taining the highest order derivative drops out of the governing equation. Then the initial conditions or boundary conditions can't be satisfied. Such a limit is often called *singular*. For example, in fluid mechanics, the limit of high Reynolds number is a singular limit; it accounts for the presence of extremely thin "boundary layers" in the flow over airplane wings. In our problem, the rapid transient played the role of a boundary layer—it is a thin layer of *time* that occurs near the boundary $t = 0$.

The branch of mathematics that deals with singular limits is called *singular perturbation theory*. See Jordan and Smith (1987) or Lin and Segel (1988) for an introduction. Another problem with a singular limit will be discussed briefly in Section 7.5.

3.6 Imperfect Bifurcations and Catastrophes

As we mentioned earlier, pitchfork bifurcations are common in problems that have a symmetry. For example, in the problem of the bead on a rotating hoop (Section 3.5), there was a perfect symmetry between the left and right sides of the hoop. But in many real-world circumstances, the symmetry is only approximate—an imperfection leads to a slight difference between left and right. We now want to see what happens when such imperfections are present.

For example, consider the system

$$\dot{x} = h + rx - x^3. \tag{1}$$

If $h = 0$, we have the normal form for a supercritical pitchfork bifurcation, and there's a perfect symmetry between x and $-x$. But this symmetry is broken when $h \neq 0$; for this reason we refer to h as an ***imperfection parameter***.

Equation (1) is a bit harder to analyze than other bifurcation problems we've considered previously, because we have *two* independent parameters to worry about (h and r). To keep things straight, we'll think of r as fixed, and then examine the effects of varying h. The first step is to analyze the fixed points of (1). These can be found explicitly, but we'd have to invoke the messy formula for the roots of a cubic equation. It's clearer to use a graphical approach, as in Example 3.1.2. We plot the graphs of $y = rx - x^3$ and $y = -h$ on the same axes, and look for intersections (Figure 3.6.1). These intersections occur at the fixed points of (1). When $r \leq 0$, the cubic is monotonically decreasing, and so it intersects the horizontal line $y = -h$ in exactly one point (Figure 3.6.1a). The more interesting case is $r > 0$; then one, two, or three intersections are possible, depending on the value of h (Figure 3.6.1b).

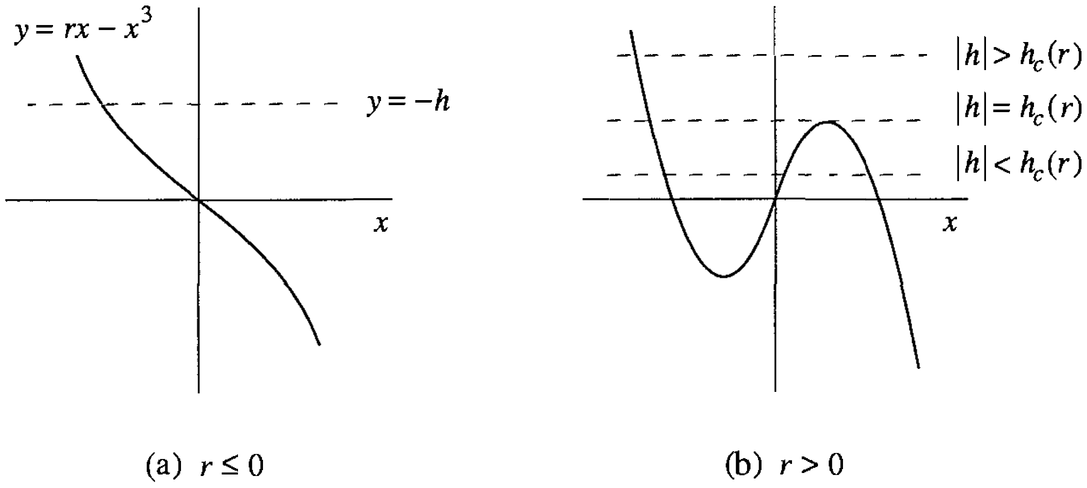

Figure 3.6.1

The critical case occurs when the horizontal line is just *tangent* to either the local minimum or maximum of the cubic; then we have a *saddle-node bifurcation*. To find the values of h at which this bifurcation occurs, note that the cubic has a local maximum when $\frac{d}{dx}(rx - x^3) = r - 3x^2 = 0$. Hence

$$x_{\text{max}} = \sqrt{\frac{r}{3}},$$

and the value of the cubic at the local maximum is

$$rx_{\text{max}} - (x_{\text{max}})^3 = \frac{2r}{3}\sqrt{\frac{r}{3}}.$$

Similarly, the value at the minimum is the negative of this quantity. Hence saddle-node bifurcations occur when $h = \pm h_c(r)$, where

$$h_c(r) = \frac{2r}{3}\sqrt{\frac{r}{3}}.$$

Equation (1) has three fixed points for $|h| < h_c(r)$ and one fixed point for $|h| > h_c(r)$.

To summarize the results so far, we plot the ***bifurcation curves*** $h = \pm h_c(r)$ in the (r, h) plane (Figure 3.6.2). Note that the two bifurcation curves meet tangentially at $(r, h) = (0, 0)$; such a point is called a ***cusp point***. We also label the regions that correspond to different numbers of fixed points. Saddle-node bifurcations occur all along the boundary of the regions, except at the cusp point, where we have a *codimension-2 bifurcation*. (This fancy terminology essentially means that we have had to tune *two* parameters, h and r, to achieve this type of bifurcation. Until now, all our bifurcations could be achieved by tuning a single parameter, and were therefore *codimension-1* bifurcations.)

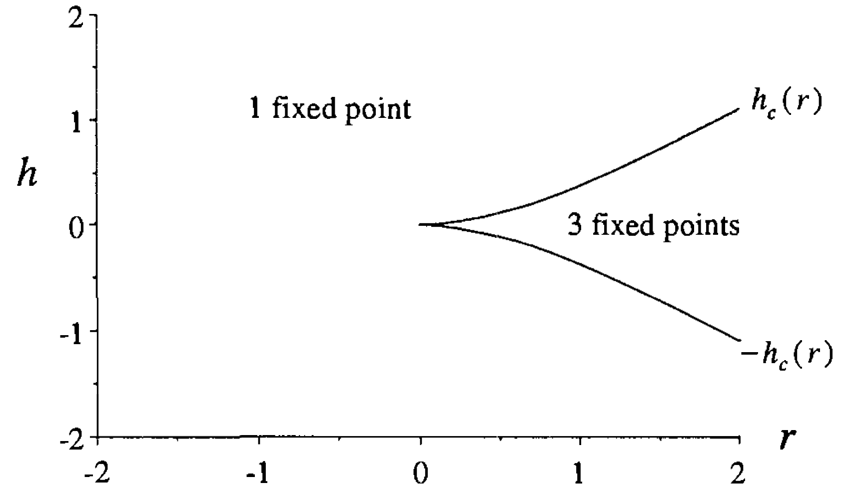

Figure 3.6.2

Pictures like Figure 3.6.2 will prove very useful in our future work. We will refer to such pictures as ***stability diagrams***. They show the different types of behavior that occur as we move around in ***parameter space*** (here, the (r,h) plane).

Now let's present our results in a more familiar way by showing the bifurcation diagram of x^* vs. r, for fixed h (Figure 3.6.3).

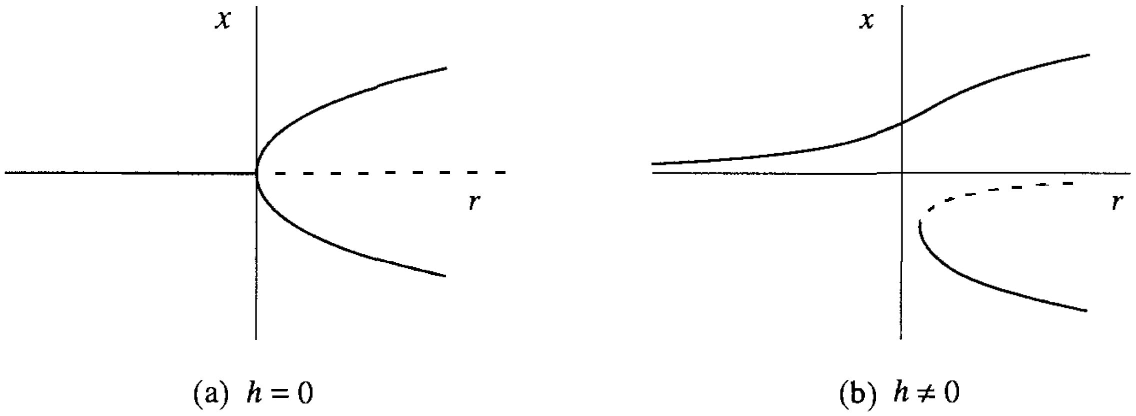

Figure 3.6.3

When $h=0$ we have the usual pitchfork diagram (Figure 3.6.3a) but when $h \neq 0$, the pitchfork disconnects into two pieces (Figure 3.6.3b). The upper piece consists entirely of stable fixed points, whereas the lower piece has both stable and unstable branches. As we increase r from negative values, there's no longer a sharp transition at $r=0$; the fixed point simply glides smoothly along the upper branch. Furthermore, the lower branch of stable points is not accessible unless we make a fairly large disturbance.

Alternatively, we could plot x^* vs. h, for fixed r (Figure 3.6.4).

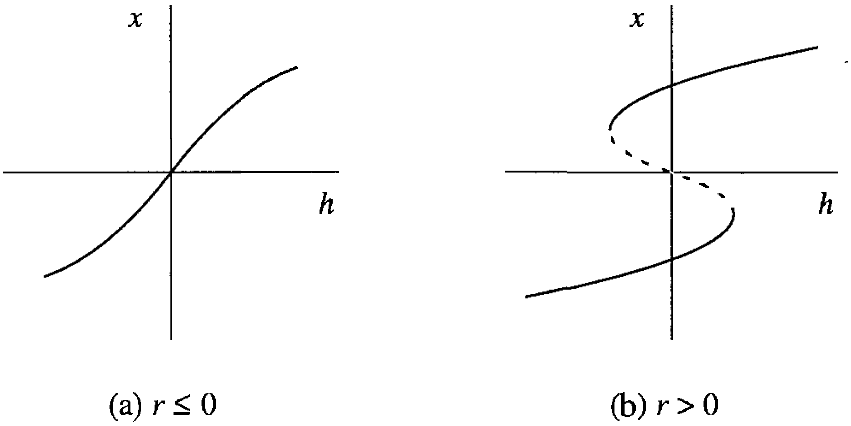

Figure 3.6.4

When $r \le 0$ there's one stable fixed point for each h (Figure 3.6.4a). However, when $r > 0$ there are three fixed points when $|h| < h_c(r)$, and one otherwise (Figure 3.6.4b). In the triple-valued region, the middle branch is unstable and the upper and lower branches are stable. Note that these graphs look like Figure 3.6.1 rotated by 90°.

There is one last way to plot the results, which may appeal to you if you like to picture things in three dimensions. This method of presentation contains all of the others as cross sections or projections. If we plot the fixed points x^* above the (r, h) plane, we get the ***cusp catastrophe*** surface shown in Figure 3.6.5. The surface folds over on itself in certain places. The projection of these folds onto the (r, h) plane yields the bifurcation curves shown in Figure 3.6.2. A cross section at fixed h yields Figure 3.6.3, and a cross section at fixed r yields Figure 3.6.4.

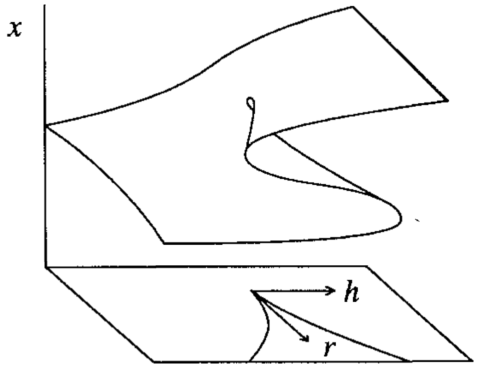

Figure 3.6.5

The term *catastrophe* is motivated by the fact that as parameters change, the state of the system can be carried over the edge of the upper surface, after which it drops discontinuously to the lower surface (Figure 3.6.6). This jump could be truly catastrophic for the equilibrium of a bridge or a building. We will see scientific examples of catastrophes in the context of insect outbreaks (Section 3.7) and in the following example from mechanics.

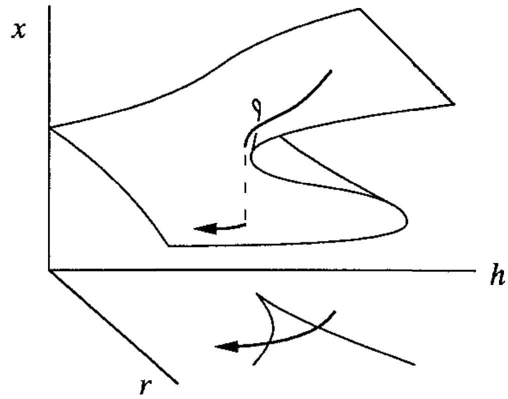

Figure 3.6.6

For more about catastrophe theory, see Zeeman (1977) or Poston and Stewart (1978). Incidentally, there was a violent controversy about this subject in the late