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Physics of Complex Networks

Sitabhra Sinha[†]
[†]The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai, 600113, India

Email: sitabhra@imsc.res.in

Abstract

The recent discovery of universal principles underlying many complex networks occurring across widely differing domains in the biological, social and technological worlds have spurred the interest of physicists in trying to understand such features using techniques from statistical physics and non-linear dynamics. In this article, we discuss some recent work from our group that looks at how network structure (e.g., its connection topology) dictates the nature of its dynamics, and conversely, how dynamical considerations constrain the network structure. These results are of obvious significance for understanding real-world systems.

INTRODUCTION .

Complex networks, comprising a large number of nodes (or elements) which are linked to each other according to specific connection topologies, occur widely across the natural, social and technological worlds [1-3]. Examples range from the intra-cellular signaling system that consist of different kinds of molecules which affect each other via enzymatic reactions, to the internet composed of servers around the world which exchange enormous quantities information packets regularly, and food webs which link via trophic relations large numbers of inter-dependent species. While the existence of such networks in various domains had been known for some time, the recent excitement among physicists working on such systems has to do with the discovery of certain universal principles among systems which had hitherto been considered very different from each other.

Reflecting the development of the modern theory of critical phenomena, the resurgent interest in the physics of complex networks has been driven by the simultaneous occurrence of detailed empirical studies of extremely large networks that were made possible by the advent of affordable high-power computing and the development of statistical mechanics tools to analyze the new network models. Prior to these developments, the networks that were looked at by physicists belonged to either the class of (i) *regular networks*, defined on geometrical lattices, where each node interacted with all the neighboring nodes belonging to a specified neighborhood, or (ii) *random networks*, where any pair of nodes had a fixed probability of being linked, i.e., interacting with each other The first work that focused public attention on the new network approach presented a new class of network models that were neither regular nor random, but exhibited properties of both [4]. Such *small-world networks*, as they were referred to, exhibited high clustering *c* (where nodes sharing a common neighbor have a higher probability of being connected to each other than to any other node) while having a very low average path length *l* (where the path length between any two nodes is defined as the shortest number of connected nodes one has to go through in order to reach one node starting from the other). As the former characterized a regular network, while the latter was typical for a random network, this new class of networks was somehow intermediate between the extremes of the two well-known network models, which was manifest in their construction procedure (Fig. 1). Several networks occurring in reality, in particular, the power grid, the actor collaboration network and the neural connection patterns of the *C elegans* worm, were shown to have the small-world property. Later, other examples have been added to this list, including the network of co-active functional brain areas [5] and the Indian railway system [6].

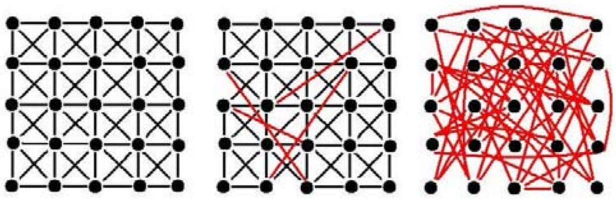

Fig 1 : Constructing a small-world network on a 2-dimensional square lattice substrate. Starting from a regular network (left) where each node is connected to its nearest and next-nearest neighbors, a fraction *p* of the links are rewired amongst randomly chosen pairs of nodes. When all the links are rewired, i.e., *p* = 1, the system is identical to a random network (right). For small *p*, the resulting network (center) still retains the local properties of the regular network (e.g., high clustering), while exhibiting global properties of a random network (e.g., short average path length).

Following this, it was discovered that the frequency distribution of node degree k (i.e., the number of links a node has) exhibits a power-law scaling form for a large variety of systems including the world wide web [7]. This further underlined the fact that most networks occurring in reality are neither regular (in which case the degree distribution would be close to a delta function) or random (which has a poisson degree distribution), as for both cases the probability of having a node with large degree would be significantly smaller than that indicated by the power law tail of empirically obtained degree distributions. In addition, it was observed that there exist non-trivial degree correlations among linked pairs of nodes. For example, a network where nodes with high degree tend to preferentially connect with other high degree nodes, the network is said to show assortative mixing [8]. On the other hand, in a disassortative network, nodes with large number of links prefer to connect with nodes having low degree. A large number of empirical studies have shown that most biological and technological networks are disassortative, while social networks tend to be assortative [2]. As assortative mixing promotes percolation and makes a network more robust to vertex removal, it is hard to understand why natural evolution in the biological world has favored disassortativity. However, in a recent study, we have shown that when one considers the stability of dynamical states of a network, disassortative networks would tend to be more robust, and this may be one of the reasons why they are preferred [9].

This brings us to the thrust of recent work in the area of complex networks which has shifted from the initial focus on purely structural aspects such as connection topology, to the role such features play in determining the dynamical processes defined on a network [10]. Our group has been involved over the past few years with a research program which seeks to understand not only how structure affects dynamics, and hence function, in a network, but also the reverse problem of how functional criteria, such as the need for dynamical stability, can constrain the topological properties of a network. In this article, we will briefly describe some of our principal results. The goal of this program is to understand the features which lead to the evolution of robust yet complex structures, viz., networks occurring in reality that are stable against perturbations and yet which can adapt to a changing environment

FROM STRUCTURE TO DYNAMICS

The role that the connection topology of a network plays in the nature of its dynamics has been extensively explored for spin models. In fact, such systems had been explored for a long time prior to the recent interest in complex networks, and many results are known regarding ordering transition in both regular as well as random structures. It is now known that for partial random rewiring in a system of sufficiently large size, any finite value of p causes a transition to the small-world regime, with the Ising model defined on such a network exhibiting a finite temperature ferromagnetic phase transition [11]. However, spin models are extremely restricted in their dynamical repertoire and therefore, researchers have looked at the effect of introducing other kinds of node dynamics in such network structures, e.g., oscillators. Motivated by recent observations that the brain may have a connection structure with small-world properties (see e.g., [5]), we have examined the effect of long range connections (i.e., non-local diffusion) over an otherwise regular network of nodes with links between nearest neighbors on a square lattice [12]. The dynamics considered is that of the excitable type, with the variable having a single stable state and a threshold. If a perturbation causes the system variable to exceed the threshold, we see a rapid transition to a metastable excited state followed by a slow recovery phase when the system gradually converges to the stable state. As a result of coupling the dynamics of individual nodes through diffusive coupling, various spatial patterns (which may be temporally varying) are observed. Such dynamics is commonly observed in a large variety of biological cells such as neurons and cardiac myocytes, as well as in nonlinear chemical systems such the Belusov-Zhabotinsky reaction.

In our simulations, by varying the probability of long-range connections, p, we have observed three categories of patterns. For $0 < p < p_c^l$, after an initial transient period where multiple coexisting circular waves are observed, the system is eventually spanned by a single or multiple rotating spiral waves whose temporal behavior is characterized by a flat power spectral density. At $p = p_c^l$, the system undergoes a transition from a regime with a temporally irregular, spatial pattern to one with a spatially homogeneous, temporally periodic pattern. The latter behavior occurs over the range $p_c^l < p < p_c^u$ as a result of the increased fraction of long range connections, whereby a large fraction of the system gets synchronously active and subsequently goes into the recovery phase. Beyond the upper critical value p_c^u, there is no longer any self-sustained activity in the system as all nodes converge to the stable state. The patterns in each regime were found to be extremely robust against even large perturbations or disorder in the system.

Our model explains several hitherto unexplained observations in experimental systems where non-local diffusion had been implemented. In addition, by identifying the long-range connections as those made by neurons and the regular network as that formed by the glial cells in the brain, our results provide a possible explanation of why evolution may have preferred to increase the number of glial cells over neurons (with a ratio of more than 10:1 for certain parts of the human brain) in order to maintain robust dynamical patterns as brain size increased. It also points towards possible functional role of small-world brain topology in the occurrence of dynamical diseases such as epileptic seizures and bursts. More generally, our work shows how non-standard network topologies can influence system dynamics by generating different kinds of spatiotemporal patterns depending on the extent of non-local diffusion.

FROM DYNAMICS TO STRUCTURE

An important functional criterion for most networks occurring in nature and society is the stability of their dynamical states. While earlier studies have concentrated on the robustness of the network when subjected to structural perturbations (e.g., removal of node or link), we have looked at the effect of perturbations given to the steady states of network dynamics. In particular, the question we ask is whether networks become more susceptible to small perturbations as their size (i.e., number of nodes N) increases, the connections between the nodes become denser (i.e., increased connectance probability C) and the average strength of interaction (s) increases. This is related to a decades-old controversy, often referred to as the stability-complexity debate. In the early 1970s, May [13] had shown that for a model ecological network, where species are assumed to interact with a randomly chosen subset of all other species, an arbitrarily chosen stable state of the system becomes unstable if any of the parameters determining the networks complexity (e.g., N, C or s) is increased. In fact, by using certain results of random matrix theory, the critical condition for the stability of the network was shown to $N\,C\,s^2 < 1$ (May-Wigner theorem) [13]. This flew against common wisdom, gleaned from large number of empirical studies as well as naïve reasoning, which dictated that increased diversity and/or stronger interactions between species results in more robust ecosystems. Thus, ever since the publication of the controversial model, there have been attempts to understand the reason behind the apparently paradoxical result, especially as this result relates not only to ecological systems but extends to all dynamical networks for which the stability of steady states have functional significance, e.g., intracellular biochemical networks where the concentrations of different molecules need to be maintained within physiological levels. Two of the common charges leveled against the theoretical model is that (i) it assumes the interaction network to be random whereas naturally occurring networks are bound to exhibit certain kind of structure, and (ii) the linear stability analysis assumes the existence of simple steady states, which may not be the case for real systems that may either be having oscillations or be in a chaotic state.

In our work on dynamical systems defined on networks, we have tried to address both of these lines of criticism. For example, focusing on the question of inadequacy of linear stability analysis, we have considered networks with non-trivial dynamics at the nodes, spanning the range from simple steady states to periodic oscillation and fully developed chaos, and measured the robustness of the full dynamics with respect to variations in N, C and s [14, 15].

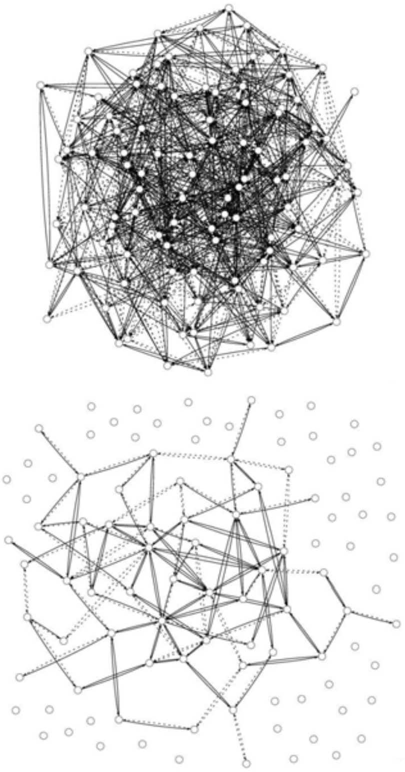

Fig 2 : Evolution of a network with non-trivial dynamics at the nodes. The initial (top) and final asymptotic (bottom) networks are shown. Only nodes having persistent activity are connected to the network.

Each node in our model network has a dynamical variable associated with it, which evolves according to a well-known class of difference equations commonly used for modeling population dynamics. By varying a nonlinear parameter, the nature of the dynamics (i.e., whether it converges to a steady state or undergoes chaotic fluctuations) at each node can be controlled. However, in the absence of coupling, each node will always have a finite, positive value for its dynamical variable. When coupled in a network (initially in a random fashion), with links that can have either positive or negative weights, it is possible that as a result of dynamical fluctuations, the variable for some nodes can become negative or zero. As this corresponds to the absence of activity, the corresponding node is considered to be extinct and thus isolated from the network. This procedure may create further fluctuations and cause more nodes to get “extinct”, resulting in gradual reduction of the size of the network (Fig. 2). The final asymptotic size of the network, relative to its initial size, is a measure of its robustness – the more robust network being one with a higher fraction of nodes having persistent activity.

Analysis showed that the network robustness (as measured by the above *global* criterion) not only decreased with N, C and s, as expected from a local stability analysis, but actually matched the May-Wigner theorem *quantitatively* [14]. In addition, the asymptotic network exhibited robust macroscopic features: (a) the number of persistently active nodes was independent of the initial network size, and (b) the asymptotic number of links between these persistently active nodes was independent of both the initial size and connectivity [15]. This is all the more surprising as the removal of nodes (and hence, links) is not guided by any explicit fitness criterion but rather emerges naturally from the nodal dynamics through fluctuations of individual node properties. Our results imply that asymptotically active networks are non-extensive: when two networks of size N are coupled to each other (with the same connectance as the individual networks), although the resulting network initially has a size $2N$, the ensuing dynamical fluctuations will reduce its size to N. This implies that simply increasing the number of redundant elements is not a good strategy for designing robust systems.

We have also looked at the effect of empirically reported structures, such as small-world connection topology and scale-free degree distribution, on the dynamical stability of networks. Our results indicate that, in general, introducing such structural features do not alter the outcome expected from the May-Wigner theorem [9, 16]. However, these details can indeed affect the nature of the stability-instability transition, for example, the transition exhibiting a cross-over from being very sharp (resembling first-order phase transition) for a random network to a more gradual change as the network becomes more regular at the small-world regime [16].

EVOLUTION OF ROBUST NETWORKS

This brings us to the issue of how complex networks can be stable at all, given that the May-Wigner theorem seems to hold even for networks that have structures similar to those seen in reality and where non-trivial dynamical situations have also been considered. The solution to this apparent paradox lies in the observation that most networks that we see around us did not occur fully formed but emerged through a process of gradual evolution, where stability with respect to dynamical fluctuations is likely to be one of the key criteria for survival. In earlier work, we have shown that a simple model where nodes are gradually added to or removed from a network according to whether this results in a (linearly) dynamically stable network or not, results in a non-equilibrium steady state where the network is extremely robust [17]. The robustness is manifested by increased resistance and resilience, as well as, decreased probability of large extinction cascades, when the network size (i.e., the system diversity) is increased. Thus, our results reconcile the apparently contradictory conclusions of the May-Wigner theorem and a large number of empirical studies.

More recently, we have shown that model networks can evolve many of the observed structural features seen among networks in the natural world, by taking into account the fact that most such systems have to optimize several simultaneous conditions – both structural and dynamical. In particular, most networks need to have high communication efficiency (i.e., low l) and low connectivity (to reduce the cost of maintaining large number of links) while being stable with respect to dynamical perturbations. Our results show that simultaneous optimization of these properties result in networks with modular structure, i.e., subnetworks with a high density of connections within themselves compared to between distinct subnetworks [18]. As such evolved systems also exhibit heterogeneous degree distribution, these findings have implications for a truly wide range of systems in the natural and technological worlds where such features have been observed.

REFERENCES

1. R. Albert and A.-L. Barabasi, Rev. Mod. Phys. **74** (2002) 47.
2. M. E. J. Newman, SIAM Review **45** (2003) 167.
3. S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks: From Biological Nets to the Internet and WWW* (Oxford Univ Press, Oxford, 2003)
4. D. Watts and S. H. Strogatz, Nature **393** (1998) 440.
5. S. Achard, R. Salvador, B. Whitcher, J. Suckling and E. Bullmore, J. Neuroscience **26** (2006) 63.
6. P. Sen, S. Dasgupta, A. Chatterjee, P. A. Sreeram, G. Mukherjee and S. S. Manna, Phys. Rev. **E67** (2003) 036106.
7. A.-L. Barabasi and R. Albert, Science **286** (1999) 509.
8. M. E. J. Newman, Phys. Rev. Lett. **89** (2002) 208701.
9. M. Brede and S. Sinha, cond-mat/0507710 (2005).
10. S. H. Strogatz, Nature **410** (2001) 268.
11. A. Barrat and M. Weigt, Eur. Phys. J. **B13** (2000) 547.
12. S. Sinha, J. Saramaki and K. Kaski, Phys. Rev. **E76** (2007) 015101(R).
13. R. M. May, *Stability and Complexity in Model Ecosystems* (Princeton Univ Press, Princeton, 1973)
14. S. Sinha and S. Sinha, Phys. Rev. **E71** (2005) 020902(R).
15. S. Sinha and S. Sinha, Phys. Rev. **E74** (2006) 066117.
16. S. Sinha, Physica **A346** (2005) 147.
17. C. C. Wilmers, S. Sinha and M. Brede, Oikos **99** (2002) 363.
18. R. K. Pan and S. Sinha, Phys. Rev. **E76** (2007) *in press*.

Sitabhra Sinha is a faculty member of the Theoretical Physics group at the Institute of Mathematical Sciences, Chennai since 2002. He did his PhD at the Indian Statistical Institute, Kolkata, on nonlinear dynamics of neural network models and postdoctoral work at Indian Institute of Science, Bangalore and Cornell Medical School, New York, on pattern formation in biological systems. His research interests include networks, pattern formation and the physics of social and economic phenomena.