Mathematical Methods I

Assignment 2

Due on September 18, 2012

1. Determine if the following are vector spaces with the usual rules for addition and multiplication by scalars and if not, state which axiom(s) they violate:
   (a) Quadratic polynomials of the form $ax^2 + bx$
   (b) Quadratic polynomials of the form $ax^2 + bx + 1$
   (c) Quadratic polynomials of the form $ax^2 + bx + c$ with $a + b + c = 0$
   (d) Quadratic polynomials of the form $ax^2 + bx + c$ with $a + b + c = 1$

2. Show that the set of all polynomials in x having degree $\leq N$ and only even powers, is a vector space and find its dimension. What if we allowed only odd powers instead ?

3. For the vectors in three dimensions: $\mathbf{a_1} = \hat{x} + \hat{y}$, $\mathbf{a_2} = \hat{y} + \hat{z}$ and $\mathbf{a_3} = \hat{z} + \hat{x}$, use the Gram-Schmidt procedure to construct an orthonormal basis starting from $\mathbf{a_1}$.

4. Consider the vector space formed by all polynomials of degree ≤ 4 satisfying $\int_{-1}^{1} dx x f(x) = 0$. Does the set $[1, x^2, x^4, 3x - 5x^3]$ form a complete basis for the space ? What is the dimensionality of the space ? If we did not have the constraint $\int_{-1}^{1} dx x f(x) = 0$ what would be the dimensionality of the space ?

5. If $f_3 = f_1 + f_2$ means $f_3(x) = Af_1(x - a) + BF_2(x - b)$ for fixed a, b, A, B, for what values of these constants is this a vector space ?

6. On the vector space of cubic polynomials, the differentiation operator d/dx is defined as the derivative of such a polynomial is also a polynomial.
   (a) Use the basis $[1, x, x^2, x^3]$ to compute components of this operator
   (b) Compute the components of the double differentiation operator d^2/dx^2
   (c) Compute the square of the first matrix and compare it with the second matrix - how are they related ?
   (d) Use another basis, viz, that of the Legendre polynomials $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3/2)x^2 - (1/2)$, $P_3(x) = (5/2)x^3 - (3/2)x$ and find the components of d/dx and d^2/dx^2 operators.

7. The force on a charge q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$. The operation $\mathbf{v} \times \mathbf{B}$ defines a linear operator on $\mathbf{v}$ stated as $f(\mathbf{v}) = \mathbf{v} \times \mathbf{B}$.
   (a) What are the components of this operator expressed in terms of the three components of the vector $\mathbf{B}$?
   (b) What are the eigenvalues and eigenvectors of this operator ? (Be careful to choose the appropriate basis to reduce the amount of algebra)

8. Write the characteristic polynomial corresponding to the eigenvalue equation of a general 2×2 matrix A. In place of the eigenvalue λ in the polynomial put the matrix A itself (the constant term will have to include the identity matrix factor I) and verify the Cayley-Hamilton Theorem, i.e., that the matrix satisfies its own characteristic equation making the polynomial in A the zero matrix.

9. Obtain the eigenvalues and eigenvectors of the rotation matrix.