,8]. It can then be expected that in chaotic ensembles [Eqs. (1) and (6)] with local maps very different from the logistic map, $P(h)$ may deviate from Gaussian.