ON ISING'S MODEL OF FERROMAGNETISM

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Ising* discussed the following model of a ferromagnetic body: Assume N elementary magnets of moment μ to be arranged in a regular lattice; each of them is supposed to have only two possible orientations, which we call positive and negative. Assume further that there is an interaction energy U for each pair of neighbouring magnets of opposite direction. Further, there is an external magnetic field of magnitude H such as to produce an additional energy of $-\mu H$ $(+\mu H)$ for each magnet with positive (negative) direction.

Ising solved the statistical problem only in the one-dimensional case and showed that his model does not behave like a ferromagnetic body. For the purpose of this discussion a ferromagnetic body may be defined as having the following property: Consider a finite region of the body, containing n_+ positive and n_- negative magnets. Then, for all distributions that occur with appreciable chance in statistical equilibrium, the expression

$$\frac{n_+ - n_-}{n_+ + n_-}$$

tends to a finite non-vanishing value as $n_+ + n_-$ tends to ∞.

In the meantime it was shown by Heisenberg† that the forces leading to ferromagnetism are due to electron exchange. Therefore the energy function is of a more complicated nature than was assumed by Ising; it depends not only on the arrangement of the elementary magnets, but also on the speed with which they exchange their places.

The Ising model is therefore now only of mathematical interest. Since, however, the problem of Ising's model in more than one dimension has led to a good deal of controversy and in particular since the opinion has often been expressed that the solution of the three-dimensional problem could be reduced to that of the linear model and would lead to similar results, it may be worth while to give its solution.

* Ising, *Zeits. für Physik*, 31 (1925), 253.

† W. Heisenberg, *Zeits. für Physik*, 49 (1928), 619; for an account of the theory of Heisenberg and the extensions by Bloch and others, see F. Bloch, *Handbuch der Radiologie*, VI, 2 (Leipzig, 1934), 375.

This solution has now become almost obvious because a number of mathematically equivalent, although physically different, problems have been studied by Bragg and Williams, Borelius, Fowler, Bethe and others*. In particular Bethe has studied equations which with a suitable change of variables may be directly applied to the Ising model without a magnetic field. The generalization required for including a magnetic field is contained in a paper by the author†. We must replace, in the formulae of that paper, empty and occupied sites by − and + magnets and $2M-N=N(2\theta-1)$ by the total moment in units of μ. If X denotes again the number of neighbouring + magnets, it is easy to see that the number of pairs of − magnets is $z(\frac{1}{2}N-M)+X$ while there are $zM-2X$ pairs of neighbouring oppositely directed magnets (z denotes the number of neighbours of each lattice point). In the Ising model we ascribe the energy U to each pair of opposite neighbouring magnets and the total energy thus becomes

$$-2XU+zMU-\mu HM.$$

Comparing this with A, equations (2) and (4), we see that we have to replace ξ, η by

$$\xi=e^{-(zU+\mu H)/kT}, \quad \eta=e^{2U/kT}.$$

Equations (9) and (10) of A then represent also the solution of the Ising problem in Bethe's approximation and the discussion of the critical condition remains unchanged. The Curie point, i.e. the temperature above which the ferromagnetic properties vanish, is, from A, equation (14),

$$kT_0=\frac{U}{\log\{z/(z-2)\}}. \tag{1}$$

For $z=2$ (linear chain), T_0 vanishes, in agreement with Ising's result.

However, the above proof is not quite rigorous, since the method of A depends on Bethe's approximation, introduced by A, equation (5), and it might be argued that this is begging the question. Although the good convergence of Bethe's method is reassuring, it is worth while to give a rigorous proof for the ferromagnetic behaviour at low temperatures.

We give this proof for a two-dimensional square array ($z=4$), for which the argument becomes particularly simple, and for $H=0$.

Consider a square array of points arranged along the x and y direction, and ascribe a sign + or − to each point. It is then possible to define boundaries which separate the + from the − signs, such that, whenever two neighbours are of opposite sign, and only in this case, a boundary passes between them (cf. Fig. 1). The boundaries determine the distribution of magnets completely except for the possibility of replacing all positives by negatives and vice versa. If there is no external field, this change makes no physical difference.

* W. L. Bragg and E. J. Williams, *Proc. Roy. Soc.* A. 145 (1934), 699; Borelius, *Ann. d. Physik*, 20 (1934), 57; R. H. Fowler, *Proc. Roy. Soc.* A. 149 (1935), 1; H. A. Bethe, *Proc. Roy. Soc.* A. 150 (1935), 552.

† Cf. the preceding paper. This will be referred to as A.

There are closed boundaries and open ones, the latter starting and ending on the edges of the array. We show that at sufficiently low temperatures the area enclosed by closed boundaries and cut off by open ones is only a small fraction of the total area. Hence the majority of the magnets must be of equal sign and the model corresponds to our definition of a ferromagnet.

We give an upper limit for the number of boundary lines passing through any one point, assuming statistical equilibrium. Each boundary that starts at a given point may be described as a succession of steps of equal length, of which each may have the direction $+x$, $-x$, $+y$ or $-y$. Since each element of the boundary separates two magnets of opposite sign, the boundary has an energy LU, if L is its length, in units of the distance between neighbouring magnets.

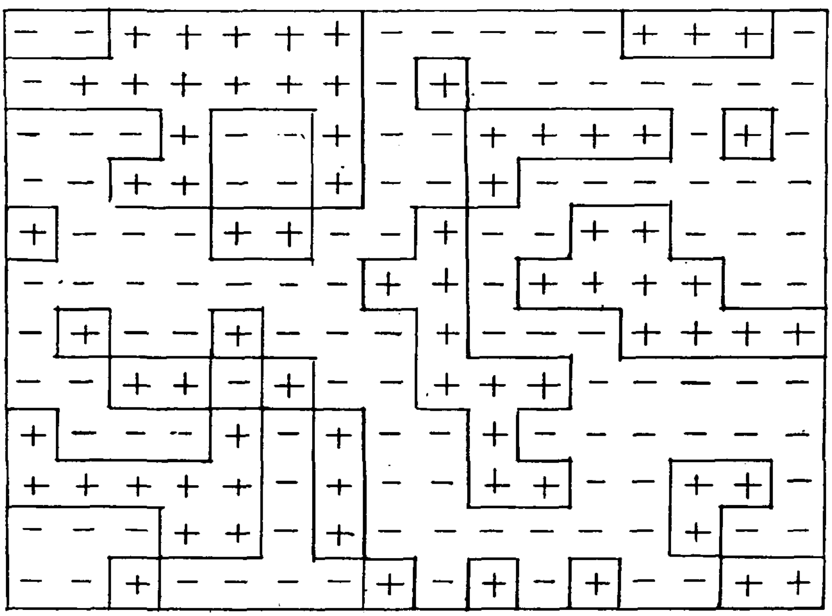

Fig. 1. Example of boundary lines.

However, not every succession of steps represents a possible boundary. Boundaries are restricted by the conditions that

(1) A boundary must either return to its starting point, or end on an edge.

(2) It passes through none of its points more than once.

(3) Not more than two boundaries can pass through any one point and, if there are two, the two steps adjoining the point on one boundary must be different from those on the other.

(4) No part of a boundary can lie outside the array of magnets.

(5) If two boundaries have two common points, it makes no physical difference which of the two branches between these points is ascribed to either boundary. Thus the same arrangement of magnets can be described by different sets of boundaries, i.e. different sets of successions, of which only one has to be counted.

We shall, however, *disregard* the conditions (1)–(5). In this way we count successions which in reality do not correspond to possible boundaries and thus we *overestimate* the number of boundaries.

If we disregard these conditions, a succession of length L consists of L *independent* steps, and, since for each of these there is a choice between four possibilities, the number of realizations is 4^L. Thus the partition function of successions is

$$f = \sum_L (4e^{-U/kT})^L = \sum_L (4\lambda)^L,$$

where

$$\lambda = \sqrt{\eta} = e^{-U/kT}.$$

If $4\lambda > 1$, this sum diverges and our upper limit becomes useless. If, however,

$$4\lambda < 1,$$

we have

$$f = (1-4\lambda)^{-1}, \tag{2}$$

and the average number of successions of length L, starting at one point, becomes

$$(4\lambda)^L (1-4\lambda). \tag{3}$$

Thus the average number of *boundaries* of length L, starting at one point, is less than (3).

Consider first the closed boundaries. Every one of them may be considered as starting from any one of its L points. Thus the total number of closed boundaries of length L in the array of N points will be less than

$$Z(L) < \frac{N}{L} (4\lambda)^L (1-4\lambda). \tag{4}$$

The area enclosed by each boundary of length L cannot exceed $(L/4)^2$ and the total area enclosed by boundary lines is less than

$$\sum_L \left(\frac{L}{4}\right)^2 Z(L) < \frac{N}{4} \frac{\lambda}{1-4\lambda}.$$

If, say, $4e^{-U/kT} < 0{\cdot}8$, this expression is less than $\frac{1}{4}N$. Thus the closed boundaries cannot enclose more than a quarter of all magnets.

As for the open boundaries, they must start and end on the edge and, since there are $4\sqrt{N}$ points on the edge, the number of open boundaries of length L is less than

$$2\sqrt{N}(1-4\lambda)(4\lambda)^L.$$

An open boundary of length L cannot cut off more than $\frac{1}{2}L^2$ atoms from the array; and thus the total number of magnets cut off is less than

$$\sqrt{N}(1-4\lambda)\Sigma L^2(4\lambda)^L,$$

which is negligible for large N.

Thus it follows rigorously that for sufficiently low temperatures the Ising model in two dimensions shows ferromagnetism and the same holds *a fortiori* also for the three-dimensional model.

Our crude estimate gives as lower limit for the value of $e^{-U/kT}$ at the Curie point a value close to $\frac{1}{4}$*, while the actual value, according to (1), is $\frac{1}{2}$ in Bethe's approximation.

The author wishes to thank H. Bethe who suggested part of the above proof.

* This value could be considerably improved without difficulty. Taking into account that according to condition (2) two successive steps must never be opposite to each other, there is only a choice of three possibilities for all but the first step, and in this way we easily obtain $\frac{1}{3}$ instead of $\frac{1}{4}$.