Emergence of universal scaling in financial markets from mean-field dynamics

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Collective phenomena with universal properties have been observed in many complex systems with a large number of components. Here we present a microscopic model of the emergence of scaling behavior in such systems, where the interaction dynamics between individual components is mediated by a global variable making the mean-field description exact. Using the example of financial markets, we show that asset price can be such a global variable with the critical role of coordinating the actions of agents who are otherwise independent. The resulting model accurately reproduces empirical properties such as the universal scaling of the price fluctuation and volume distributions, long-range correlations in volatility, and multiscaling.

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I. INTRODUCTION

Universal scaling behavior is an emergent property of many complex systems [1]. In such systems, the interactions between a large number of individual components yields macroscale collective behavior with features that are almost invariant across different spatial and temporal scales [2]. A financial market provides a general and useful paradigm of such a system, since it involves a large number of agents whose actions are subject to internal and external influences, such as information about the state of the market as provided by market indices [3]. Despite this complexity, the availability of a large volume of high-quality data for analysis has enabled the identification of well-characterized statistical properties [4,5]. These properties, including the fat-tailed distribution of relative price changes [6,7] and intermittent bursts of large fluctuations that characterize volatility clustering [8], appear to be universal: they are invariant across different markets, types of assets traded, and periods of observation [9]. More generally, the question of how universal features emerge from collective behavior in systems with many components is not restricted to the purely economic domain. Thus, new approaches to understanding the behavior of financial markets may contribute to the understanding of the physics of nonequilibrium steady states in general.

Mainstream economic theories for price fluctuations of financial assets typically assume the *efficient market hypothesis* [10]. According to this, price variations reflect changes in the fundamental (or "true") value of the assets. However, detailed analysis of data from actual markets show that much of the observed price variation cannot be explained solely in terms of changes in economic fundamentals [11]. The absence of a strong correlation between large market fluctuations and purely economic factors leaves unresolved the question of why markets are so volatile. As the dynamics of markets are a result of the collective behavior of many interacting constituents, models based on statistical physics have been proposed to explain the observed universal behavior [12–15]. Most such models consider explicit interactions between agents to reproduce a very limited set of the universal empirical features. However, it is possible that the observed complex behavior is a result of a mean-field-like global variable mediating the dynamics of components, which are therefore coupled only indirectly. Such a potential simplification in analyzing the nonequilibrium steady state for markets holds promise as a general descriptive framework for the dynamics of many complex systems.

In this article, we present a model for market dynamics where the action of each agent is governed solely by global information about the system, viz., the price p_t of the single asset being traded. At each time step, every agent goes through a two-step stochastic process, analogous to decision making in an uncertain environment. Based on the deviation of the instantaneous price from its long-term average (representing the notional fundamental value of the asset) and the direction of price movement, each agent decides (a) whether to initiate a trade and (b), if yes, whether to buy or sell. The price in turn evolves as a function of the net demand, measured as the difference between the number of buyer- and seller-initiated trades. Thus, our approach falls broadly within the theoretical framework that treats markets as a system of spins, but it differs from earlier models in not having direct Ising-like interactions between the agents [14,16]. Further, the fluctuations in the model variables are endogenous to the system and are not responses to external noise simulating the arrival of news or information [12]. Despite its simplicity the model reproduces the observed universal properties of markets. These include the scaling behavior of the distribution of price fluctuation measured by the relative logarithmic change, viz., the return, $R_{t,\Delta t} = \ln(p_{t+\Delta t}/p_t)$ defined over a time interval Δt. The cumulative distribution of $R_{t,\Delta t}$ shows a power-law tail with a characteristic exponent $\alpha \sim 3$ for many different markets—a robust property referred to as the *inverse cubic law* [17,18]. Our model, which displays power-law scaling in the return distribution over a large region of the parameter space, can quantitatively reproduce the inverse cubic law on introducing heterogeneity among the agents. For the same parameters, the scaling exponent ζ_V for the distribution of trading volume in a given interval of time, V_t, agrees with the empirical values reported in Ref. [19]. Moreover, the time series generated by the model exhibits multifractal statistics [20] and the autocorrelation of absolute returns decay slowly, a signature of volatility clustering seen in actual markets [21]. We also give an analytical derivation of the relation between the scaling exponents for return and volume distributions generated by the model, which is in good agreement with the empirical literature.

II. THE MODEL

We consider the market to comprise N agents, each of whom are in one of three possible states at time t, viz., $S_i(t) = 0$ (not trading), $+1$ (buying), and -1 (selling) ($i = 1, \ldots, N$) [22]. For simplicity, we assume that an agent can trade a unit quantity of asset at a given instant. The change in the price of the asset is driven by the net demand, as measured by the global order parameter $M_t = \sum_i S_i(t)/N$. Thus, after time instant t, the asset price changes to $p_{t+1} = [(1 + M_t)/(1 - M_t)]p_t$, with $p_0 > 0$, which ensures that the price is always positive and rises (falls) when there are relatively more buyer- (seller-) initiated trades, with the inactive agents, i.e., $S_i(t) = 0$, not affecting the process. A price equilibrium ($p_{t+1} = p_t$) is achieved when supply equals demand ($M_t = 0$), while in extreme cases, when all N agents wish to buy (sell), the price diverges (crashes to 0). We have verified that the exact form of the price function is not critical to obtain the results described here.

In our model, the net demand M_t is driven by the collective behavior of agents, with each individual's state S_i in turn evolving as a result of fluctuations in the instantaneous price p_t around the asset's fundamental value p_t^* as perceived by an agent. As a potential buyer (seller) who initiates the trading process needs to find a potential seller (buyer) in order to complete the transaction, the trading frequency would tend to decrease as the deviation $|p_t - p_t^*|$ increases. This is because when the price p_t differs significantly from the fundamental value of the asset, even though the initiating agent may be extremely eager to trade, the other agent would be unwilling to engage in what it sees as an unfavorable transaction (assuming all agents possess the same information about the fundamental value). As the "true" value of p_t^* is privileged information and therefore inaccessible to an agent, it is estimated based on the observed price time series as $p_t^* \simeq \langle p_t \rangle_\tau$, a long-term moving average measured over a time interval ending at the present instant t and beginning τ time units earlier ($\tau = 10^4$ for the results shown in the article). Note that previous studies have shown that several features of empirical market dynamics are determined by an effective potential defined in terms of the long-term moving average of price [23]. Given the price history, an agent i will trade at time t according to the probability

$$P[|S_i(t)| = 1] = 1 - P[S_i(t) = 0] = \exp\left(-\mu \left| \frac{p_t - \langle p_t \rangle_\tau}{\langle p_t \rangle_\tau} \right| \right), \quad (1)$$

where the parameter μ is a measure of the sensitivity of an agent to the magnitude of deviation of the price from its perceived fundamental value. For $\mu = 0$, the system reduces to a two-state model where every agent trades at all time instants.

Any agent i who initiates a trade at time t, has to either buy $[S_i(t) = +1]$ or sell $[S_i(t) = -1]$. Using the simple assumption that this is a random process, we allow each trade to be either buyer initiated or seller initiated with equal probability, independent of the price movement. We have verified that introducing more complicated rules based on consideration of supply and demand, where the decision to buy or sell depends on the instantaneous price fluctuations (e.g., as measured by the return), do not qualitatively change the results reported here.

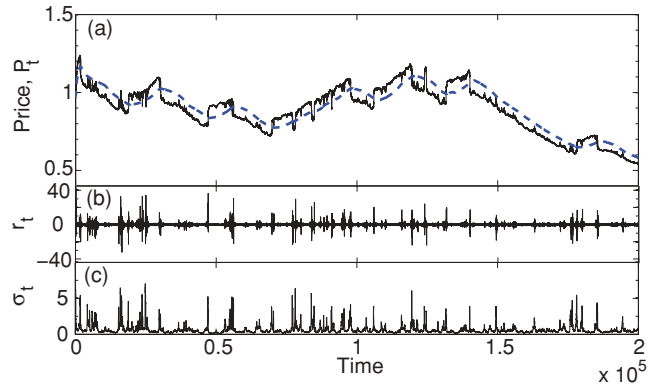

FIG. 1. (Color online) Time evolution. (a) The time series of price p_t (solid line) and its moving average $\langle p_t \rangle_\tau$ (broken line). The corresponding (b) normalized returns, r_t, and (c) volatility, σ_t, calculated as the standard deviation of returns in a moving window of interval $\delta t = 100$, show intermittent bursts of large price fluctuations. This indicates the presence of volatility clustering. Varying δt does not change the result qualitatively. The model parameters are $N = 10^4$ and $\mu = 100$.

III. RESULTS

Time evolution of the asset price, p_t, shown in Fig. 1(a), is qualitatively similar to the time series of stock prices or indices observed in real markets. The moving average of p_t (broken line) which is the agents' perceived fundamental value of the asset, tracks a smoothed pattern of price fluctuations coarse-grained over a time scale τ. The normalized returns r_t for $\Delta t = 1$ time unit, obtained from R_t by subtracting the mean and dividing by the standard deviation of the entire return time series, exhibits significantly large deviations relative to that expected from a Gaussian distribution [Fig. 1(b)]. These intermittent bursts of large fluctuations have a tendency to aggregate together. This is seen more clearly from the volatility σ_t, which is a measure of risk (the unpredictable change in the value of an asset) and may be calculated as the standard deviation of r_t over a moving window. The clustering of volatility seen in Fig. 1(c) is a universal feature of financial markets.

The nature of price fluctuations can be examined in more detail by focusing on the cumulative distribution $P_c(r_t > x)$. When the agents are homogeneous (i.e., having the same sensitivity μ), this distribution shows power-law tails having exponent $\alpha \simeq 2$ for a large range of values of μ (viz. $\mu > 50$). For lower values of μ (<10) the distribution is exponential. In reality, agents will differ in their responses to the same stimulus. This heterogeneity in agent behavior is modeled by a distribution of the sensitivity parameter μ that measures the degree of risk aversion in an individual. Figure 2(a) shows that the cumulative distribution for r_t quantitatively reproduces the inverse cubic law ($\alpha \simeq 3$) when μ for each agent is randomly selected from an interval. To accurately determine the numerical value of the return exponent α, we use the Hill estimator, $\gamma_{k,n}$, for a time series of length n, whose inverse

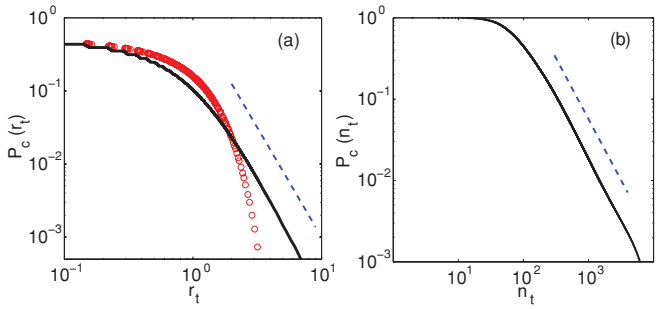

FIG. 2. (Color online) Distributions. (a) Cumulative distribution of normalized returns for heterogeneous agents with μ distributed uniformly over [10,200]. The broken line indicates a power-law exponent of -3 and the circles represent the standard normal distribution. (b) The corresponding cumulative distribution of the number of agents trading at a given time instant t, n_t, with the broken line indicating a power-law exponent of -1.5. The results are obtained by averaging over multiple realizations of the model with $N = 10^4$ agents simulated over 2×10^5 time units.

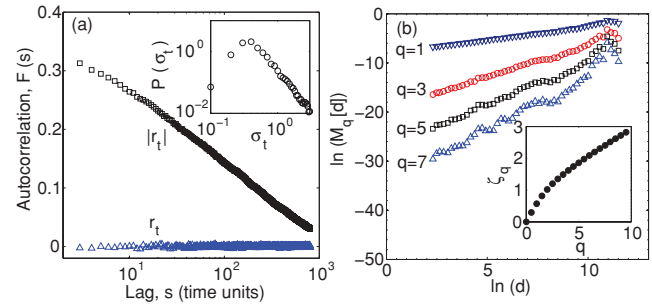

FIG. 3. (Color online) Correlations. (a) The autocorrelation of r_t (triangle) rapidly falls to noise level but its absolute value (squares) shows long-term memory. (Inset) Probability distribution of the volatility, σ_t. (b) The q-th moments of absolute value of the fluctuations in return have a power-law scaling relation with respect to the time scale d. The inset shows the nonlinear variation of the corresponding power-law exponent, ζ_q, indicating multifractality. The model parameters are $N = 10^4$ and $\mu = 100$.

approaches the true value of α as the order statistic $k \to \infty$ with $\frac{k}{n} \to 0$ [24]. To avoid the bias arising from finite length of the time series, we have used a subsample bootstrap method to estimate the optimal k [25]. Using this method, the estimated value of the exponent α is 3.11 for the positive tail of the return distribution [shown in Fig. 2(a)] and 3.12 for the negative tail. We have verified that this long-tailed behavior of returns is robust with respect to variations in the interval and the nature of the distribution for μ.

As the model assumes that each trade involves a unit quantity of the asset, the total number of traders at any instant t, viz., $n_t = \sum_i |S_i(t)|$, is equivalent to the trading volume V_t. The distribution of this variable also exhibits a power-law scaling, with the exponent $\zeta_V \simeq 1$ when the agents are homogeneous. The heavy-tailed nature of n_t distribution is even more robust than that of r_t, as we observe a power-law tail also for lower values of μ (where the return distribution is exponential). On introducing heterogeneity among agents as explained before, the cumulative distribution of n_t is seen to be a power law, whose exponent is evaluated by the Hill estimator to be $\zeta_V \simeq 1.63$ (using the same parameters for which $\alpha \simeq 3$) [Fig. 2(b)]. This is almost identical to the trading volume exponents reported for different markets [26]. In order to check the sensitivity of our results on the assumption that an agent can trade only a unit quantity, we have verified that a Poisson distribution of the number of units traded by an agent does not change the results qualitatively. Thus, our model suggests that heterogeneity in agent behavior is a key factor for explaining the quantitative properties of the observed distributions. It implies that when the behavior of agents become more homogeneous, e.g., during a market crash, the return exponent α will tend to decrease. This is intriguing in light of earlier work [27] showing that the power-law exponent for the distribution of relative prices during a crash has a value that significantly differs from that seen at other times.

Turning now to the correlation properties of the return time series, we see that r_t is uncorrelated, as expected from the efficient market hypothesis [10]. However, the absolute values, which are a measure of the volatility, show a slow logarithmic decay in their autocorrelation [Fig. 3(a)], which is a signature of long-memory effects operating in actual markets [21]. Figure 3(a) (inset) shows the bulk of the volatility distribution which has a log-normal form as found empirically [28]. To understand better the temporal organization of price fluctuations than is possible with the two-point correlations, we consider the n-point correlations as reflected in the multifractal spectrum [29,30]. Figure 3(b) shows the power-law scaling of the q-th moment $M_q(d)$ of the absolute value of fluctuations as a function of the time scale (d) being considered. The resulting power-law exponents ζ_q do not have a simple linear relation to q (inset), indicating that the process is multifractal. Thus, our model also reproduces the multifractal nature of financial markets [20,21].

The genesis of the power-law scaling relations in the model is strongly connected to the dynamics by which agents decide to trade, which results in a variable number of buyer- or seller-initiated trades at a given instant. This is illustrated by the absence of power-law scaling when the number of trading agents do not change with time (viz., $\mu = 0$ for which $n = N$). Further, it is the long-tailed nature of the distribution of n_t that is responsible for the heavy tails of the returns. For instance, imposing a log-normal form on n_t rather than generating it by using Eq. (1), again results in fat tails for r_t. This dependence of the long-tailed nature of the returns on the distribution of number of trading agents can be analytically derived as follows. First, we note that if the number of trading agents is a constant ($n_t = n$), the returns follow a Gaussian distribution with mean 0 and variance, $\sigma^2 \sim n$. Therefore, when the number of traders changes over time, with n_t following a distribution $P(n)$, the corresponding return distribution $P(r)$ can be expressed as a sum over many conditional distributions $P(r|n)$:

$$P(r) = \sum_{n=1}^{N} P(r|n)P(n) = \sum_{n=1}^{N} \frac{1}{\sqrt{2\pi n}} \exp(-r^2/2n)P(n), \tag{2}$$

where N is the maximum number of agents who can trade. If the cumulative distribution for n_t follows a power law with exponent ζ_V as obtained from our model, Eq. (2) can be rewritten as $P(r) \sim [1/\sqrt{2\pi}] \sum_{n=1}^{N} n^{-(\zeta_V+\frac{3}{2})} \exp(-r^2/2n)$. Replacing the sum by an integral and taking the upper limit $N \to \infty$, we get a closed form solution

$$P(r) = C_{\zeta_v} K(0.5+\zeta_V, 1.5+\zeta_V, -r^2/2), \tag{3}$$

where $K(.)$ is the Kummer confluent hypergeometric function and the normalization constant $C_{\zeta_v} = \frac{\Gamma(0.5+\zeta_V)\Gamma(1+\zeta_V)}{\sqrt{2\pi}\Gamma(\zeta_V)\Gamma(1.5+\zeta_V)}$. Numerically evaluating $K(.)$ gives a power-law distribution for r. For half-integral values of the exponent ζ_V, Eq. (3) simplifies to a form where the power-law nature of the return distribution is evident, e.g., for $\zeta_V = 3/2$, as obtained in our model for heterogeneous distribution of agents, $P(r) = C_{\zeta_V=3/2}(1/r^4)[4 - 2e^{-r^2}(2+r^2)]$, with $C_{\zeta_V=3/2} \simeq 1.67$. For large r, $P(r) \sim r^{-4}$, indicating that the cumulative distribution of returns will have a power-law tail with exponent $\alpha = 3$ (i.e., the inverse cubic law).

IV. CONCLUSIONS

In this article we have presented a model for the dynamics of complex systems which quantitatively reproduces the observed universal properties of markets without considering explicit interactions among agents or prior assumptions about individual trading strategies (e.g., chartists vs. fundamentalists) [12]. Recent work on other aspects of financial markets have shown that coherent collective behavior can emerge in a system through components responding to the same global signal [31]. We show that the price of an asset can play the role of such a mediator that generates effective interactions between agents, resulting in a nonequilibrium steady state characterized by scaling distributions. Heterogeneity of agent behavior is seen to be critical for obtaining the inverse cubic law, suggesting that in normal circumstances agents differ significantly in terms of their response to similar market signals. On the other hand, when the agents are more homogeneous in their behavior (as during a crash), the model exhibits even fatter tails. Possible extensions of our model include the introduction of a volume dynamics that decides the quantity of assets traded by an agent at a particular time instant, the inclusion of multiple assets and considering the effect of external news. The framework presented here can be applied to many other complex systems whose emergent phenomena can be explained in terms of indirect interactions between components mediated by a mean-field-like variable.

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