An efficient control algorithm for nonlinear systems

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We suggest a scheme to step up the efficiency of a recently proposed adaptive control algorithm, which is remarkably effective for regulating nonlinear systems. The technique involves monitoring of the "stiffness of control" to get maximum gain while maintaining a predetermined accuracy. The success of the procedure is demonstrated for the case of the logistic map, where we show that the improvement in performance is often factors of tens, and for small control stiffness, even factors of hundreds.

Realistic models of a variety of physical, chemical and biological systems are given by coupled nonlinear equations, and display a wide repertoire of dynamics – ranging from fixed points to chaos. It is usually possible to identify certain physical quantities in the system as state variables, and others (that are relatively more invariant) as parameters. Generically the the nature of the dynamics is governed by values of these parameters and in real systems they may be quantities like electric fields, temperature, pressure gradients, pH, molarity etc. Now the parameters, in principle, can vary, driven by fluctuations in the environment, and this may push the system to drastically different kinds of dynamic behaviour. Thus, it is of considerable interest to develop mechanisms of self-regulation or control, in systems intrinsically capable of very complicated dynamics, so that it is guaranteed to maintain a fixed activity (the "goal") even when subject to environmental fluctuations [1–3].

A simple adaptive control algorithm was recently proposed in ref [1], and developed and extended in ref. [2]. It was demonstrated that the algorithm was a powerful and robust tool for regulating multidimensional, multiparameter, strongly nonlinear systems. The procedure utilizes an error signal proportional to the difference between the goal output and the actual output of the system. This error signal drives the evolution of parameters which readjust so as to reduce the error to zero. For a general N-dimensional system

$$\dot{X}=F(X;\mu;t)\,, \tag{1}$$

where $X\equiv(X_1, X_2, ..., X_N)$ are the variables and $\mu\equiv(\mu_1, \mu_2, ..., \mu_M)$ are the parameters whose values determine the nature of the dynamics. The prescription for adaptive control is through the additional dynamics,

$$\dot{\mu}=\epsilon(X-X_S)\,, \tag{2}$$

where X_S is the desired steady state value and ϵ indicates the "stiffness of control". This technique is very effective in bringing the system back to its original dynamical state after a sudden perturbation in the system parameters changes its dynamical behaviour drastically. We call this scheme "adaptive" as in this algorithm the parameters (which determine the nature of the dynamics) self adjust or "adapt" themselves to yield the desired dynamics [#1]. Recovery time (defined as the time required to reach the desired state within finite precision) is crucially dependent on the value of ϵ. Numerical experiments showed that for small ϵ the recovery time was *always* inversely

[#1] A similar type of procedure is also referred to as "dynamic feedback control" in the literature.

proportional to the stiffness of control [#2].

In this Letter we introduce a scheme to enhance the efficiency of the above mentioned adaptive control algorithm. The idea is as follows: we would like the algorithm to exert some "adaptive control" over its own progress, by making frequent changes in the "stiffness of control", ϵ. The purpose is to achieve some predetermined accuracy in the minimum time. Ideally the algorithm should ensure that the system tip-toes by many small steps through treacherous parameter regimes and in a few great strides speed through smooth safe terrains. The resulting gains in efficiency (versus an algorithm where the ϵ is fixed throughout) cn be factors of two and sometimes, tens or more.

To achieve this, we propose the following method: we monitor at each step in the algorithm, how far we can safely increase the value of ϵ for the next step. Implementation of this involves a test which returns information on the error incurred in taking higher ϵ. If this is within acceptable limits of accuracy desired (and after all, in real and numerical experiments one can only demand *finite* accuracy) we increase the stiffness of control for the next adaptive control step. The computational effort required for the test is repaid handsomely in terms of decrease in time required for recovery.

The most straightforward flowchart for this principle is given below:

(1) Double the value of ϵ at step n.

(2) Evaluate μ via control equation (2) with 2ϵ.

(3) Evaluate μ via control equation (2) with ϵ, for *two* successive steps.

(4) Compare the values obtained in step 2 and 3.

(5) If the difference between the two is smaller than a given accuracy (usually taken to be the accuracy used to define "recovery") then go to step 1 and repeat the procedure.

(6) If not, the above iteration stops (as the value of ϵ can no longer be increased without compromising with demands of accuracy).

What one achieves by the above is that when the parameter space is smoothly and gently varying one can take jumps, via large stiffness parameters, towards the desired state (see fig. 1). This decreases the time required for recovery, *enormously*. When there are more than one parameter we get a vector at step 2 and 3, and can implement step 5 with the "worst offender" parameter, i.e. if any one of the parameters violates the accuracy bar the iteration stops.

This "variable ϵ" adaptive control algorithm is tested on the logistic map which is given as

$$X_{n+1} = \alpha X_n(1+X_n) . \tag{3}$$

The results are shown in table 1. It is clear that this

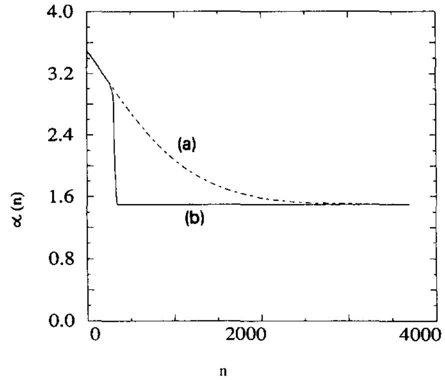

Fig. 1. The evolution of parameter α under control dynamics given by (a) the "fixed ϵ" algorithm (- - - -), and (b) the "variable ϵ" algorithm (——).

Table 1
Recovery times τ_1 and τ_2 are obtained from the "fixed ϵ" [1,2] and "variable ϵ" adaptive control algorithms respectively. Here ϵ denotes the "stiffness of control" in the control equation (2). it is evident that the "variable ϵ" scheme yields much faster recovery, as manifested in the enormously smaller values of τ. The representative example of the logistic map is considered here.

ϵ	τ_1	τ_2
0.1	180	70
0.05	364	89
0.01	1839	217
0.005	3682	345
0.001	18428	394
0.0005	36860	394
0.0001	184313	426

[#2] An argument to account for the universality of the linear relationship between recovery time and stiffness of control, observed in a wide class of systems of varying complexity (in ref. [2]), was pointed out by Haake. The key point is that when ϵ is small compared to the time scales in the original dynamical system, we can use an *adiabatic* approximation, as $\dot{\mu}\to 0$. So eq. (1) yields $X(\mu)$ as a solution, plugging which into eq. (2) gives $\dot{\mu}=\epsilon[X(\mu)-X_S]$, from where it simply follows that recovery time will be proportional to $1/\epsilon$.

scheme yields much faster recovery than the simple "fixed ϵ" method. This is particularly true when ϵ values are low. In fact, this feature enhances the utility of the proposed technique for the following reason: for a real system the dynamics is seldom well known. So eq. (1) is essentially, often, a black box. One of the powerful features of adaptive control is that it does not, in the control equation (2), require explicit knowledge of the dynamical equations of the system it is trying to control. It requires as input only the desired set of state variables. Now it has been seen in extensive numerical experiments [2] that stepping up the control stiffness beyond a critical value actually retards recovery and with very high ϵ the system fails to recover. So when the time scales in the dynamical system to be controlled are not well known, to prevent breakdown of recovery, one should keep the value of ϵ as *small* as possible. So, the fact that the "variable ϵ" algorithm works remarkably well for low ϵ should prove very useful in practical terms.

In summary, we introduce an adaptive control algorithm using variable (adaptively regulated) stiffness of control, ϵ. This greatly enhances the efficiency of the older "fixed ϵ" algorithm. The gain in performance is drastic (factors of hundreds!) for low ϵ values. This procedure may then be of utility in designing more powerful control tools.

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Appendix

Here we present an analytical argument which guarantees that the control scheme will work for sufficiently small ϵ. We concentrate on a one-dimensional system,

$$\dot{X}=F(X;\mu)\,, \tag{4}$$

with one parameter μ, an example of which is the logistic map in eq. (3) above [#3]. Now, the control algorithm given by eq. (2) leads to an augmented dynamical system consisting of the original dynamical system and an additional control equation, which is coupled to the original system by feedback. First, one must ensure by construction that the fixed point of the control system is the desired fixed point. This is ensured by the most general form of the control equation,

$$\dot{\mu}=\epsilon g(X)\,, \tag{5}$$

where $g(X=X_S)=0$ and X_S is the desired state. The explicit form $g(X)=X-X_S$ used in eq. (2) satisfies this. Other forms of $g(X)$ have been examined in ref. [2]. (Of course, X_S must also be a solution of $\dot{X}=0$, i.e. it must be a fixed point of the original system as well.) This formulation assures us that the control is directed towards the desired dynamics. But it still does not guarantee that the dynamics is stable under (nonlinear) control, near the goal dynamics. It is easy to see that the value of ϵ has direct bearing on the stability characteristics. To determine this exactly we examine the eigenvalues of the Jacobian matrix

$$J=\begin{vmatrix} \partial F/\partial X & \partial F/\partial\mu \\ \epsilon\partial g/\partial X & \epsilon\partial g/\partial\mu \end{vmatrix}\,. \tag{6}$$

Now $\partial g/\partial\mu=0$, and so in the limit $\epsilon\to 0$, we obtain a triangular matrix. The eigenvalues of J then are 0 and $\partial F/\partial X$, and $|\partial F/\partial X|<1$ at X_S if the desired state is a stable fixed point of the original system. *So, in the very low ϵ limit, if the goal dynamics is a stable state of the original dynamics, then the control dynamics is also stable.*

So, if we can ensure that the value of ϵ is small compared to the time scales in the original dynamical system, we are guaranteed that the algorithm will work. Usually one does not have a good estimate of how small ϵ should be, to be small enough. The procedure suggested in this Letter gets around this problem by "experimentally" finding out how large a value of ϵ is acceptable. Hence the utility of the scheme.

[#3] The argument can be extended to higher dimensions as well, and will be presented in detail in a subsequent work.

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