Optimal interdependence enhances the dynamical robustness of complex systems

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Although interdependent systems have usually been associated with increased fragility, we show that strengthening the interdependence between dynamical processes on different networks can make them more likely to survive over long times. By coupling the dynamics of networks that in isolation exhibit catastrophic collapse with extinction of nodal activity, we demonstrate system-wide persistence of activity for an optimal range of interdependence between the networks. This is related to the appearance of attractors of the global dynamics comprising disjoint sets ("islands") of stable activity.

Many complex systems that occur in biological [1], technological [2], and socioeconomic [3] contexts are strongly influenced by the behavior of other systems [4]. Such interdependence can result in perturbations in one system propagating to others, potentially resulting in a cascading avalanche through the network of networks [5,6]. Recent studies of percolation of failure processes in a system of two [7–9] or more [10,11] connected networks have suggested that interdependence makes the entire system fragile. However, a proper appraisal of the role of interdependence on the stability of complex systems necessarily needs to take into account the dynamical processes occurring on them [12,13]. Compared to a purely structural approach (such as percolation, which considers the effect of removing nodes or links), a dynamical system perspective provides a richer framework for assessing the robustness of systems [14,15]. Indeed, investigating how fluctuations from equilibrium in a local region of a complex system can propagate to other regions forms the basis for addressing the dynamical stability of systems [16]. Extending this framework to the context of interdependent networks can potentially offer us insights into why such systems are ubiquitous in the real world in spite of their structural fragility.

In this Rapid Communication we show that strong interdependence between networks can *increase* the robustness of the system in terms of its dynamical stability. In particular, we show for a pair of networks that there exists an optimal range of interdependence which substantially enhances the persistence probability of active nodes. By contrast, decreasing the internetwork coupling strength so that the networks are effectively independent results in a catastrophic collapse with extinction of activity in the system almost in its entirety. The increased persistence at optimal coupling is seen to be related to the appearance of attractors of the global dynamics comprising disjoint sets of stable activity. Our results also suggest that the nature of internetwork interactions is a crucial determinant of the role of interdependence on the dynamical robustness of complex systems. For example, increasing the intensity of nonlinear interactions between nodes leads to loss of stability and subsequent transition to a quiescent state, while stronger diffusive coupling between the networks can make a global state corresponding to persistent activity extremely robust.

Let us consider a model system comprising G interdependent networks. Each network has N dynamical elements

connected to each other through a sparse random topology of nonlinear interactions. Interdependence is introduced by diffusively coupling an element i in a network to the corresponding ith element of other network(s). This framework can be used to represent, for instance, dispersal across G neighboring habitat patches of N interacting species in an ecological system. A continuous dynamical variable z_i^μ ($i = 1, \ldots, N$; $\mu = 1, \ldots, G$) is associated with each node of the coupled networks. In the above-mentioned example, it can be interpreted as the relative mass density of the ith species in the μth patch. We consider generalized Lotka-Volterra interactions between the nodes within a network as this is one of the simplest and ubiquitous types of nonlinear coupling [17,18]. The dynamical evolution of the system can then be described in terms of the GN coupled equations:

$$
z_i^\mu(n+1) = (1-D)F[z_i^\mu(n)]\left[1 + \sum_{j=1}^N J_{ij}^\mu z_j^\mu(n)\right]
$$
$$
+ \frac{D}{(G-1)}\sum_{\nu \neq \mu}^G F[z_i^\nu(n)]\left[1 + \sum_{j=1}^N J_{ij}^\nu z_j^\nu(n)\right].
$$
(1)

Here $\mathbf{J}^\mu$ is the interaction matrix for the μth network, while D is a measure of the strength of interdependence via diffusive coupling between networks. The range of the variable z_i^μ is decided by the function $F(\cdot)$ governing the dynamics of individual elements in the system. Here we consider F to be a smooth unimodal nonlinear map defined over a finite support and having an absorbing state. This class of dynamical systems is quite general and are capable of exhibiting a wide range of behavior including equilibria, periodic oscillations, and chaos [19]. For the results shown here we have used the logistic form [20]: $F(z) = rz(1-z)$ if $0 < z < 1$, and $= 0$ otherwise, such that $z = 0$ is the absorbing state, and r is a nonlinearity parameter that determines the nature of the dynamics.

Unlike most studies with logistic map where $r \in [0,4]$, we specifically choose $r > 4$ such that $F(\cdot)$ maps a finite subinterval within $[0,1]$ directly to the absorbing state. Iterative application of $F(\cdot)$ implies that only a a set of measure zero will remain in the unit interval [21], resulting in a leaky dynamical system [22]. Thus, an isolated node will almost

always converge to the absorbing state, corresponding to its extinction. Interaction with other nodes can, however, maintain a node in the active state [$z \in (0,1)$] indefinitely. We define a measure for the global stability of the system as the asymptotic fraction of nodes in each network that have not reached the absorbing state, viz., $f_{active} = Lt_{n \to \infty} f_{active}(n)$, where $f_{active}(n) = \sum_{i=1}^{N} \Theta\{F[z_i^{\mu}(n)]\}/N$ (with $\Theta[x] = 1$ for $x > 0$, and 0 otherwise). Thus, we explicitly investigate conditions under which interdependence between networks can result in persistent activity in at least a subset of the nodes comprising the system. Using an ecological analogy, our focus is on the long-term survival of a finite fraction of the ecosystem as a function of the degree of dispersal between neighboring patches rather than the intrinsic stability of individual species populations.

The degree of interdependence between the networks can be varied by changing the number of pairs of corresponding nodes M ($0 \leqslant M \leqslant N$) that are linked via dispersion. The interaction matrix $\mathbf{J}^{\mu}$ in each network is considered to be sparse, such that only C fraction of the matrix elements are nonzero with their interaction strengths chosen randomly from a Normal($0,\sigma^2$) distribution. For simplicity, we shall focus on a pair of interdependent networks (i.e., $G = 2$) schematically shown in Fig. 1(a), both networks being chosen from the same ensemble so as to have identical parameters r, C, and σ. We distinguish between the variables z of the N nodes in the two networks by denoting them as x_i and y_i ($i = 1, \ldots, N$), respectively, their initial values being chosen at random from the uniform distribution [0,1].

Figures 1(b) and 1(c) show the time evolution of the state of the dynamical variables x_i and the global stability measure $f_{active}(n)$ for one of the networks ($N = 256$, $C = 0.1$, $\sigma = 0.01$) where the nonlinearity parameters r_i are distributed uniformly in [4.0,4.1]. As mentioned above, this distribution of r_i implies that the individual node dynamics would almost certainly converge to the absorbing state, and this is indeed what is observed when the networks are isolated, i.e., $D = 0$. However, when the interdependence is increased, e.g., to $D = 0.15$, we observe that a finite fraction of nodes persist in the active state, although for much lower (e.g., $D = 0.1$) and higher (e.g., $D = 0.2$) interdependence the system exhibits complete extinction of activity [Figs. 1(b) and 1(c)]. Thus an optimal diffusive coupling between corresponding nodes in the two networks enhances the global stability of the system. This suggests, for instance, that ecological niches which in isolation are vulnerable to systemic collapse resulting in mass extinction, can retain species diversity if connected to neighboring habitats through species dispersal. Indeed, for this to happen, it is not even required that all species in the network be capable of moving between the different habitats. As seen from Fig. 1(d), if only a subset of M nodes (out of N) are coupled between the two networks through diffusion, the system exhibits enhanced persistence with f_{active} increasing with M. However, enhancing the intensity of nonlinear interactions within each network by increasing either their connectivity C or range of interaction strengths (measured by the dispersion σ), as well as amplifying the intrinsic nonlinear dynamics of the nodes by increasing the range of r, decreases the survival probability of active nodes. This is also evident from the variation with C and σ of the probability that a node
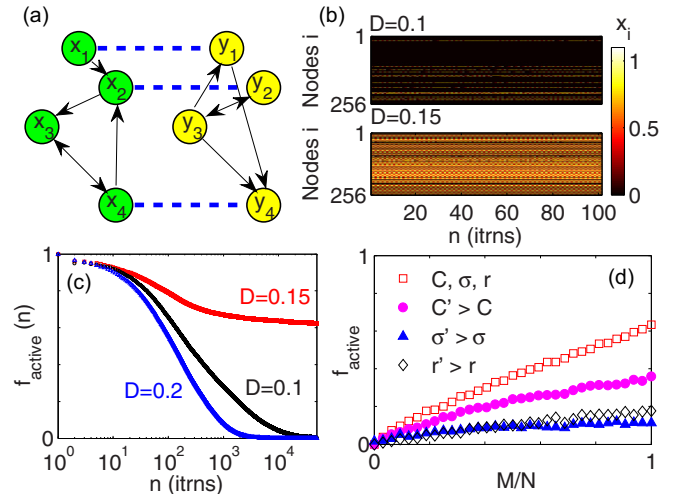

FIG. 1. (a) Schematic diagram representing two interdependent networks, each comprising N nodes that have intranetwork directed nonlinear interactions (indicated by arrows) and internetwork diffusive coupling between M ($\leqslant N$) pairs of corresponding nodes (broken lines). (b) Pseudocolor representation of the spatiotemporal evolution of dynamical state x_i for each node i in one of the networks at two different values of the internetwork diffusive coupling strength, viz., (top) $D = 0.1$ and (bottom) $D = 0.15$, with black representing the absorbing state $x_i = 0$, i.e., extinction of activity. Increased interdependence between the networks allows more nodes to maintain persistent activity, i.e., $x \neq 0$. Increasing D further can result in a decrease in the fraction of active nodes f_{active} with time as seen in (c), indicating that long-term persistent activity occurs only within an optimal range of interdependence. (d) Increasing intranetwork interactions either in terms of the connection density (C) or their strength (σ) for a given internetwork diffusive coupling strength (e.g., $D = 0.15$) results in a decrease in the fraction f_{active} of nodes with persistent activity. A similar decrease is also observed on increasing the range of the nonlinearity parameter r. However, increasing the number of corresponding node pairs M in the two networks that are coupled diffusively is seen to increase f_{active}, pointing to a fundamental distinction between intra- and internetwork interactions in their contribution to the overall dynamical robustness of the system. Results shown here are obtained for $N = 256$, $C = 0.1, C' = 0.3, \sigma = 0.01, \sigma' = 0.05, r \in [4.0,4.1], r' \in [4.1,4.2]$ and averaged over 100 realizations.

persists in the active state asymptotically [Figs. 2(a) and 2(b)] and is in agreement with earlier studies of global stability of independent networks [18,23].

Figure 2(c) shows in detail the contrasting contribution of intra- and internetwork interactions to the robustness of the network in terms of maintaining persistent activity. The probability that a node persists in the active state asymptotically is seen to vary nonmonotonically with increasing interdependence D between the networks at different values of the parameters C, σ, and r that determine intranetwork dynamics. For reference let us focus on the curve for $C = 0.1$, $\sigma = 0.01$, and $r \in [4.0,4.1]$ [shown using circles in Fig. 2(c)]. We observe that when diffusion is either too low ($D < 0.09$) or high ($D > 0.2$) all activity in the network ceases within the duration of simulation. However, for the intermediate range of values of D, activity continues in at least a part
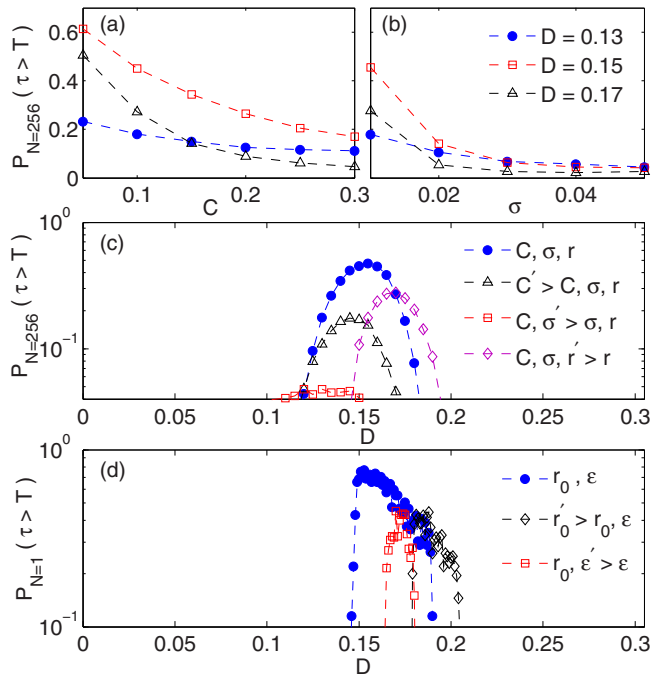
FIG. 2. (a)–(c) Probability that nodal activity persists for longer than the duration of simulation $P(\tau > T)$ for an interdependent system of two networks decreases monotonically with increasing connection density C (a) and dispersion of interaction strengths σ (b) as shown for three different values of internetwork coupling strength D [indicated by same symbols in (a) and (b)]. (c) shows that the probability of persistent nodal activity has a nonmonotonic dependence on D but decreases with increasing C, σ, and r. Each of the networks comprise $N = 256$ nodes. Parameter values used are $C = 0.1$, $\sigma = 0.01$, $r \in [4.0, 4.1]$, $C' = 0.3$, $\sigma' = 0.05$, and $r' \in [4.1, 4.2]$. (d) Probability of persistent activity in a system of two diffusively coupled elements ($N = 1$) whose nonlinearity parameters fluctuate about r_0 due to a noise of strength ϵ. Nonmonotonic dependence on coupling strength D is seen, similar to that for the large networks shown in (a). Parameter values are $r_0 = 4.05$, $\epsilon = 0.005$, $r_0' = 4.2$, and $\epsilon' = 0.01$. For all panels, simulation duration is $T = 5 \times 10^4$ iterations and results shown are averaged over 100 realizations.

of the network with the persistence probability reaching a peak around $D \simeq 0.16$. Varying the other parameters, such as network connectivity C, intranetwork interaction strength σ, or the nonlinearity parameter r, has a simpler outcome, viz., a decrease in the probability that activity will persist in the network at long times. This is shown by the other curves where we increase in turn C (triangles), σ (squares), and r (diamonds). Thus, our results indicate that there exists an attractor corresponding to persistent activity in the network for an optimal range of interdependence (in the neighborhood of $D = 0.15$) which coexists with the attractor corresponding to the extinction of network activity, relatively independent of other parameters.

To understand this in detail, we first note that even when $N = 1$, this much simpler system of two diffusively coupled elements exhibits qualitatively similar features when subjected to noise [Fig. 2(d)]. The multiplicative noise of strength ϵ in the nonlinearity parameter, viz., $r = r_0(1 + \epsilon\eta)$, where η is a

Gaussian random process with zero mean and unit variance, is introduced in lieu of the perturbations that each map will feel when connected to a much larger network through nonlinear interactions [Eq. (1)] [24]. As in the case of the network, we choose $r_0 > 4$ so that an isolated node will almost always converge to the absorbing state, resulting in its extinction. Upon coupling two nodes, however, we observe that the probability of long-term survival of activity in the system becomes finite at an intermediate range of diffusive coupling strength (around $D = 0.15$), similar to that observed for a $N = 256$ network in Fig. 2(c). Thus, understanding the genesis of diffusion-induced persistence for a pair of coupled logistic maps subject to noise [25], may provide an explanation for the same phenomenon observed in the system of interdependent networks described earlier.

The evolution equation for each node in the coupled system comprises two terms, the first representing the intrinsic dynamics of the node with the nonlinearity parameter r effectively reduced by a factor of $(1 - D)$ and the second being the contribution from the other node diffusively coupled to it. Note that the system converges to the absorbing state if the sum of the two terms exceeds 1. A lower bound for the range of D where persistence can occur is obtained by observing that in a persistent system the effective parameter governing the intrinsic dynamics has to necessarily be lower than 4, implying that $D_{c1} = 1 - (4/r)$. The upper bound for persistence is obtained by noting that when $D > D_{c2} = [1 - (1/r)]/2$, the dynamics of the two nodes become synchronized asymptotically, effectively making them identical to the uncoupled node that almost surely converges to the absorbing state. In the regime $D_{c1} < D < D_{c2}$, persistence results from out-of-phase oscillations of the two nodes, each alternately visiting disjoint intervals in $(0,1)$ such that the sum of the terms in their evolution equations never exceeds 1. Thus, regions in the $(0,1) \times (0,1)$ domain giving rise to in-phase oscillation converge to the absorbing state (extinction), while the ones mapping to out-of-phase solution persist, resulting in a complex basin of attraction for the persistent activity state of the system [26].

The bifurcation diagrams shown in Figs. 3(a)–3(c) indicate how the range of diffusive coupling strengths over which persistent activity is observed changes as we move from the simple case of two coupled maps ($N = 1$) to interdependent networks ($N \gg 1$). As already discussed, diffusively coupling two logistic maps having $r > 4$ allow their states to remain in the unit interval (corresponding to the nodes being active) provided the strength of coupling D remains within an optimal range [Fig. 3(a)]. Note that within this range there exists a region, approximately between $(0.11, 0.18)$, in which the attractor of the dynamical state of the node occupies a much smaller region of the available phase space $I : (0,1)$. It is intuitively clear that for such values of D, introducing noise is much less likely to result in the system dynamics going outside the unit interval (thereby making the node inactive). If we now introduce multiplicative noise of low intensity (i.e., small ϵ), the range of D over which persistent activity occurs shrinks [Fig. 3(b)]. However, noise does not completely alter the nature of the system dynamics even though the bifurcation structure is now less crisp. The system appears to be particularly robust in the region referred to earlier where
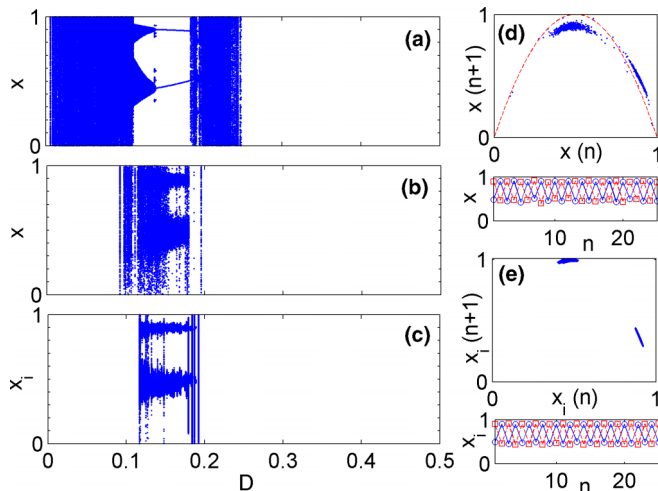
FIG. 3. Bifurcation diagrams showing the attractor of the dynamical state x of a representative element as a function of the diffusive coupling strength between [(a),(b)] two maps and (c) two networks each comprising $N = 256$ nodes. (a),(b) The range of D over which there is long-term persistence of activity in two coupled maps for $r = 4.0025$ (a) is reduced when multiplicative noise of strength $\epsilon = 0.0125$ is introduced (b). The bifurcation structure resembles that of coupled networks shown in (c) for $M = 24$, $C = 0.1, \sigma = 0.01, r \in [4.0, 4.1]$. The contribution of intranetwork interactions is qualitatively similar to multiplicative noise, resulting in a similar range of D for which persistence is observed in (b) and (c). (d),(e) This similarity is reinforced by comparing the return maps (upper panels) and time series (lower panels) of the asymptotic dynamical states for (d) two coupled maps with noise [as in (b)] and (e) two networks [as in (c)] for $D = 0.15$. The broken curve in panel (d) represents the return map of an uncoupled logistic map for $r = 4.0025$ shown for comparison.

the attractor covers only a small volume of the unit interval. We can compare this case with that of two interdependent networks, each comprising a large number of nodes [Fig. 3(c)] where the intranetwork interactions are considered effectively to be "noise." We observe a reasonable similarity between their bifurcation structures, with the region of persistent activity spanning approximately the same range of D. As in the case of coupled maps with noise, in the case of networks also the system is most robust in the region where the attractor for the unperturbed system of two diffusively coupled maps is confined within a small subinterval inside $I \times I$. The validity of considering the dynamics of coupled nodes embedded within a network as equivalent to the pair being perturbed by an effective noise is further established by the strong resemblance between the return maps and time series for the two cases [Figs. 3(d) and 3(e)]. As mentioned earlier, to survive indefinitely the dynamical state of each map switches alternately between two disjoint intervals of the unit interval in

an out-of-phase arrangement [see the time series in the lower panels of Figs. 3(d) and 3(e)], corresponding to a trajectory that jumps between two "islands" of the basin for the attractor corresponding to persistent activity in the coupled system [26].

The above analysis, apart from explaining why populations that go extinct rapidly in isolation will survive for long times upon being coupled optimally, also helps us understand how the persistence behavior in the system will be affected by increasing the number of interacting components. As can be observed from Eq. (1), increasing N keeping C, σ unchanged corresponds to the summation in the interaction term being performed over more components. This suggests that there will be stronger fluctuations, which can be interpreted as a larger effective noise applied to the individual elements resulting in a higher probability of reaching the absorbing state and thereby lowering the survival fraction f_{active}. We have confirmed this through explicit numerical calculations in which N is systematically increased. To ensure that the results reported here are not sensitively dependent on the specific details of the model that we have considered here, we have also carried out simulations with (i) different forms of unimodal nonlinear maps, e.g., $F(x) = (x - l)e^{r(1-x)}$ for $x > l$; 0 otherwise [27], and (ii) different types of connection topologies for the initial network, e.g., those with small-world properties [28,29] or having scale-free degree distribution [30]. We find in all such cases that the qualitative features reported here are unchanged, with the network connecting the surviving nodes becoming homogeneous asymptotically irrespective of the nature of the initial topology, suggesting that the enhanced persistence of activity in optimally interdependent networks is a generic property.

To conclude, we have investigated the role of interdependence between constituent networks on the stability of the entire system in a dynamical framework. Unlike percolation-based approaches where failure is often identified exclusively with breakdown of connectivity so that increasing interdependence necessarily enhances fragility [5], our dynamical perspective leads to a strikingly different conclusion. In particular, we show that the system has a much higher likelihood of survival for an optimal interdependence, with both networks facing almost certain catastrophic collapse in isolation. Such enhancement of persistence of activity in a critical range of coupling is analogous to the promotion of synchronization among self-propelled agents for an optimal interaction strength [31]. Our results suggest that interdependence may be essential in several natural systems for maintaining diversity in the presence of fluctuations that are potentially destabilizing. Thus, interdependence need not always have negative repercussions. Instead its impact may depend strongly on the context, e.g., the nature of coupling and the type of dynamics being considered.

[1] M. J. Pocock, D. M. Evans, and J. Memmott, Science **335**, 973 (2012).

[2] S. M. Rinaldi, J. P. Peerenboom, and T. K. Kelly, IEEE Control Syst. **21**, 11 (2001).

[3] F. Schweitzer, G. Fagiolo, D. Sornette, F. Vega-Redondo, A. Vespignani, and D. R. White, Science **325**, 422 (2009).

[4] D. Helbing, Nature (London) **497**, 51 (2013).

[5] A. Vespignani, Nature (London) **464**, 984 (2010).

[6] J. Gao, S. V. Buldyrev, S. Havlin, and H. E. Stanley, Phys. Rev. Lett. **107**, 195701 (2011).

[7] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin, Nature (London) **464**, 1025 (2010).

[8] R. Parshani, S. V. Buldyrev, and S. Havlin, Phys. Rev. Lett. **105**, 048701 (2010).

[9] W. Li, A. Bashan, S. V. Buldyrev, H. E. Stanley, and S. Havlin, Phys. Rev. Lett. **108**, 228702 (2012).

[10] J. Gao, S. V. Buldyrev, H. E. Stanley, and S. Havlin, Nat. Phys. **8**, 40 (2012).

[11] J. Gao, S. V. Buldyrev, H. E. Stanley, X. Xu, and S. Havlin, Phys. Rev. E **88**, 062816 (2013).

[12] B. Barzel and A.-L. Barabási, Nat. Phys. **9**, 673 (2013).

[13] M. M. Danziger, A. Bashan, and S. Havlin, New. J. Phys. **17**, 043046 (2015).

[14] J. M. Carlson and J. Doyle, Proc. Natl. Acad. Sci. USA **99**, 2538 (2002).

[15] C. C. Wilmers, Trends Ecol. Evol. **22**, 504 (2007).

[16] R. M. May, Nature (London) **238**, 413 (1972).

[17] N. S. Goel, S. C. Maitra, and E. W. Montroll, Rev. Mod. Phys. **43**, 231 (1971).

[18] S. Sinha and S. Sinha, Phys. Rev. E **74**, 066117 (2006).

[19] M. J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978).

[20] R. M. May, Nature (London) **261**, 459 (1976).

[21] M. Brin and G. Stuck, *Introduction to Dynamical Systems* (Cambridge University Press, Cambridge, 2002).

[22] E. G. Altman, J. S. E. Portela, and T. Tél, Rev. Mod. Phys. **85**, 869 (2013).

[23] S. Sinha and S. Sinha, Phys. Rev. E. **71**, 020902 (2005).

[24] The choice of Gaussian noise is motivated by the strength of intranetwork interactions, J_{ij}, being drawn from a normal distribution of zero mean and a finite variance. Moreover, if we focus on a single time-evolution step, the perturbations arising from intranetwork interactions can be approximated by Gaussian distributed random variables for large networks by using the central limit theorem. Subsequent evolution of the network may introduce correlations among the perturbations arising from the interactions which will not be accounted for in this Gaussian noise approximation.

[25] S. J. Linz and M. Lucke, Phys. Rev. A **33**, 2694 (1986).

[26] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.96.020301 for details about the basin of attraction for the persistent activity state of the system.

[27] Similar in form to the population dynamics model proposed in W. E. Ricker, J. Fish. Res. Board Can. **11**, 559 (1954).

[28] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).

[29] S. Sinha, Physica A **346**, 147 (2005).

[30] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).

[31] H. Peng, D. Zhao, X. Liu, and J. Gao, PLoS One **10**, e0144153 (2015).