In the absence of damping and external driving, the motion of a pendulum is governed by

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0 \qquad (1)$$

where θ is the angle from the downward vertical, g is the acceleration due to gravity, and L is the length of the pendulum (Figure 6.7.1).

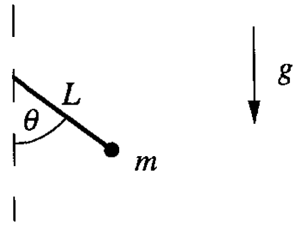

Figure 6.7.1

We nondimensionalize (1) by introducing a frequency $\omega = \sqrt{g/L}$ and a dimensionless time $\tau = \omega t$. Then the equation becomes

$$\ddot{\theta} + \sin\theta = 0 \qquad (2)$$

where the overdot denotes differentiation with respect to τ. The corresponding system in the phase plane is

$$\dot{\theta} = v \qquad (3a)$$
$$\dot{v} = -\sin\theta \qquad (3b)$$

where v is the (dimensionless) angular velocity.

The fixed points are $(\theta^*, v^*) = (k\pi, 0)$, where k is any integer. There's no physical difference between angles that differ by 2π, so we'll concentrate on the two fixed points $(0,0)$ and $(\pi,0)$. At $(0,0)$, the Jacobian is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so the origin is a linear center.

In fact, the origin is a *nonlinear* center, for two reasons. First, the system (3) is *reversible*: the equations are invariant under the transformation $\tau \to -\tau$, $v \to -v$. Then Theorem 6.6.1 implies that the origin is a nonlinear center.

Second, the system is also *conservative*. Multiplying (2) by $\dot{\theta}$ and integrating yields

$$\dot{\theta}(\ddot{\theta} + \sin\theta) = 0 \Rightarrow \tfrac{1}{2}\dot{\theta}^2 - \cos\theta = \text{constant}.$$

The energy function

$$E(\theta, v) = \tfrac{1}{2}v^2 - \cos\theta \qquad (4)$$

has a local minimum at (0,0), since $E \approx \frac{1}{2}(v^2 + \theta^2) - 1$ for small (θ, v). Hence Theorem 6.5.1 provides a second proof that the origin is a nonlinear center. (This argument also shows that the closed orbits are approximately *circular*, with $\theta^2 + v^2 \approx 2(E+1)$.)

Now that we've beaten the origin to death, consider the fixed point at $(\pi, 0)$. The Jacobian is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic equation is $\lambda^2 - 1 = 0$. Therefore $\lambda_1 = -1$, $\lambda_2 = 1$; the fixed point is a saddle. The corresponding eigenvectors are $\mathbf{v}_1 = (1, -1)$ and $\mathbf{v}_2 = (1, 1)$.

The phase portrait near the fixed points can be sketched from the information obtained so far (Figure 6.7.2).

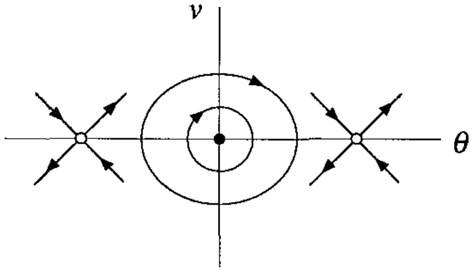

Figure 6.7.2

To fill in the picture, we include the energy contours $E = \frac{1}{2}v^2 - \cos\theta$ for different values of E. The resulting phase portrait is shown in Figure 6.7.3. The picture is periodic in the θ-direction, as we'd expect.

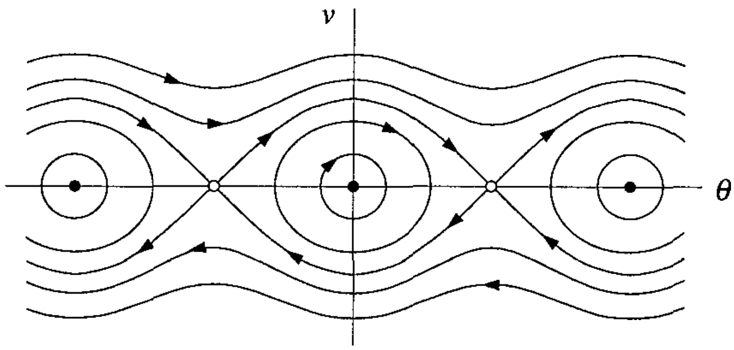

Figure 6.7.3

Now for the physical interpretation. The center corresponds to a state of neutrally stable equilibrium, with the pendulum at rest and hanging straight down. This is the lowest possible energy state $(E = -1)$. The small orbits surrounding the center represent small oscillations about equilibrium, traditionally called ***librations***. As E increases, the orbits grow. The critical case is $E = 1$, corresponding to the heteroclinic trajectories joining the saddles in Figure 6.7.3. The saddles represent an *inverted* pendulum at rest;

hence the heteroclinic trajectories represent delicate motions in which the pendulum slows to a halt precisely as it approaches the inverted position. For $E > 1$, the pendulum whirls repeatedly over the top. These ***rotations*** should also be regarded as periodic solutions, since $\theta = -\pi$ and $\theta = +\pi$ are the same physical position.

Cylindrical Phase Space

The phase portrait for the pendulum is more illuminating when wrapped onto the surface of a cylinder (Figure 6.7.4). In fact, a cylinder is the *natural* phase space for the pendulum, because it incorporates the fundamental geometric difference between v and θ: the angular velocity v is a real number, whereas θ is an *angle*.

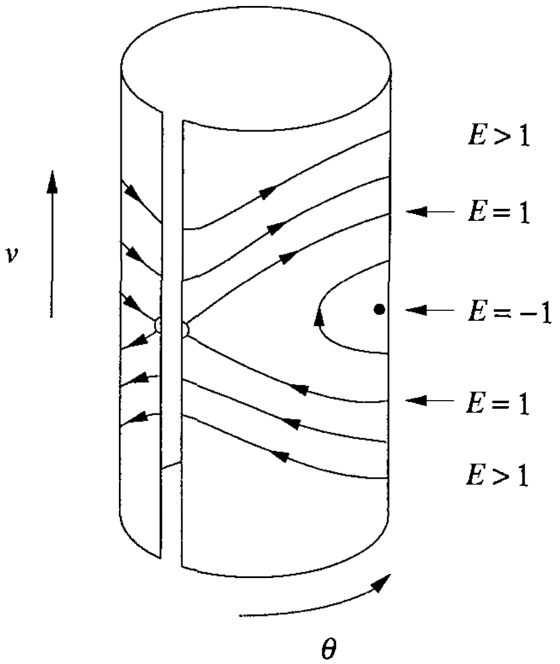

Figure 6.7.4

There are several advantages to the cylindrical representation. Now the periodic whirling motions *look* periodic—they are the closed orbits that encircle the cylinder for $E > 1$. Also, it becomes obvious that the saddle points in Figure 6.7.3 are all the same physical state (an inverted pendulum at rest). The heteroclinic trajectories of Figure 6.7.3 become homoclinic orbits on the cylinder.

There is an obvious symmetry between the top and bottom half of Figure 6.7.4. For example, both homoclinic orbits have the same energy and shape. To highlight this symmetry, it is interesting (if a bit mind-boggling at first) to plot the *energy* vertically instead of the angular velocity v (Figure 6.7.5). Then the orbits on the cylinder remain at constant height, while the cylinder gets bent into a ***U-tube***. The two arms of the tube are distinguished by the sense of rotation of the pendulum, either clockwise or counterclock-

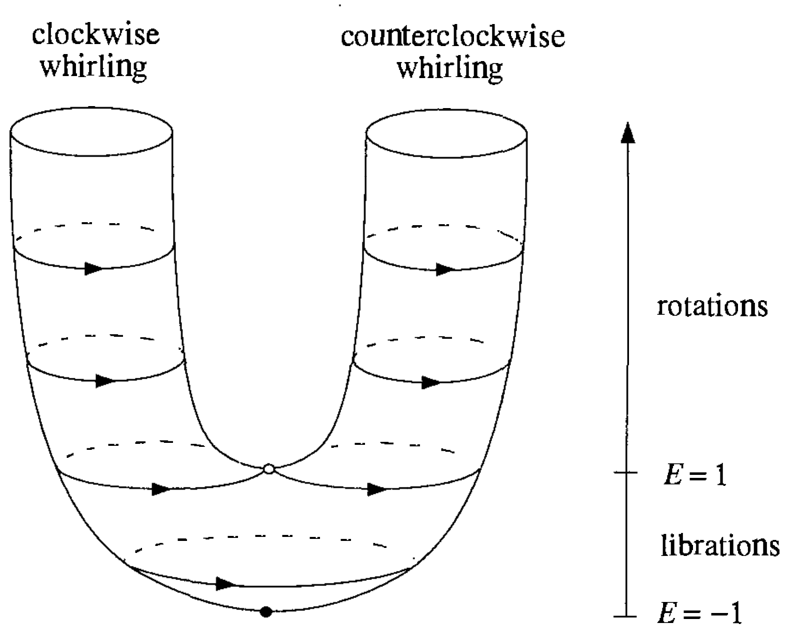

Figure 6.7.5

wise. At low energies, this distinction no longer exists; the pendulum oscillates to and fro. The homoclinic orbits lie at $E = 1$, the borderline between rotations and librations.

At first you might think that the trajectories are drawn incorrectly on one of the arms of the U-tube. It might seem that the arrows for clockwise and counterclockwise motions should go in *opposite* directions. But if you think about the coordinate system shown in Figure 6.7.6, you'll see that the picture is correct.

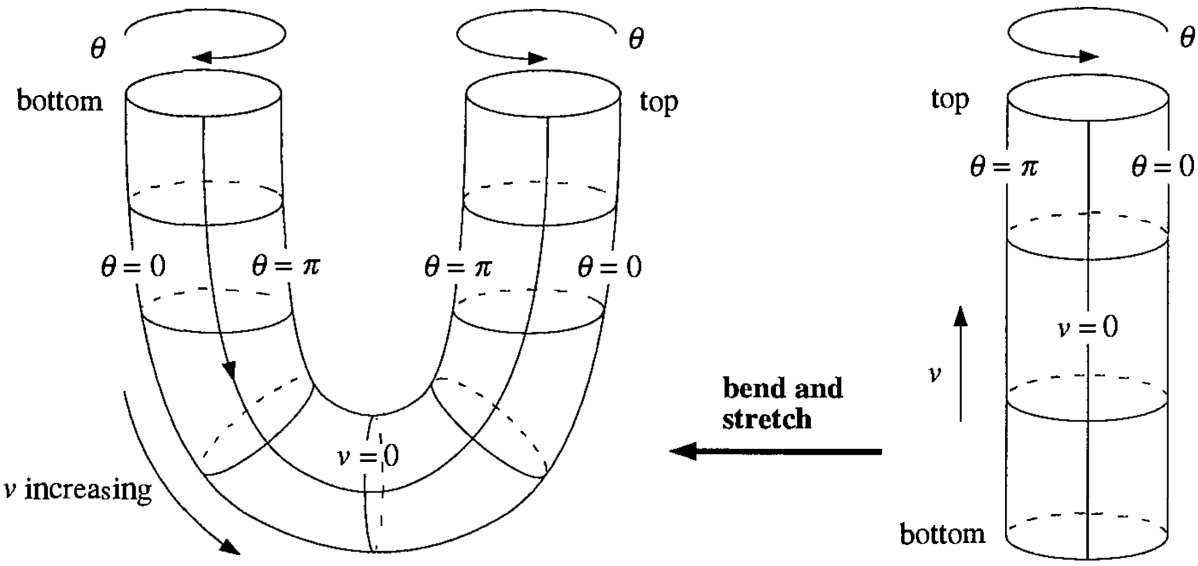

Figure 6.7.6

The point is that the direction of increasing θ has reversed when the bottom of the cylinder is bent around to form the U-tube. (Please understand that Figure 6.7.6 shows the coordinate system, not the actual trajectories; the trajectories were shown in Figure 6.7.5.)

Damping

Now let's return to the phase plane, and suppose that we add a small amount of linear damping to the pendulum. The governing equation becomes

$$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$$

where $b > 0$ is the damping strength. Then centers become stable spirals while saddles remain saddles. A computer-generated phase portrait is shown in Figure 6.7.7.

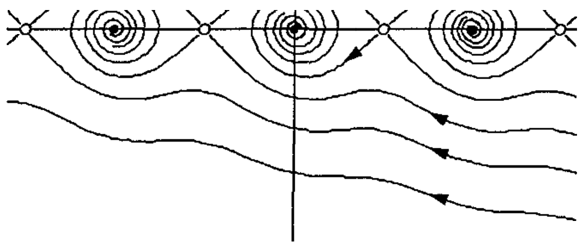

Figure 6.7.7

The picture on the U-tube is clearer. *All trajectories continually lose altitude,* except for the fixed points (Figure 6.7.8).

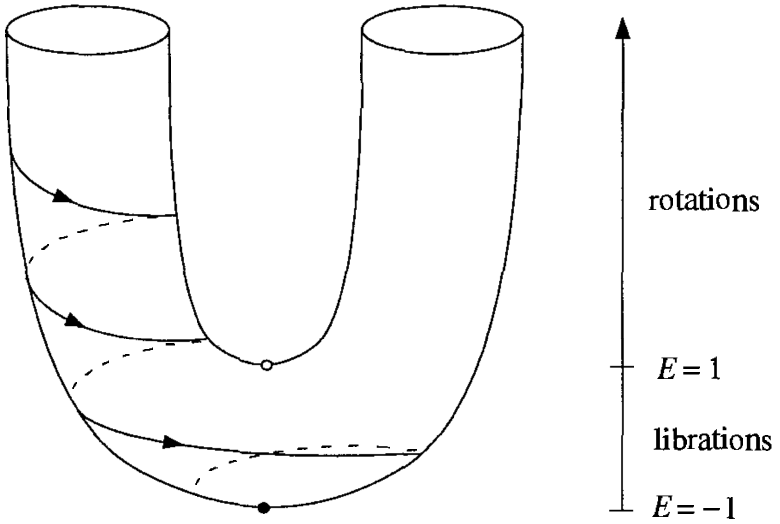

Figure 6.7.8

We can see this explicitly by computing the change in energy along a trajectory:

$$\frac{dE}{d\tau}=\frac{d}{d\tau}\left(\tfrac{1}{2}\dot{\theta}^2-\cos\theta\right)=\dot{\theta}\left(\ddot{\theta}+\sin\theta\right)=-b\dot{\theta}^2\leq 0.$$

Hence E decreases monotonically along trajectories, except at fixed points where $\dot{\theta}\equiv 0$.

The trajectory shown in Figure 6.7.8 has the following physical interpretation: the pendulum is initially whirling clockwise. As it loses energy, it has a harder time rotating over the top. The corresponding trajectory spirals down the arm of the U-tube until $E<1$; then the pendulum doesn't have enough energy to whirl, and so it settles down into a small oscillation about the bottom. Eventually the motion damps out and the pendulum comes to rest at its stable equilibrium.

This example shows how far we can go with pictures—without invoking any difficult formulas, we were able to extract all the important features of the pendulum's dynamics. It would be much more difficult to obtain these results analytically, and much more confusing to interpret the formulas, even if we *could* find them.

6.8 Index Theory

In Section 6.3 we learned how to linearize a system about a fixed point. Linearization is a prime example of a *local* method: it gives us a detailed microscopic view of the trajectories near a fixed point, but it can't tell us what happens to the trajectories after they leave that tiny neighborhood. Furthermore, if the vector field starts with quadratic or higher-order terms, the linearization tells us nothing.

In this section we discuss index theory, a method that provides *global* information about the phase portrait. It enables us to answer such questions as: Must a closed trajectory always encircle a fixed point? If so, what types of fixed points are permitted? What types of fixed points can coalesce in bifurcations? The method also yields information about the trajectories near higher-order fixed points. Finally, we can sometimes use index arguments to rule out the possibility of closed orbits in certain parts of the phase plane.

The Index of a Closed Curve

The index of a closed curve C is an integer that measures the winding of the vector field on C. The index also provides information about any fixed points that might happen to lie inside the curve, as we'll see.

This idea may remind you of a concept in electrostatics. In that subject, one often introduces a hypothetical closed surface (a "Gaussian surface") to probe a configuration of electric charges. By studying the behavior of the electric field on the surface, one can determine the total amount of charge *inside* the surface. Amazingly, the behavior *on* the surface tells us what's happening far away *inside* the surface! In the present context, the electric field is analogous to our vector field, the Gaussian surface is analogous to the curve C, and the total charge is analogous to the index.

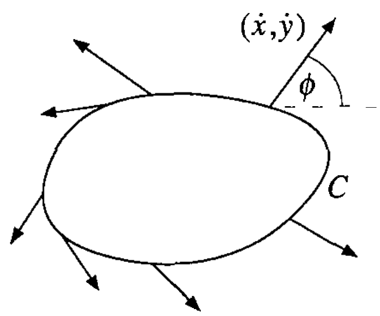

Figure 6.8.1

Now let's make these notions precise. Suppose that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a smooth vector field on the phase plane. Consider a closed curve C (Figure 6.8.1). This curve is *not* necessarily a trajectory—it's simply a loop that we're putting in the phase plane to probe the behavior of the vector field. We also assume that C is a