When $N_0 < k/G$, the fixed point at $n^* = 0$ is stable. This means that there is no stimulated emission and the laser acts like a lamp. As the pump strength N_0 is increased, the system undergoes a transcritical bifurcation when $N_0 = k/G$. For $N_0 > k/G$, the origin loses stability and a stable fixed point appears at $n^* = (GN_0 - k)/\alpha G > 0$, corresponding to spontaneous laser action. Thus $N_0 = k/G$ can be interpreted as the ***laser threshold*** in this model. Figure 3.3.3 summarizes our results.

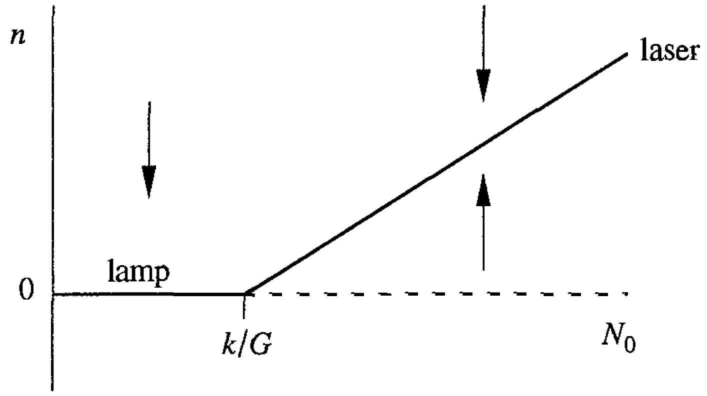

Figure 3.3.3

Although this model correctly predicts the existence of a threshold, it ignores the dynamics of the excited atoms, the existence of spontaneous emission, and several other complications. See Exercises 3.3.1 and 3.3.2 for improved models.

3.4 Pitchfork Bifurcation

We turn now to a third kind of bifurcation, the so-called pitchfork bifurcation. This bifurcation is common in physical problems that have a ***symmetry.*** For example, many problems have a spatial symmetry between left and right. In such cases, fixed points tend to appear and disappear in symmetrical pairs. In the buckling example of Figure 3.0.1, the beam is stable in the vertical position if the load is small. In this case there is a stable fixed point corresponding to zero deflection. But if the load exceeds the buckling threshold, the beam may buckle to *either* the left or the right. The vertical position has gone unstable, and two new symmetrical fixed points, corresponding to left- and right-buckled configurations, have been born.

There are two very different types of pitchfork bifurcation. The simpler type is called *supercritical*, and will be discussed first.

Supercritical Pitchfork Bifurcation

The normal form of the supercritical pitchfork bifurcation is

$$\dot{x} = rx - x^3. \qquad (1)$$

Note that this equation is ***invariant*** under the change of variables $x \to -x$. That is, if we replace x by $-x$ and then cancel the resulting minus signs on both sides of the equation, we get (1) back again. This invariance is the mathematical expression of the left–right symmetry mentioned earlier. (More technically, one says that the vector field is *equivariant*, but we'll use the more familiar language.)

Figure 3.4.1 shows the vector field for different values of r.

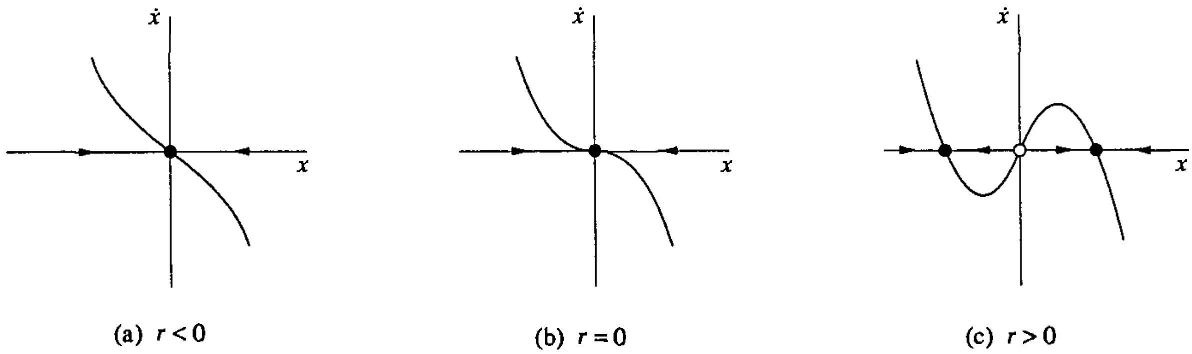

Figure 3.4.1

When $r < 0$, the origin is the only fixed point, and it is stable. When $r = 0$, the origin is still stable, but much more weakly so, since the linearization vanishes. Now solutions no longer decay exponentially fast—instead the decay is a much slower algebraic function of time (recall Exercise 2.4.9). This lethargic decay is called ***critical slowing down*** in the physics literature. Finally, when $r > 0$, the origin has become unstable. Two new stable fixed points appear on either side of the origin, symmetrically located at $x^* = \pm\sqrt{r}$.

The reason for the term "pitchfork" becomes clear when we plot the bifurcation diagram (Figure 3.4.2). Actually, pitchfork trifurcation might be a better word!

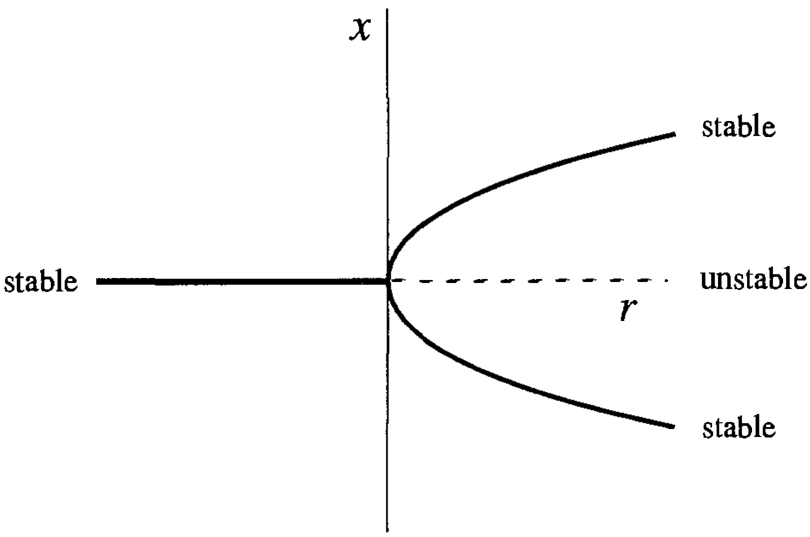

Figure 3.4.2

EXAMPLE 3.4.1:

Equations similar to $\dot{x} = -x + \beta \tanh x$ arise in statistical mechanical models of magnets and neural networks (see Exercise 3.6.7 and Palmer 1989). Show that this equation undergoes a supercritical pitchfork bifurcation as β is varied. Then give a *numerically accurate* plot of the fixed points for each β.

Solution: We use the strategy of Example 3.1.2 to find the fixed points. The graphs of $y = x$ and $y = \beta \tanh x$ are shown in Figure 3.4.3; their intersections correspond to fixed points. The key thing to realize is that as β increases, the tanh curve becomes steeper at the origin (its slope there is β). Hence for $\beta < 1$ the origin is the only fixed point. A pitchfork bifurcation occurs at $\beta = 1$, $x^* = 0$, when the tanh curve develops a slope of 1 at the origin. Finally, when $\beta > 1$, two new stable fixed points appear, and the origin becomes unstable.

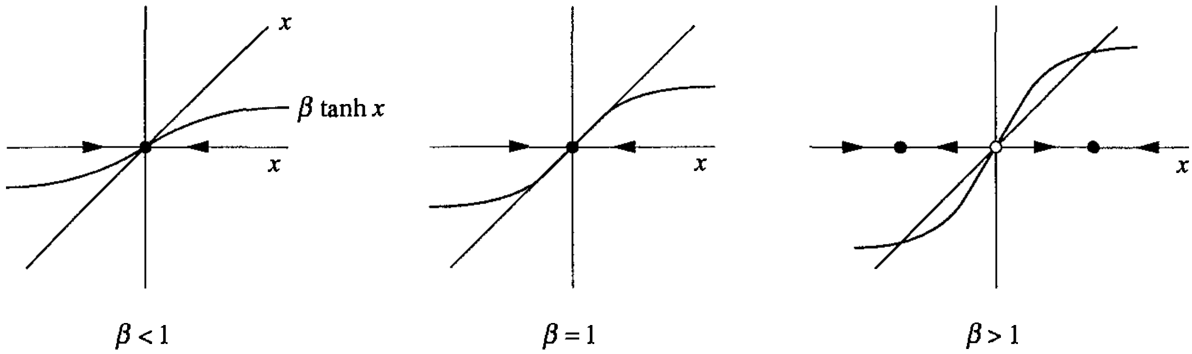

Figure 3.4.3

Now we want to compute the fixed points x^* for each β. Of course, one fixed point always occurs at $x^* = 0$; we are looking for the other, nontrivial fixed points. One approach is to solve the equation $x^* = \beta \tanh x^*$ numerically, using the Newton–Raphson method or some other root-finding scheme. (See Press et al. (1986) for a friendly and informative discussion of numerical methods.)

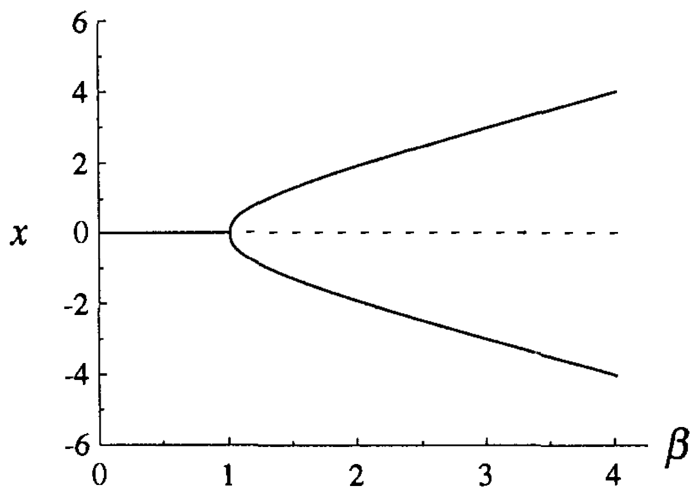

Figure 3.4.4

But there's an easier way, which comes from changing our point of view. Instead of studying the dependence of x^* on β, we think of x^* as the *independent* variable, and then compute $\beta = x^*/\tanh x^*$. This gives us a table of pairs (x^*, β). For each pair, we plot β horizontally and x^* vertically. This yields the bifurcation diagram (Figure 3.4.4).

The shortcut used here exploits the fact that $f(x,\beta) = -x + \beta \tanh x$ depends more simply on β than on x. This is frequently the case in bifurcation problems—the dependence on the control parameter is usually simpler than the dependence on x. ■

EXAMPLE 3.4.2:

Plot the potential $V(x)$ for the system $\dot{x} = rx - x^3$, for the cases $r < 0$, $r = 0$, and $r > 0$.

Solution: Recall from Section 2.7 that the potential for $\dot{x} = f(x)$ is defined by $f(x) = -dV/dx$. Hence we need to solve $-dV/dx = rx - x^3$. Integration yields $V(x) = -\frac{1}{2}rx^2 + \frac{1}{4}x^4$, where we neglect the arbitrary constant of integration. The corresponding graphs are shown in Figure 3.4.5.

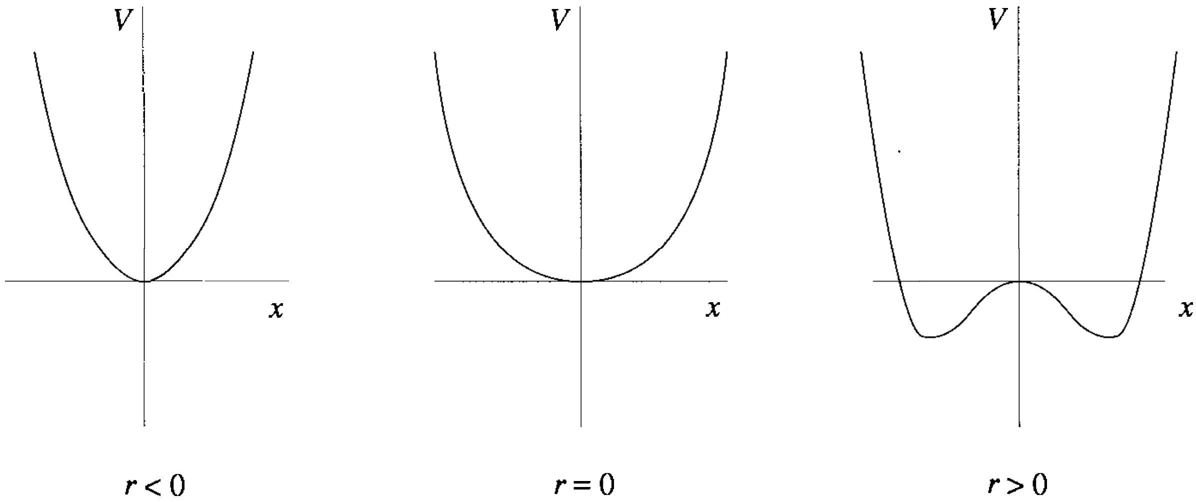

Figure 3.4.5

When $r < 0$, there is a quadratic minimum at the origin. At the bifurcation value $r = 0$, the minimum becomes a much flatter quartic. For $r > 0$, a local *maximum* appears at the origin, and a symmetric pair of minima occur to either side of it. ■

Subcritical Pitchfork Bifurcation

In the supercritical case $\dot{x} = rx - x^3$ discussed above, the cubic term is *stabilizing*: it acts as a restoring force that pulls $x(t)$ back toward $x = 0$. If instead the cubic term were *destabilizing*, as in

$$\dot{x} = rx + x^3, \qquad (2)$$

then we'd have a ***subcritical*** pitchfork bifurcation. Figure 3.4.6 shows the bifurcation diagram.