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Abstract – It has recently been shown that a population of oscillators having identical environments can exhibit a heterogeneous phase topology termed as *chimera state*. We extend this phenomenon to the broader perspective of order-disorder transitions in physical systems with discrete states. By an exact analytic treatment we show that chimera states can occur in a system of Ising spins in thermal equilibrium. We also numerically establish the existence of chimera ordering in 3-dimensional models of layered magnetic materials (such as manganites) suggesting possible means of experimentally observing it.

Transition to states characterized by simple or complex ordered patterns is a phenomenon of central importance in equilibrium statistical physics as well as in dynamical systems far from equilibrium [1,2]. Examples of simple ordering at thermal equilibrium include the aligned orientation of spins in Ising-like systems, while, in the context of nonlinear dynamics, this may be observed in the phase synchronization of coupled oscillators. However, more complex ordering behavior may also occur in various systems under different conditions, especially in the presence of heterogeneities. A surprising recent finding is that even *homogeneous* dynamical systems can exhibit a robust, partially ordered state characterized by the coexistence of incoherent, desynchronized domains with coherent, phase-locked domains [3]. Such *chimera* states have so far been observed only in different types of oscillator populations, including complex Ginzburg-Landau equations, phase oscillators, relaxation oscillators, etc., arranged in various connection topologies [4–13]. Given that "chimera" refers to the co-occurrence of incongruous elements, one can extend the concept of chimera-like states to include those characterized by the simultaneous existence of strongly and weakly ordered regions in an otherwise homogeneous system. If such a state can occur as the global free-energy minimum of a system in thermal equilibrium, it may widen the scope of experimentally observing chimera-like order in physical situations.

It is with this aim in mind that we investigate chimera-like ordering in systems at thermal equilibrium. Specifically, we consider spin-models as they are paradigmatic for different complex systems comprising interacting components which can be in any of multiple discrete states. For example, simple Ising-like models consisting of binary-state elements are versatile enough to be used for understanding processes operating in a wide range of physical (*e.g.*, magnetic materials [14–16]), biological (*e.g.*, neural networks [17]) and social (*e.g.*, opinion formation [18,19]) systems. The nature and connection topology of the interactions between the spins decide whether the entire population reaches a consensus corresponding to a highly ordered state, or is in a weakly ordered state (including the case of complete disorder) that corresponds to the stable coexistence of contrary orientations. The existence of a chimera state in such situations would imply that even though every spin is in an identical environment, different regions of the system will exhibit widely different degrees of ordering.

In this paper we report for the first time the occurrence of chimera order in spin systems. This is characterized for a system of Ising spins by the simultaneous occurrence of strongly and weakly ordered regions, as measured by the magnitude of local magnetizations. The specific system we consider in detail is globally coupled and comprises two sub-populations (or *modules*) with the nature of interactions between spins depending on whether they belong to the same or different groups. Our central result is that when subjected to a uniform magnetic field at a

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finite temperature, one of the sub-populations can become *highly ordered* while the other remains *weakly ordered*. This is surprising as both the interactions as well as the external field for the two modules are *identical*. Moreover, the chimera state is not a metastable state, but rather the global minimum of free energy for the system. The critical behavior of the system associated with the onset of chimera ordering is established in this paper by an exact analytical treatment. We also demonstrate by Monte Carlo (MC) simulations the existence of similar complex ordering phenomena in three-dimensional spin systems with nearest neighbor interactions. This opens up the possibility of experimentally observing chimera states in layered magnetic systems, *e.g.*, manganites [15,16]. While the effect of noise on chimera state in coupled oscillators is not well understood, the chimera order in spin systems reported here arises in the presence of (thermal) noise; thus, it is robust and likely to be seen in real physical situations.

We consider a system of $2N$ globally coupled Ising spins arranged into two sub-populations, each having N spins, at a constant temperature T and subjected to a uniform external magnetic field H (>0). A dynamical system analogous to our model has recently been analyzed by Abrams *et al.* [8] where two clusters of identical oscillators, each maintaining a fixed phase difference with the others, was shown to possess a chimera state. The interaction between two spins belonging to the same module is ferromagnetic, having strength J (>0), while that between spins belonging to different modules is antiferromagnetic with strength $-J'$ (where $J' > 0$). It is obvious that in the absence of an external field, the modules will be completely ordered in opposite orientations at zero temperature. As temperature is increased, the magnitude of the magnetizations for the two modules will decrease by the same amount, eventually becoming zero at a critical temperature, T_c. In the presence of an external field H that favors spins with +ve orientation, the module having negative magnetization will be subjected to competition between i) the field H which attempts to align the spins along the +ve direction and ii) the antiferromagnetic interaction J' which is trying to do the opposite. In the presence of a strong field $H > H_0$ (where H_0 is a threshold field), as the temperature is increased from zero, the spins in both modules initially remain ordered and are oriented in the *same* direction. We show below that beyond a certain critical temperature T_{c1}, one module becomes more disordered relative to the other. As the temperature increases further beyond a second critical temperature T_{c2}, the two modules again attain the same magnetization, which decreases gradually with T (fig. 1(a)). The phase transitions at T_{c1} and T_{c2} are continuous and are characterized by critical exponents α and β which are derived exactly below. For $H < H_0$, the spins in the two modules are oriented at $T = 0$ in opposite directions, although having the same magnitude. At any finite temperature below T_{c2}, the module whose spins were initially oriented opposite to the direction of
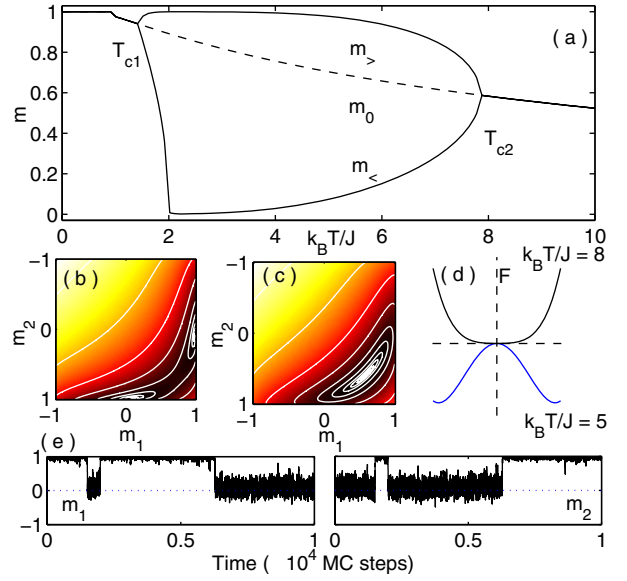
Fig. 1: (Colour on-line) (a) Variation of magnetization per spin of the two modules (m_1, m_2) with temperature for $H > H_0$. Between the temperatures T_{c1} and T_{c2}, the magnetizations of the two modules are different, with the smaller value being called $m_<$ and the larger one $m_>$. The broken line indicates the saddle point of the free-energy function corresponding to $m_1 = m_2 = m_0$ (see text). The free-energy landscape corresponding to chimera order at $k_B T/J = 5$ (b) shows that there are two free-energy minima for $m_1 \neq m_2$ (the curves are iso-energy contours and darker shades correspond to lower energy), whereas outside the range $[T_{c1}, T_{c2}]$ there is only one free-energy minimum (m_0) on the $m_1 = m_2$ line as is seen for $k_B T/J = 8$ (c). This is seen explicitly in (d) when the free energy per spin F is observed along the curve of steepest descent from m_0 (for $T_{c1} < T < T_{c2}$) or along the curve of slowest ascent (for $T < T_{c1}$ or $T > T_{c2}$). All the results shown in (a–d) are obtained analytically. (e) Time-evolution of the magnetizations per spin of the two modules, m_1 and m_2, shown for MC simulations with $N = 100$ at $k_B T/J = 5$. In the chimera ordered state, the system switches due to thermal noise between the two free-energy minima corresponding to the two modules exchanging their magnetization states between $m_>$ and $m_<$. In all cases, $a = 1$ and $b = H = 10$.

the field is seen to be more disordered than the other module. The same critical exponents as in the case of $H > H_0$ are observed for the transition at T_{c2}, beyond which the magnetization of the two modules are the same in magnitude and orientation.

For the system described above, the energy for a given configuration of spins is

$$E = -J \sum_{\substack{i,j,s \\ i \neq j}} \sigma_{is}\sigma_{js} + J' \sum_{\substack{i,j,s,s' \\ s \neq s'}} \sigma_{is}\sigma_{js'} - H \sum_{i,s} \sigma_{is}, \quad (1)$$

where $\sigma_{is} = \pm 1$ is the Ising spin on the i-th node ($i, j = 1, 2, \ldots N$) in the s-th module ($s, s' = 1, 2$) and $J, J' > 0$. Since each spin is connected to every other spin, mean-field treatment is exact for our effectively infinite-dimensional

system. Thus, the total free energy of the system can be expressed as

$$F(m_1, m_2) = -aN(m_1^2 + m_2^2) + bNm_1m_2 - HN(m_1 + m_2)$$
$$+ Nk_BT\left[S(m_1) + S(m_2)\right], \qquad (2)$$

where the magnetizations per spin of the two modules $(m_s = \frac{1}{N}\sum_{i=1}^{N}\sigma_{is})$ are the order parameters for the system, $S(m) = \frac{1}{2}[(1+m)\log(1+m) + (1-m)\log(1-m)] - \log 2$ is the entropy term, and $a = J(N-1)/2$, $b = J'N$ are system parameters (k_B being the Boltzmann constant).

To find the condition for equilibrium at a temperature T, the free energy can be minimized with respect to m_1 and m_2 to obtain

$$-2am_1 + bm_2 - H + \frac{k_BT}{2}\log\frac{1+m_1}{1-m_1} = 0, \qquad (3)$$

$$-2am_2 + bm_1 - H + \frac{k_BT}{2}\log\frac{1+m_2}{1-m_2} = 0. \qquad (4)$$

Equations (3), (4) may be expressed in the form $m_{1,2} = g(m_{2,1})$, where we define the one-dimensional map

$$g(x) = \frac{1}{b}\left[2ax + H - \frac{k_BT}{2}\log\frac{1+x}{1-x}\right]. \qquad (5)$$

Solutions of $g^2(x) \equiv g(g(x)) = x$ give the extrema m_1^* and m_2^* of the free-energy F (eq. (2)). Numerical solution for the extrema values shows that for suitable parameter values and $H > H_0$, the system has two critical temperatures T_{c1} and T_{c2}. For temperatures lower than T_{c1} and above T_{c2} the only fixed-point of the map g^2 is the unstable fixed point, $g(x) = x$, of eq. (5). Thus, this solution corresponds to $m_1 = m_2 \equiv m_0$, where the free energy $F(m_1, m_2)$ has a minimum. The value for m_0 is obtained from

$$(-2a + b)m_0 - H + \frac{k_BT}{2}\log\frac{1+m_0}{1-m_0} = 0. \qquad (6)$$

However, in the temperature range $T_{c1} < T < T_{c2}$, there are *two* types of fixed points of g^2: i) a stable fixed point $m_1 = m_2 = m_0$ (obtained from eq. (6)) corresponding to a saddle point of the free-energy function (shown by a broken line in fig. 1(a)), and ii) the pair of unstable fixed points $m_1 \neq m_2$ which form a period-2 orbit of eq. (5) corresponding to a minimum of the free energy F (shown by solid lines in fig. 1 (a)). As one of (m_1, m_2) is higher and the other low, we obtain a chimera state where one module is disordered $(m_<)$ relative to the other module $(m_>)$. The chimera state occurs through subcritical pitchfork bifurcations of the map g^2 as the temperature is increased above T_{c1} or decreased below T_{c2} (see footnote [1]). For $H < H_0$, the system exhibits chimera ordering for $T > 0$ and it has a single critical temperature at T_{c2} above which the magnetizations of the two modules become the same.

By observing the free-energy $F(m_1, m_2)$ landscape in the range $0 \leqslant m_1, m_2 \leqslant 1$, we obtain a clear physical picture of the transition to chimera ordering (fig. 1(b), (c)). The homogeneous state $m_1 = m_2 = m_0$ is a local extremum (*i.e.*, $\partial F/\partial m_{1,2} = 0$) for the range of parameters considered here. However, its nature changes from a free-energy minimum to a saddle point as the temperature is increased beyond T_{c1} and again changes back to a minimum when temperature exceeds T_{c2}. This is seen by looking at the matrix of the second derivatives of free energy per site with respect to m_1, m_2:

$$\mathcal{H}|_{m_0} = \begin{pmatrix} A & b \\ b & A \end{pmatrix}, \qquad (7)$$

where $A = -2a + k_BT\frac{1}{1-m_0^2}$. The eigenvalues of this matrix are $\lambda_+ = A + b$ along the $m_1 = m_2$ line and $\lambda_- = A - b$ in the direction perpendicular to it (parallel to $m_1 = -m_2$ line). Below T_{c1} and above T_{c2} both eigenvalues are positive indicating that m_0 is a minimum (fig. 1(c)). The transition to chimera ordering occurs in the range $T_{c1} < T < T_{c2}$ when the smaller eigenvalue λ_- becomes negative while the other eigenvalue remains positive, indicating that m_0 is now a saddle point. This gives us an implicit relation for T_c as the temperature where $\lambda_- = 0$, which gives

$$k_BT_c = 2(2a + b)(1 - m_0^2).$$

Numerical investigation of the landscape indicates that this transition is accompanied by the creation of two minima away from the $m_1 = m_2$ line (fig. 1(b)). These minima are symmetrically placed about the $m_1 = m_2$ line (as $F(m_1, m_2) = F(m_2, m_1)$) and correspond to the two coexisting chimera states $C_1 : m_1 = m_>, m_2 = m_<$ and $C_2 : m_1 = m_<, m_2 = m_>$. The two minima are separated by an energy barrier $\Delta = F(m_0, m_0) - F(m_>, m_<)$ which for a finite system can be crossed by thermal energy (fig. 1(d)). This switching behavior between the two chimera states has a characteristic time $\tau \sim \exp(\Delta/k_BT)$ which is indeed observed from MC simulations in small systems (fig. 1(e))[2]. Note that each minimum corresponds to a state having a specific arrangement of both highly ordered and weakly ordered regions, and hence is unlike the minima seen in phase-coexistence state of systems such as metamagnets, where each of the minima corresponds to a homogeneous phase (ordered or disordered)[3].

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[1]It is of interest to note that the chimera state in oscillator arrays occur via saddle-node bifurcation [4,8,20].

[2]For $N = 100$, we observe that $\tau > 10^3$ MC steps for a range of temperatures. For larger systems, *e.g.*, $N = 1000$, switching was not observed for the duration of our simulations, which is consistent with the exponential divergence of τ as one approaches the thermodynamic limit. This follows from the exponential dependence of τ on the energy barrier height, and the fact that free energy is proportional to N for given values of a, b, H and T as can be seen from eq. (2).

[3]See, *e.g.*, sect. 4 of ref. [15]. Note that the spins in the weakly ordered module change their state continuously with time and the auto-correlation function of each spin in this region rapidly decays to zero.
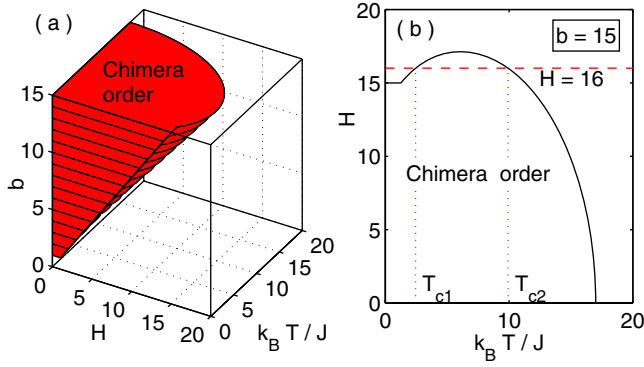
Fig. 2: (Colour on-line) (a) Phase diagram in the magnetic field (H), temperature ($k_B T/J$) and anti-ferromagnetic coupling (b) parameter space obtained by numerical minimization of free energy for $a = 1$ with the region in which chimera ordering occurs being indicated. A cross-section along the H-T plane for $b = 15$ is shown in (b). The broken line indicates $H = 16$, for which the critical temperatures are shown by dotted lines.

Figure 2 shows the region in (H-T-b) parameter space where chimera ordering is observed in our system as obtained by numerical minimization of the free energy. Temperature-induced transitions are always continuous whose exponents are analytically derived below. To investigate the critical behavior of the system around T_{c1} and T_{c2}, we shall use the order parameters

$$p_1 = m_1 - m_2 \quad \text{and} \quad p_2 = 2m_0 - (m_1 + m_2).$$

For $T_{c1} < T < T_{c2}$ where the chimera ordering is observed, as mentioned earlier the free-energy minima are at m_1 and m_2, while m_0 corresponds to a saddle point. The order parameters p_1 and p_2 are non-zero in this region and zero elsewhere. When p_1, p_2 are small, we solve for them using eqs. (3), (4) and (6) by expressing m_1 and m_2 in terms of p_1, p_2, and obtain

$$p_1 \propto |T - T_c|^{1/2} \quad \text{and} \quad |p_2| \propto |T - T_c|. \tag{8}$$

Thus, as $T \to T_{c1}^+$ or $T \to T_{c2}^-$, the order parameters vanish continuously with exponents $\beta = 1/2$ for p_1 and $\beta = 1$ for p_2. Similar calculations for the field-induced transition at finite temperature yield identical critical exponents. Note that at zero temperature the field-induced transition is of first order and its discontinuous nature can be shown exactly by analyzing the free energy. The values of the exponents for all continuous transitions have been confirmed by us numerically.

We have also analyzed the critical behavior of the specific heat $C = -T \frac{\partial^2 F_0}{\partial T^2}$, where F_0 is the equilibrium free energy at a given a, b, H and T. Although it involves both first- and second-order derivatives of p_1 and p_2, as the most dominant term is $\partial^2 p_1/\partial T^2$, the divergence at critical temperature is characterized by exponent $\alpha = 3/2$: $C \propto |T - T_{c1,c2}|^{-3/2}$.
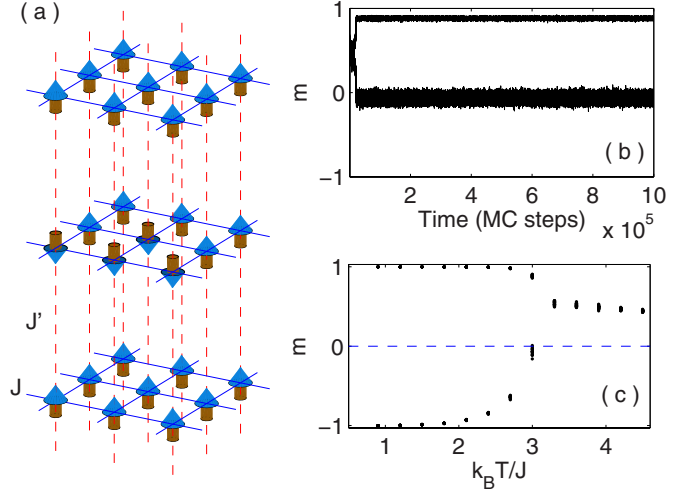

Fig. 3: (Colour on-line) (a) Schematic diagram of the 3-dimensional layered spin system with ferromagnetic (anti-ferromagnetic) interactions within layers, J (between layers, J') indicated by continuous (broken) lines. In the chimera state alternate layers show strong and weak order. (b) The time-evolution in MC steps of the magnetization of each of the 32 layers of a 3-dimensional system, with every layer having 128×128 spins, showing chimera ordering for $k_B T/J = 3$. (c) The magnetizations of different layers of the $128 \times 128 \times 32$ spin system at different temperatures. Chimera ordering is manifested as different values of $|m|$ for alternate layers (*e.g.*, at $k_B T/J = 3$). In all cases $J = J' = 1$ and $H = 1.8$.

While the system we have considered so far has the advantage of being amenable to exact analytical treatment, we have also numerically analyzed spin models which are closer to real magnetic materials. We have performed MC simulation studies of a three-dimensional Ising spin model with nearest-neighbor interactions having an anisotropic nature (fig. 3(a)). The system emulates a layered magnetic system comprising multiple layers of two-dimensional spin arrays stacked on top of each other, with interactions along a plane being ferromagnetic (J) and those between planes anti-ferromagnetic ($-J'$). One example of such a system is $FeCl_2$ where the exchange integral between Fe electron clouds is such that the Fe atoms within the same layer interact ferromagnetically while those in different layers interact antiferromagnetically [15]. Figure 3(b), (c) shows chimera ordering in such a 3-dimensional spin system with periodic boundary conditions and starting from random initial spin configurations. As seen from fig. 3(b), the chimera state appears relatively rapidly and persists for the duration of the simulation. Similar behavior was observed in other systems having different sizes, parameters and interaction structure, indicating that chimera ordering is a robust phenomenon that should be possible to observe in an experimental magnetic system. Note that in this system the chimera ordering is observed with nearest-neighbor

interactions, while, for systems of coupled oscillators, chimera states have so far been observed only with spatially non-local coupling.

In summary, we have shown the existence of a novel complex ordering behavior that we term chimera order in analogy with the simultaneous occurrence of coherent and incoherent behavior in dynamical systems. For a system of two clusters of Ising spins, where the spins are coupled ferromagnetically (anti-ferromagnetically) to all spins in the same (other) cluster, subjected to a uniform external magnetic field at a given temperature, chimera ordering is manifested as a much higher magnetization in one cluster compared to the other. To illustrate the wider implication of our result we can use the analogy of two communities of individuals who are deciding between a pair of competing choices. The interactions of an agent with other members of its own community strongly favor consensus while that with members of a different community are antagonistic. Thus, given that every individual is exposed to the same information or external environment, we would expect that unanimity about a particular choice in one community will imply the same for the contrary choice in the other community. However, the occurrence of the chimera order suggests that under certain conditions, when given the same external stimulus we may observe consensus in one community while the other is fragmented. The extension of the concept of chimera state as defined in our paper suggests that it may be experimentally observed in physical systems. Our demonstration of chimera order in a three-dimensional spin system with nearest-neighbor interactions indicate that a possible experimental example can be layered magnetic materials (*e.g.*, manganites) having different types of interactions between and within layers [15,16]. Although the present paper looks at the case of two competing choices, it is possible to extend the analysis to q-state Potts spin dynamics. Given the wider applicability of spin models for studying ordering in different contexts, one can consider other connection topologies as well as mesoscopic features such as the occurrence of multiple modules (>2) and hierarchical organization [21].

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