The Institute of Mathematical Sciences, Chennai

Advanced Statistical Mechanics (IIIrd Semester Course)
END-TERM EXAMINATION
November 29, 2017
Duration: 3 hours

ATTEMPT ALL QUESTIONS

1. Combinatorics:

(a) Consider a language which has m different vowels and n different consonants (e.g., English will have m=5 and n=21). Let us assume that for ease of pronunciation, a word cannot have two vowels or two consonants occurring next to each other. Given this constraint how many different words of length q (i.e., consisting of q letters) are possible in the language? Give the number of distinct words for q=2, 3,4 and 5 and write down the general expression for any q.

(b) Now remove the restriction that two vowels or consonants cannot be next to each other, so that any letter can occur next to any other letter. In such a situation, find the number of possible words of length 7 which have exactly 4 consonants and 3 vowels.

(c) In problem (b), in how many arrangements will we have a cluster of 3 consonants [i.e., 3 consonants that are next to each other]? Note that the cluster of 3 consonants cannot have another consonant as a neighbor as in that case it will be a cluster of 4 consonants.

2. Distributions:

A star typically produces a large number N (>>1) of photons during the period in which it is under observation by an astronomical observatory that detects only a tiny fraction p (<<1) of all the photons that are being emitted in every direction by the star.

(a) If the astronomer makes many observations of the star, each observation being of the same duration, what will be the frequency distribution of photons collected from the star per observation?

(b) What will be the mean number of photons detected per observation?

3. Conditional probability:

You have gone to the airport to surprise your friend who you have to come to know is coming by an early morning flight. There have been a few incidents in the past in which your friend has overslept and missed important appointments – so that there is a small (5%) chance that he/she may miss the flight (*prior* probability). As you watch passengers from the flight come out of the airport terminal, there is no sight of your friend. Unwilling to spoil the surprise by calling your friend on his/her cellphone, you keep vigil hoping to see him/her soon. After what fraction of passengers from the flight have left the terminal building, will you conclude that there is a higher than 50% probability (*posterior* probability) that your friend was actually not on the flight?

4. Energy function:

Plot the free energy of the mean-field Ising model as a function of magnetization for the two cases (A) $T<T_c$ and (B) $T>T_c$ where T_c is the critical temperature. For each case give plots for the three sub-cases (i) $H<0$, (ii) $H=0$ and (iii) $H>0$ where H is the applied magnetic field.

5. Low-temperature behavior of Ising model:
Consider the 2-dimensional Ising model with N spins in the absence of magnetic field and at extremely low temperature (close to T=0). The Hamiltonian of the system is written as
$H = -J \; \Sigma_{i,j} \; s_i \; s_j$ where the sum is over nearest neighbor sites *i,j* in the lattice.

(a) The ground state corresponds to all spins aligned in the same direction (i.e., all up or all down, so that the degeneracy of this state is 2). What is the ground state energy E_0? (Remember to avoid double counting of the same bond)

(b) To get the first excited state (lowest energy excitation from the ground state), we can flip a single spin in the direction opposite to all other spins in the lattice. What is (i) the energy E_1 of this state and (ii) its degeneracy? Also say (iii) how many misaligned *pairs* of spins (i.e., the two spins in the pair point in opposite direction) are there?

(c) The next higher energy state (second excited state) is obtained by going back to the ground state and flipping two *neighboring* spins in the direction opposite to all other spins in the lattice. What is (i) the energy E_2 of this state and (ii) its degeneracy? Also say (iii) how many misaligned *pairs* of spins (i.e., the two spins in the pair point in opposite direction) are there?

(d) Consider now the third excited state. One configuration for this state corresponds to two non-neighboring spins flipped in the direction opposite to all other spins in the lattice. What is (i) the energy E_2 of this state and (ii) its degeneracy? Also say (iii) how many misaligned *pairs* of spins (i.e., the two spins in the pair point in opposite direction) are there?

(e) What other different configurations are possible which have the same energy E_3? [Note that such configurations will also have the same number of misaligned spin pairs]. Draw the possible configurations and indicate what their degeneracies are.

(f) Using the information gained from (a-e) write down the partition function *Z* of the system in powers of $e^{-\beta J}$ (<<1) upto the third excited state. [Remember that the ground state has a degeneracy of 2, corresponding to all up or all down spins].

6. Renormalization Group (RG) transformation:

Consider the RG equations in momentum space for the Landau-Wilson model:

$dx/d\tau = 2x + 12y(1 - x)$, $dy/d\tau = \varepsilon y - 36\, y^2$. [$\varepsilon$ is a parameter]

(a) How many fixed points (x^*,y^*) does this system of coupled equations have? Write down all the fixed points. [Recall that at the fixed points $dx/d\tau = 0$, $dy/d\tau = 0$]

(b) Linearize the RG equations around each of the fixed points to obtain the matrix (i.e., Jacobian) corresponding to the linearized recursion relations.

(c) What will be the eigenvalues for the matrix in the neighborhood of each of the fixed points?

(d) Based on the eigenvalues can you comment on the nature (stability/instability) of the different fixed points for different choices of the value of the parameter ε?

7. Percolation clusters:

In class we have worked out the possible configurations of percolation clusters upto 4-clusters in a 2-dimensional lattice (i.e., all possible arrangements of 4 nearest neighbor sites that are occupied). Show how many possible configurations are possible for 5-clusters.

Note: Actually this problem is equivalent to finding out the number of all possible *pentominoes* (counting rotations of asymmetric pentominoes as distinct configurations) familiar to all players of the computer game *Tetris*.

8. Spin model on Bethe lattice:

Consider the Ising model on a Bethe lattice where each spin is interacting with spins on q nearest neighbor sites. Start from a given site as the origin (call it 0), i.e., assume it is the root of the Cayley tree which forms the Bethe lattice. The partition function is written as:
$Z = \Sigma_{\{s\}} \exp(K\, \Sigma_{i,j}\, s_i\, s_j + h\, \Sigma_i\, s_i)$
[$\Sigma_{\{s\}}$ represents sum over all spin configurations and $\Sigma_{i,j}$ represents sum over nearest neighbor spin pairs] where $K=J/k_B\, T$ and $h = H/\, k_B\, T$, with J and H being strength of exchange interaction coupling and magnetic field, respectively.
If the tree is cut at 0, it splits into q identical disconnected pieces, each of which is tree rooted at 0. Thus,
$Z = \Sigma_{\{s\}} \exp(hs_0)\, \Pi_{j=1..q}\, Q_n\, (s_0 \mid s^{(j)}\,)$
where $s^{(j)}$ denotes all spins (other than s_0 that are on the j-th subtree) and
$Q_n\, (s_0 \mid s^{(j)}\,) = \exp(K\, \Sigma_{p,q}\, s_p^{(j)}\, s_q^{(j)} + K\, s_1^{(j)}\, s_0 + h\, \Sigma_i\, s_i^{(j)}\,)$
where $s_i^{(j)}$ is the i-th spin on the j-th subtree and $s_1^{(j)}$ is the spin of the immediately neighboring site of the origin 0.

(a) Cut a sub-tree at site 1 thereby fragmenting it into q parts. Write $Q_n\, (s_0 \mid s^{(j)}\,)$ in terms of Q_{n-1} $(s_1 \mid t^{(m)}\,)$ where $t^{(m)}$ denotes all spins (other than s_1) lying on the m-th branch of the sub-tree.

(b) If $g_n(s_0) = \Sigma_{\{s\}}\, Q_n\, (s_0\, |s)$ show that $Z = \Sigma_{s0} \exp\, (h\, s_0)\, (g_n(s_0))^q$.

(c) Write the expression of the local magnetization $M = \langle s_0 \rangle$ using the above expression for *Z*.

(d) If $x_n=g_n(-)/g_n(+)$, show that the magnetization $M = [e^h - e^{-h} x_n^q]/ [e^h + e^{-h} x_n^q]$.

(e) To obtain the recursion relation for x_n, we note that summing $Q_n (s_0 |s)$ over all spin s – or effectively, over s_1 | and $t^{(m)}$ – we will get $g_n(s_0)= \Sigma_{s1} \exp (K s_0 s_1 + h s_0) (g_{n-1}(s_1))^{q-1}$ [you don't need to show this].
Recalling that s_0 and s_1 are single spins and each can take values +1 or –1, show that the recursion relation is $x_n = F (x_{n-1})$ where $F(z) = [e^{-K+h} - e^{K-h} z^{q-1}]/ [e^{K+h} - e^{-K-h} z^{q-1}]$.
[As $x_0 = g_0 (s_0) = 1$, we can now find x_n for any n and thus determine M.]