Complex Network Formation with Localized Payoffs

Acknowledgements

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Summary of the Results

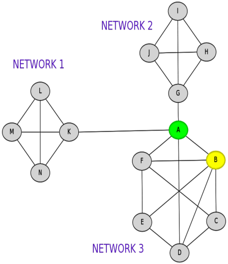
- Analyze a network formation game in a strategic setting where payoffs of individuals depend only on their immediate neighbourhood. We call these payoffs as localized payoffs .
- In this network formation game, the payoff of each individual captures the gain from immediate neighbors , the bridging benefits , and cost to form links .
- Analytically prove the pairwise stability (PS) of several interesting network structures.
- Analytically characterize topologies of efficient networks.
- Simulations validate our analysis and reveal additional insights on PS topologies.
- Our price of stability(PoS) analysis indicate the emergence of efficient PS networks.

Motivation for the Model

- Complex networks are large in size and hence, it is realistic to use *only* the local neighbourhood information to characterize such networks.
- Incorporate bridging benefits to model complex networks.

Importance of Bridging Benefits

- Benefit to a node :- Direct Link Benefit + Bridging Benefits .
- Node A has a bridging role in the social network.
- Node A is a structurally advantageous position than Node B.
- Empirical evidence show bridging benefit is insignificant if bridged paths are of length greater than two .



Formulation of Utility Function Using Local Neighbourhood Information

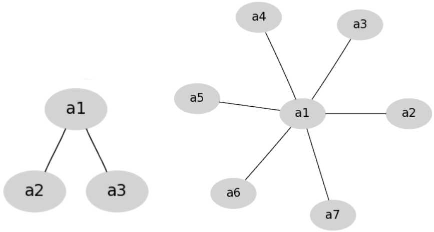
Sparsity of neighbourhood of node i with degree d_i is given by

$$s_i = \left(1 - \frac{\text{Number of links between neighbours}}{\text{Total possible links between neighbours}}\right)$$

- Interpretation for Sparsity: Higher $s_i \implies$ There are *less* links among the neighbours of node $i \implies$ Node i is *more* important for communication among neighbours.

Utility Function Formulation

$$u_i = d_i(\delta - c) + d_i s_i \delta^2$$

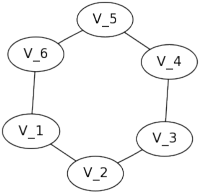


Sparsity of a1 in both networks = 1 !

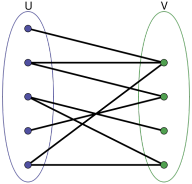
Analytical Deductions on Topologies of Pairwise Stable Networks

- A network is stable if no node has an incentive to add/delete a link.
- Some standard topologies considered for pairwise stability are given below.

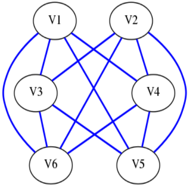
Cycle Network:



Bi-partite Network:



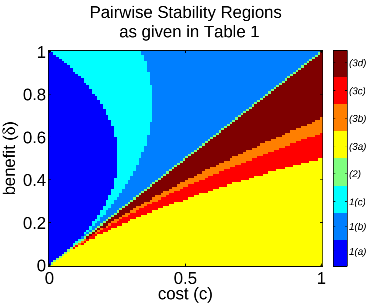
Equi-partitioned k-partite complete network ($k = 3$):



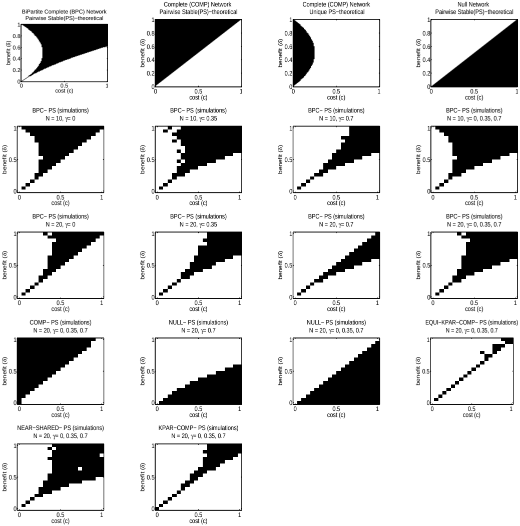
Analytical Deductions on Topologies of Pairwise Stable Networks

Parameter Region	Additional Conditions	P.S.[1] networks
(1) $\delta > c$	**(1a)** $(\delta - c) \geq \delta^2$	Complete
	(1b) $(\delta - c) < \delta^2$	Complete C.B.P [4]
	(1c) $(\delta - c) < 2/3\delta^2$	C.E.T.P [6] Complete C.B.P
(2) $\delta = c$		Complete, Null, C.B.P, C.E.K.P[5]
(3) $\delta < c$	**(3a)** $(c - \delta) > 2\delta^2$	Null
	(3b) $(c - \delta) \leq \delta^2$	C.B.P Null
	(3c) $\delta^2 \leq (c - \delta) \leq 2\delta^2$	Cycle Null
	(3d) $(c - \delta) < 2/3\delta^2$	C.E.T.P Null C.B.P

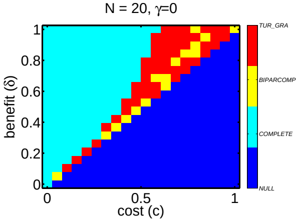
[1]P.S: Pairwise Stable [4]C.B.P: Complete BiPartite
[5]C.E.K.P: Complete Equi K-Partite
[6]C.E.T.P: Complete Equi Tri-Partite

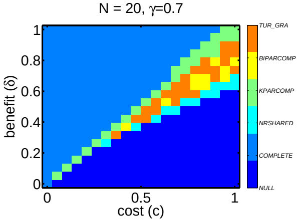


Validation of theoretical results

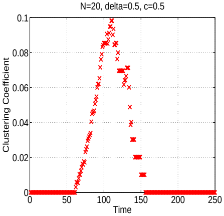


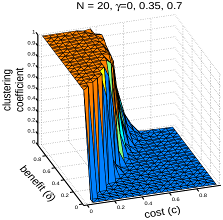
Emergent Topologies during Simulations



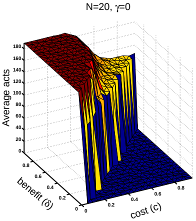


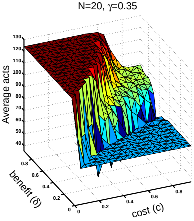
Study of Clustering Coefficient in the Network Formation Game

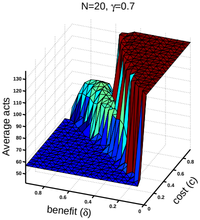




Convergence to Pairwise Stable Network







Topologies of Efficient Networks

Efficiency ($v(G)$)- The sum of individual utilities of the nodes in the network G i.e., $v(G) = \sum_{i=1}^{n} u_i$.

Results from Classical Extremal Graph Theory

From Turan's theorem, we know that

$$T \geq \begin{cases} \frac{n(4e-n^2)}{9} & \text{if } e > \left\lfloor \frac{n^2}{4} \right\rfloor \\ \\ 0 & \text{if } e \leq \left\lfloor \frac{n^2}{4} \right\rfloor \end{cases} \tag{1}$$

e : Number of edges, n : Number of vertices, T : Number of triangles of a graph.

Turan Graph Efficiency

Theorem

When $\delta = c$, the Turan graph is the unique efficient graph.

Proof sketch It can be shown

$$u(G) \leq \delta^2 \sum_{i=1}^{n} d_i - \frac{\delta^2}{(n-2)}(2 \times 3 \times T_3(G))$$

where, $T_3(G)$ is the number of triangles in the graph G.
Let $\overline{G}$ have $\lfloor \frac{n^2}{4} \rfloor + x$ edges where $x > 0$. We can also get

$$\Delta u = u(\overline{G}) - u(G_{Turan}) \leq 2\delta^2 \left(x - \frac{n}{(n-2)} \frac{4x}{3} \right)$$

Efficient Networks Emerging From Proposed Utility Model

Parameter Range	Efficient Topologies
$\delta < c$ and $\delta^2 < (c - \delta)$	Null network
$\delta < c$ and $\delta^2 > (c - \delta)$	Turan network
$\delta = c$	Turan network
$\delta > c$ and $\delta^2 > 3(\delta - c)$	Turan network
$\delta > c$ and $(\delta - c) > 2\delta^2$	Complete network

Price of stability (PoS) is the ratio of the sum of payoffs of the players in a best pairwise stable network to that of an efficient network.

PoS is 1 in each of the following scenarios:

(i) $\delta > c$ and $(\delta - c) > 2\delta^2$

(ii) $\delta > c$, $\delta^2 > (\delta - c)$ and $\delta^2 \geq 3(\delta - c)$

(iii)$\delta = c$

(iv) $\delta < c$ and $\delta^2 > (c - \delta)$

PoS is at least $1/2$ in the following case:

$\delta > c$ and $(\delta - c) \leq \delta^2 < 3(\delta - c)$