Epidemic and media campaigns: Understanding the interplay from mathematical standpoint

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Outline

Motivation

Preliminaries
- Epidemic models
- Behavioral epidemiology

The mathematical model
- Basic model: model for infectious diseases
- Model for vector borne diseases
- Model for HIV/AIDS

Conclusions

Future research

2009 H1N1 pandemic

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- **How awareness propagated by mass media affects the aftermath of an epidemic outbreak ?**

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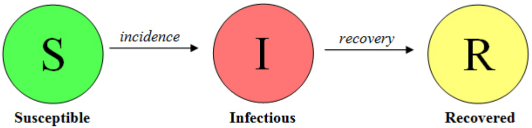
- **Can we capture this interplay using mathematical models?**

Kermack–McKendrick model



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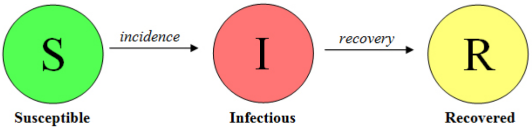

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$$\dot{S} = -\beta S I,$$
$$\dot{I} = \beta S I - \gamma I,$$
$$\dot{R} = \gamma I.$$

Basic reproduction number R_0

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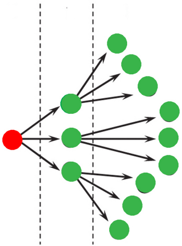

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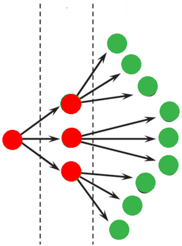

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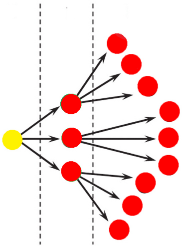

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$$\begin{cases} \dfrac{dS}{dt} = r - dS - \left(\beta_1 - \beta_2 \dfrac{I}{m+I}\right) SI + \delta R, \\ \dfrac{dI}{dt} = \left(\beta_1 - \beta_2 \dfrac{I}{m+I}\right) SI - (d+\gamma+\alpha)I, \\ \dfrac{dR}{dt} = \gamma I - (d+\delta)R, \end{cases}$$

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Modeling the effect of media-induced preventive behavior on an epidemic outbreak

Variable considered

- Susceptible population, X

Variable considered

- Susceptible population, X
- Infected population, Y

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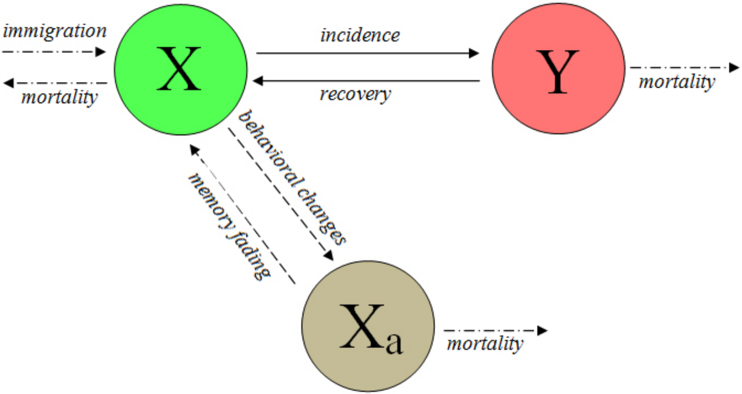
- Awareness programs by media, M

Schematic diagram



The model

$$
\begin{aligned}
\frac{dX}{dt} &= A - \beta XY - \lambda XM + \nu Y + \lambda_0 X_a - dX, \\
\frac{dY}{dt} &= \beta XY - \nu Y - \alpha Y - dY, \\
\frac{dX_a}{dt} &= \lambda XM - \lambda_0 X_a - dX_a, \\
\frac{dM}{dt} &= \phi Y - \phi_0 M,
\end{aligned} \tag{1.1}
$$

where, $X(0) > 0,\ Y(0) > 0,\ X_a(0) \geq 0,\ M(0) \geq 0.$

The model contd...

Using $X + Y + X_a = N$, model system (1.1) becomes,

$$
\begin{aligned}
\frac{dY}{dt} &= \beta(N - Y - X_a)Y - (\nu + \alpha + d)Y, \\
\frac{dX_a}{dt} &= \lambda(N - Y - X_a)M - \lambda_0 X_a - dX_a, \\
\frac{dN}{dt} &= A - dN - \alpha Y, \\
\frac{dM}{dt} &= \phi Y - \phi_0 M.
\end{aligned}
\tag{1.2}
$$

Region of attraction

The total population $N(t)$ is variable with

$$\frac{dN}{dt} = A - \alpha Y - dN \leq A - dN. \tag{1.3}$$

For the solution of equation (1.3), we have

$$0 < N(t) \leq N(0)e^{-dt} + \frac{A}{d}(1 - e^{-dt}). \tag{1.4}$$

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- $t \to \infty$, $N \to \frac{A}{d}$.
- If $N(0) \leq \frac{A}{d}$, then $\frac{A}{d}$ is the upper bound of N.
- If $N(0) > \frac{A}{d}$, then the solution enters or approaches asymptotically to the feasible region Ω defined by

$$\Omega = \left\{(Y, X_a, N, M) \in \mathbb{R}^4_+ : 0 \leq Y + X_a \leq N \leq \frac{A}{d}, 0 \leq M \leq \frac{\phi A}{\phi_0 d} = M_R\right\},$$

Analytical results

The model system (1.2) has two non-negative equilibria as follows:

- (i) Disease free equilibrium (DFE) $E_0(0, 0, \frac{A}{d}, 0)$.

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E^* exists only when $\beta A - d(\nu + \alpha + d) > 0$.

$$R_0 = \frac{\beta A}{d(\nu + \alpha + d)},$$

Equilibrium analysis

Analytical results contd...

Theorem

The equilibrium E_0 is stable whenever $R_0 < 1$ and is unstable for $R_0 > 1$. The endemic equilibrium E^* exists for $R_0 > 1$ and is locally stable provided,

$$A_1 A_2 A_3 - A_3^2 - A_1^2 A_4 > 0, \tag{1.5}$$

where, A_i's are the coefficients of characteristic equation of Jacobian matrix evaluated at E^*.

Proof

Analytical results contd...

Theorem

The endemic equilibrium E^* is globally stable in Ω provided,

$$\frac{3\lambda^2\phi^2}{(\lambda_0 + d)^2\phi_0^2} < \min\left\{\frac{d^2}{3A^2}, \frac{d^3}{\alpha A^2}, \frac{2}{9(N^* - Y^* - X_a^*)^2}\right\}. \quad (1.6)$$

Proof

Analytical results contd...

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▸ Proof

λ and ϕ have destabilizing effect on the system.

Numerical simulation

- **Set of parameter values :**

$$A = 400, \quad \beta = 0.00002, \quad \lambda = 0.0002, \quad \lambda_0 = 0.2, \quad \nu = 0.6,$$
$$\alpha = 0.02, \quad d = 0.01, \quad \phi = 0.0005, \quad \phi_0 = 0.06.$$

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- **Components of endemic equilibrium :**

$$Y^* = 2615, \; X_a^* = 653, \; N^* = 34769, \; M^* = 21.$$

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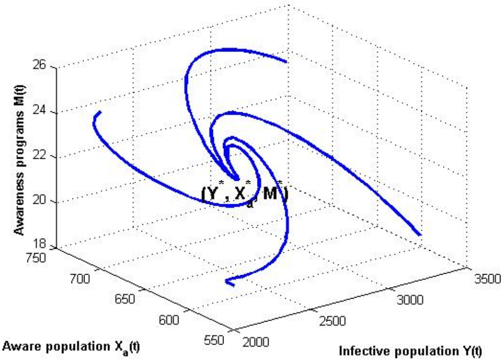
- **Eigenvalues :**

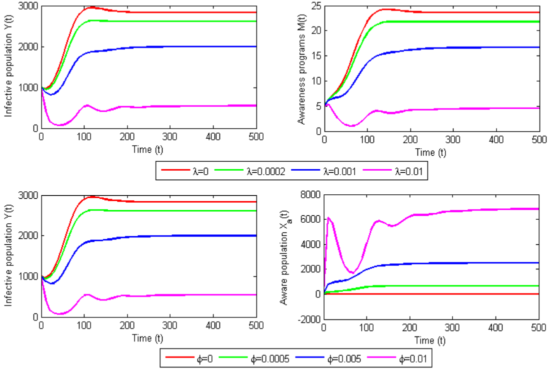
$$-0.2214, \; -0.03014, \; -0.0426 - 0.0375i \text{ and } -0.0426 + 0.0375i.$$

Numerical simulation contd...



Global stability of E^* in $Y - X_a - M$ space.

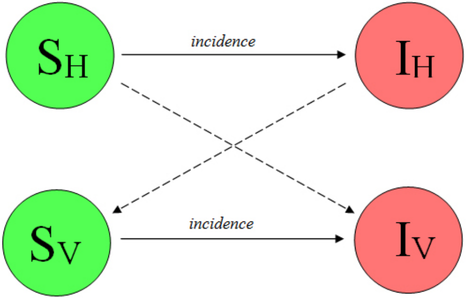
Numerical simulation contd...



Variation of variables with time for different values of parameters

Modeling the effect of media-induced awareness on the prevention of vector borne diseases

Criss-cross interaction



Variables considered

Total human population N_H

- Susceptible human population, X_H
- Infected human population, Y_H
- Aware human population, A_H

Variables considered

Total human population N_H

- Susceptible human population, X_H
- Infected human population, Y_H
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Total mosquito population N_V

- Susceptible vector population, X_V
- Infected vector population, Y_V

Variables considered

Total human population N_H

- Susceptible human population, X_H
- Infected human population, Y_H
- Aware human population, A_H

Total mosquito population N_V

- Susceptible vector population, X_V
- Infected vector population, Y_V

- Awareness programs by media, M

The model

$$
\begin{aligned}
\frac{dX_H}{dt} &= \Lambda - \beta_{HV} X_H Y_V - \lambda X_H M - d_H X_H + \nu Y_H + \lambda_0 A_H, \\
\frac{dY_H}{dt} &= \beta_{HV} X_H Y_V - \nu Y_H - \alpha Y_H - d_H Y_H, \\
\frac{dA_H}{dt} &= \lambda X_H M - \lambda_0 A_H - d_H A_H, \\
\frac{dX_V}{dt} &= b_V N_V - r \frac{N_V^2}{K} - \beta_{VH} X_V Y_H - \theta A_H X_V - d_V X_V, \\
\frac{dY_V}{dt} &= \beta_{VH} X_V Y_H - \theta A_H Y_V - d_V Y_V, \\
\frac{dM}{dt} &= \phi Y_H - \phi_0 (M - M_0),
\end{aligned}
\tag{2.1}
$$

where, $X_H(0) > 0, Y_H(0) \geq 0, A_H(0) \geq 0, X_V(0) \geq 0, Y_V(0) > 0, M(0) \geq M_0$.

The model contd...

As $N_H = X_H + Y_H + A_H$ and $N_V = X_V + Y_V$, the model system (2.1) can also be written as,

$$\begin{aligned}
\frac{dY_H}{dt} &= \beta_{HV}(N_H - Y_H - A_H)Y_V - (\nu + \alpha + d_H)Y_H, \\
\frac{dA_H}{dt} &= \lambda(N_H - Y_H - A_H)M - (\lambda_0 + d_H)A_H, \\
\frac{dN_H}{dt} &= \Lambda - \alpha Y_H - d_H N_H, \\
\frac{dY_V}{dt} &= \beta_{VH}(N_V - Y_V)Y_H - (\theta A_H + d_V)Y_V, \\
\frac{dN_V}{dt} &= rN_V\left(1 - \frac{N_V}{K}\right) - \theta A_H N_V, \\
\frac{dM}{dt} &= \phi Y_H - \phi_0(M - M_0).
\end{aligned} \tag{2.2}$$

Region of attraction

$$\Omega = \Big\{ (Y_H, A_H, N_H, Y_V, N_V, M) \in \mathbb{R}^6_+ : 0 \le Y_H + A_H \le N_H \le \frac{\Lambda}{d_H},$$
$$0 \le Y_V \le N_V \le K_R, 0 \le M \le M_R \Big\},$$

where, $K_R = \dfrac{K\left(r - (\theta p \Lambda / d_H)\right)}{r}$ and $M_R = \dfrac{\phi(\Lambda/d_H) + \phi_0 M_0}{\phi_0}$.

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where, $K_R = \dfrac{K\left(r - (\theta p \Lambda / d_H)\right)}{r}$ and $M_R = \dfrac{\phi(\Lambda/d_H) + \phi_0 M_0}{\phi_0}$.

- p is a dimensionless quantity defined as,

$$p = \frac{\lambda M_0}{\lambda M_0 + \lambda_0 + d_H}$$

Equilibria obtained

The model system (2.2) exhibits three non-negative equilibria:

- Disease and vector free equilibrium (DVFE)
  $E_0(0, p\Lambda/d_H, \Lambda/d_H, 0, 0, M_0)$

Equilibrium analysis

Equilibria obtained

The model system (2.2) exhibits three non-negative equilibria:

- Disease and vector free equilibrium (DVFE)
  $E_0(0, p\Lambda/d_H, \Lambda/d_H, 0, 0, M_0)$
- Disease free equilibrium (DFE)
  $E_1(0, p\Lambda/d_H, \Lambda/d_H, 0, K_R, M_0)$

Equilibrium analysis

Equilibria obtained

The model system (2.2) exhibits three non-negative equilibria:

- Disease and vector free equilibrium (DVFE)
  $E_0(0, p\Lambda/d_H, \Lambda/d_H, 0, 0, M_0)$
- Disease free equilibrium (DFE)
  $E_1(0, p\Lambda/d_H, \Lambda/d_H, 0, K_R, M_0)$
- Endemic equilibrium
  $E^*(Y_H^*, A_H^*, N_H^*, Y_V^*, N_V^*, M^*)$

◂ Equilibrium analysis

Basic reproduction number

- $R_0 = \dfrac{\beta_{HV}\beta_{VH}(1-p)\Lambda K(r-\theta p(\Lambda/d_H))}{rd_H(\theta p(\Lambda/d_H)+d_V)(\nu+\alpha+d_H)}$

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- It is cumulation of $\frac{K(r-\theta p(\Lambda/d_H))\beta_{HV}}{r(\nu+\alpha+d_H)}$ and $\frac{(1-p)\Lambda\beta_{VH}}{d_H(\phi p(\Lambda/d_H)+d_V)}$

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- ${R_0}'(p) < 0$

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- ${R_0}'(p) < 0$
- $\dfrac{\partial p}{\partial \lambda} > 0$ and $\dfrac{\partial p}{\partial M_0} > 0.$

Critical coverage

$$p_c = \frac{R_{00}\left(1+\frac{\theta\Lambda}{rd_H}\right)+\frac{\theta\Lambda}{d_V d_H}+\sqrt{\left[R_{00}\left(1+\frac{\theta\Lambda}{rd_H}\right)+\frac{\theta\Lambda}{d_H d_V}\right]^2-\frac{4\Lambda\theta R_{00}(R_{00}-1)}{rd_h}}}{\frac{2\Lambda\theta R_{00}}{rd_H}}$$

where $R_{00} = \dfrac{\beta_{HV}\beta_{VH}\Lambda K}{d_H d_V(\nu+\alpha+d_H)}$.

Stability analysis

Theorem

The DVFE always exists and is locally asymptotically stable if $r < \theta p(\Lambda/d_H)$. Whenever r is greater than $\theta p(\Lambda/d_H)$, DVFE becomes unstable and DFE exists which is stable until $R_0 < 1$.

Proof

Stability analysis contd...

Theorem

The endemic equilibrium, if exists, is locally asymptotically stable provided the following conditions hold,

$$\frac{\beta_{HV}^2 {Y_V^*}^2 \lambda^2}{(\beta_{HV} Y_V^* + \nu + \alpha + d_H)(\lambda M^* + \lambda_0 + d_H)^2} < \frac{4}{45} \min \left\{ \frac{(\beta_{HV} Y_V^* + \nu + \alpha + d_H)}{5{M^*}^2}, \frac{\beta_{HV} Y_V^* d_H}{\alpha {M^*}^2}, \frac{\phi_0^2(\beta_{HV} Y_V^* + \nu + \alpha + d_H)}{6\phi^2 (N_H^* - Y_H^* - A_H^*)^2} \right\}, \tag{2.3}$$

$$\frac{5\beta_{HV}^2 (N_H^* - Y_H^* - A_H^*)^2}{(\beta_{HV} Y_V^* + \nu + \alpha + d_H)(\beta_{VH} Y_H^* + \theta A_H^* + d_V)^2} < \min \left\{ \frac{(\beta_{HV} Y_V^* + \nu + \alpha + d_H)}{5\beta_{VH}^2 (N_V^* - Y_V^*)^2}, \frac{(\lambda M^* + \lambda_0 + d_H) k_1}{6\theta^2 {Y_V^*}^2} \right\}, \tag{2.4}$$

$$\frac{\beta_{VH}^2 {Y_H^*}^2}{\beta_{VH} Y_H^* + \theta A_H^* + d_V} k_3 < \frac{(\lambda M^* + \lambda_0 + d_H) r^2}{6K^2 \theta^2} k_1 . \tag{2.5}$$

Proof

Stability analysis contd...

Theorem

The endemic equilibrium is globally asymptotically stable in Ω, provided the following inequalities hold,

$$\frac{\beta_{HV}^2 Y_V^{*2}\lambda^2}{(\beta_{HV}Y_V^* + \nu + \alpha + d_H)(\lambda_0 + d_H)^2} < \frac{4}{45}\min\left\{\frac{(\beta_{HV}Y_V^* + \nu + \alpha + d_H)}{5M_R^2}, \frac{\beta_{HV}Y_V^* d_H}{\alpha M_R^2}, \frac{\phi_0^2(\beta_{HV}Y_V^* + \nu + \alpha + d_H)}{6\phi^2(N_H^* - Y_H^* - A_H^*)^2}\right\}, \tag{2.6}$$

$$\frac{5\beta_{HV}^2\Lambda^2}{(\beta_{HV}Y_V^* + \nu + \alpha + d_H)d_V^2 d_H^2} < \min\left\{\frac{(\beta_{HV}Y_V^* + \nu + \alpha + d_H)}{5\beta_{VH}^2(N_V^* - Y_V^*)^2}, \frac{(\lambda_0 + d_H)p_1}{6\theta^2 Y_V^{*2}}\right\}, \tag{2.7}$$

$$\frac{\beta_{VH}^2\Lambda^2}{d_H^2 d_V} p_3 < \frac{(\lambda_0 + d_H)r^2}{6K^2\theta^2} p_1. \tag{2.8}$$

◂ Proof

Numerical simulation

- **Parameter values**:
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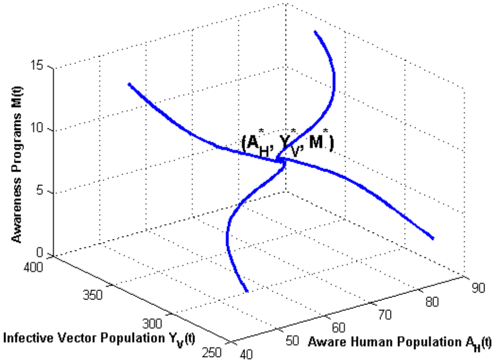
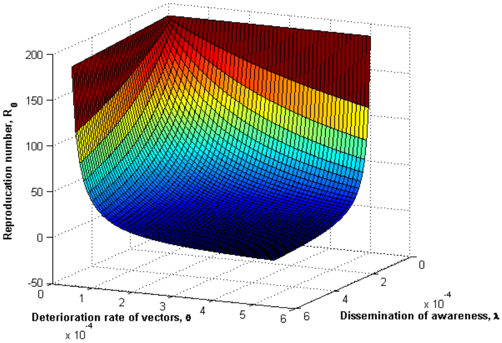
- $R_0 = 168.47$ and $p_c = 0.9944$.

Numerical simulation contd...



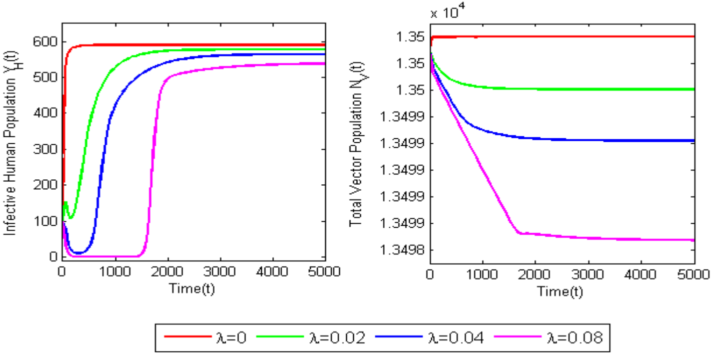
Global stability of E^* in $A_H - Y_V - M$ space

Numerical simulation contd...



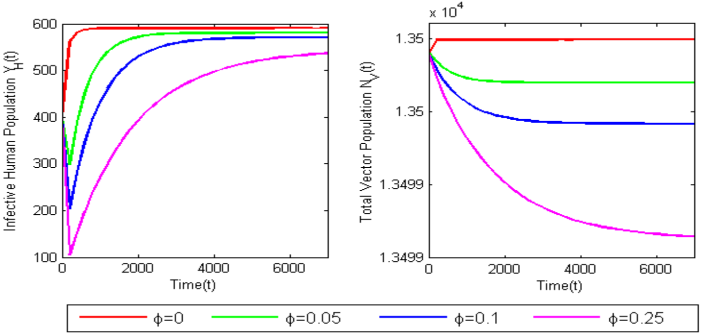
Variation of reproduction number, R_0 w.r.t. λ and θ.

Numerical simulation contd...



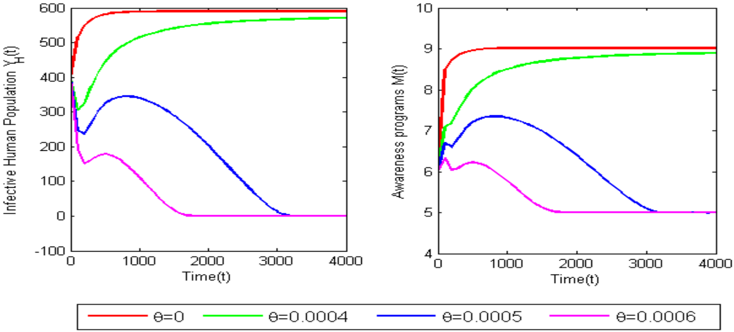
Variation of variables w.r.t. time for different values of λ

Numerical simulation contd...



Variation of variables w.r.t. time for different values of ϕ

Numerical simulation contd...



Variation of variables w.r.t. time for different values of θ

Modeling the interplay between transmission of HIV/AIDS and awareness with a case study of India

Scenario HIV/AIDS epidemic in India

- PLWHA: 23.9 lakh

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- Nearly 68% of Indian population lives in rural areas.
- Interpersonal communication through *Self Help Groups, Anganwadi workers, ANM, ASHA, NGOs*, etc.

Variable considered

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- Susceptible population, X
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- Aware susceptible population, X_a
- Aware infected population, Y_a
- Awareness programs by media, M

The model

$$
\begin{aligned}
\frac{dX}{dt} &= \mu N - \beta \frac{X(Y + pY_a)}{N} - \lambda_1 XM - \gamma_1 \frac{X(X_a + Y_a)}{N} - \mu X, \\
\frac{dY}{dt} &= \beta \frac{X(Y + pY_a)}{N} - \lambda_2 YM - \gamma_2 \frac{Y(X_a + Y_a)}{N} - (\mu + \alpha) Y, \\
\frac{dX_a}{dt} &= \lambda_1 XM + \gamma_1 \frac{X(X_a + Y_a)}{N} - \beta \frac{pX_a(Y + pY_a)}{N} - \mu X_a, \quad (3.1) \\
\frac{dY_a}{dt} &= \beta \frac{pX_a(Y + pY_a)}{N} + \lambda_2 YM + \gamma_2 \frac{Y(X_a + Y_a)}{N} - (\mu + \alpha) Y_a, \\
\frac{dM}{dt} &= \phi \frac{Y}{N} - \phi_0 M,
\end{aligned}
$$

where, $X(0) > 0,\ Y(0) > 0,\ X_a(0) > 0,\ Y_a(0) \geq 0,\ M(0) \geq 0.$

The model contd...

Using $\frac{X}{N} = S$, $\frac{Y}{N} = I$, $\frac{X_a}{N} = S_a$, and $\frac{Y_a}{N} = I_a$, we obtain the scaled system comprising the fractions of populations as,

$$
\begin{aligned}
\frac{dS}{dt} &= \mu - \beta SI - \lambda_1 SM - \gamma_1 S(S_a + I_a) - \mu S, \\
\frac{dI}{dt} &= \beta SI - \lambda_2 IM - \gamma_2 I(S_a + I_a) - \mu I, \\
\frac{dS_a}{dt} &= \lambda_1 SM + \gamma_1 S(S_a + I_a) - \mu S_a, \\
\frac{dI_a}{dt} &= \lambda_2 IM + \gamma_2 I(S_a + I_a) - \mu I_a, \\
\frac{dM}{dt} &= \phi I - \phi_0 M.
\end{aligned}
\tag{3.2}
$$

Region of attraction:

$$\Omega = \left\{ (I, S_a, I_a, M) \in \mathbb{R}^4_+ : 0 \leq I + S_a + I_a < 1, 0 \leq M < \frac{\phi}{\phi_0} \right\}$$

Equilibria obtained

The model system exhibits three equilibria namely:

- **Disease and awareness free equilibrium** $E_0(0, 0, 0, 0)$,

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This equilibrium exists provided $\gamma_1 > \mu$.

Equilibria obtained

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This equilibrium exists provided $\gamma_1 > \mu$.

- **Endemic equilibrium** $E^*(I^*, S_a^*, I_a^*, M^*)$,

It exists if $\beta > \mu$ and $(\gamma_1\gamma_2 + \mu\gamma_1 - \mu\beta - \mu\gamma_2) < 0$.

Basic reproduction number

- $R_0^i = \dfrac{\beta}{\mu}$ and $R_0^a = \dfrac{\gamma_1}{\mu}$

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- $R_0^i = \dfrac{\beta}{\mu}$ and $R_0^a = \dfrac{\gamma_1}{\mu}$
- **Relation:**

$$R_0^a < \zeta R_0^i,$$

where $\zeta = \left(\dfrac{1 + \frac{\gamma_2}{\beta}}{1 + \frac{\gamma_2}{\mu}} \right)$

Stability analysis

Theorem

1. The Disease and awareness free equilibrium is locally stable iff $R_0^i < 1$ and $R_0^a < 1$.
2. Disease free equilibrium exits iff $R_0^a > 1$ and is locally stable provided $\zeta R_0^i < R_0^a$.

Theorem

The endemic equilibrium E^*, whenever exists, is locally asymptotically stable provided, $A_1 > 0$, $A_3 > 0$, $A_1 A_2 > A_3$.

A case study of India

According to 2006 technical report of NACO, the adult prevalence in the initial phase of HIV/AIDS epidemic are given in the following table:

Year	Adult prevalence (%)
1989	0.010
1990	0.017
1991	0.03
1992	0.05
1993	0.09
1994	0.16
1995	0.26
1996	0.35
1997	0.43
1998	0.47
1999	0.48

A case study of India contd...

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$$\begin{cases} \dot{X} = & \mu N - \beta XY/N - \mu X, \\ \dot{Y} = & \beta XY/N - (\alpha + \mu)Y. \end{cases} \tag{3.3}$$

A case study of India contd...

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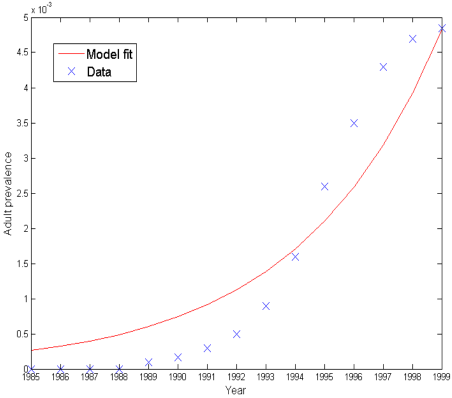
- $\mu = 1/34 = 0.0294$ and $\alpha = 0.0706$.

A case study of India contd...



HIV/AIDS model fitting for the adult prevalence in India (initial phase).

A case study of India contd...

- $\beta = 0.3089$

A case study of India contd...

- $\beta = 0.3089$

- *Best fit* (with R-squared $= 0.98$) for parameter values:

  $\lambda_1 = 0.8,\ \lambda_2 = 0.5,\ \gamma_1 = 0.5,\ \gamma_2 = 0.25,\ \phi = 50,$
  $\phi_0 = 0.02,\ p = 0.4.$

A case study of India contd...

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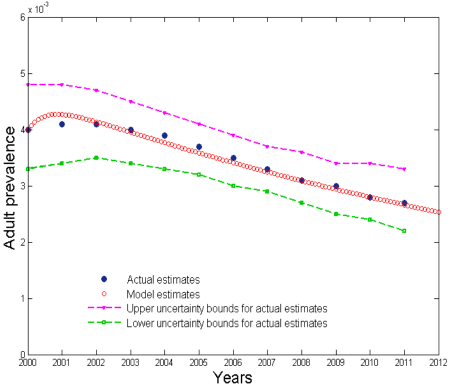
- Initial start:

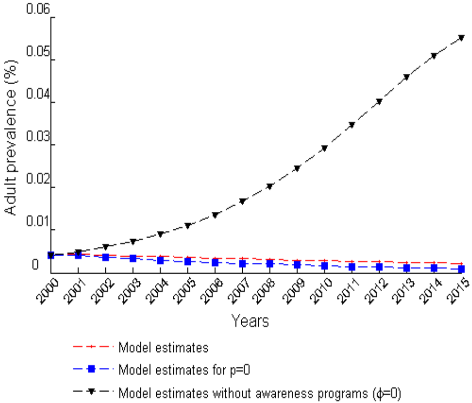
$$I(0) = 0.004, S_a(0) = 0.001, I_a(0) = 0, M(0) = 2.$$

A case study of India contd...



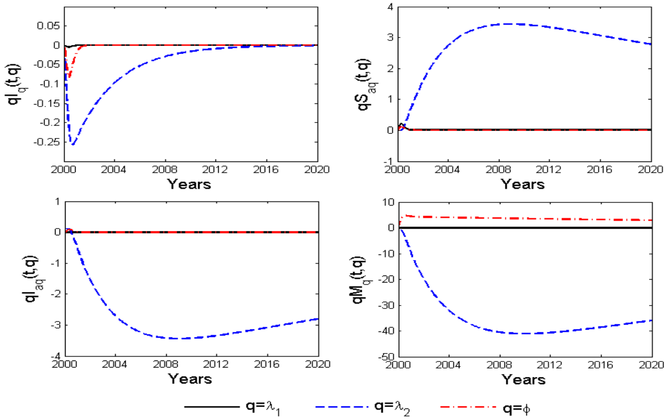
HIV/AIDS model fitting for the adult prevalence in India.

A case study of India contd...



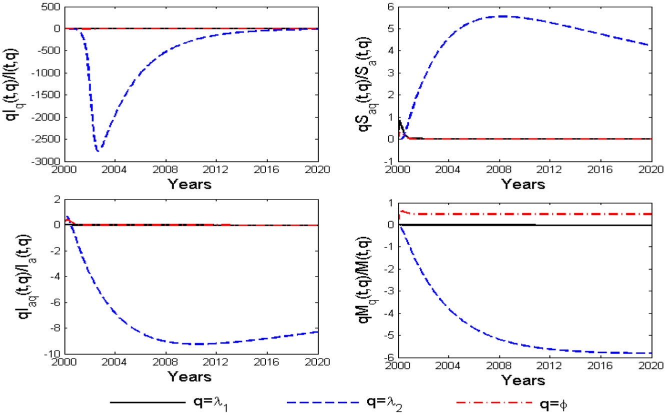
HIV/AIDS prevalence simulation under different scenarios

A case study of India contd...



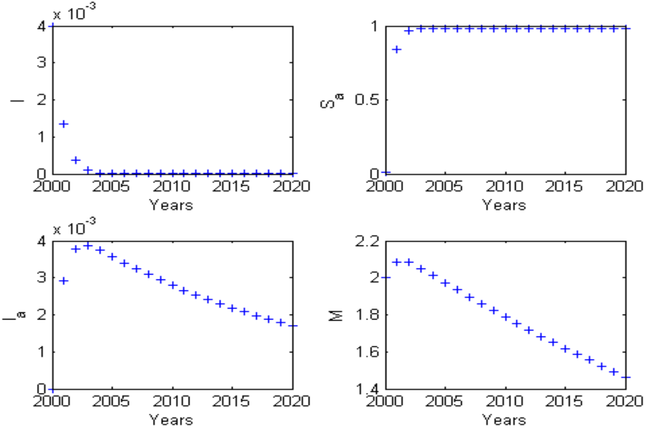
Semi-relative sensitivity solutions.

A case study of India contd...



Logarithmic sensitivity solutions.

A case study of India contd...



Future projections of HIV epidemic in India.

- Prevalence-elastic media campaigns can only control the epidemic not eradicate it.

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- Swift dissemination of awareness via word-of-mouth can also eradicate disease.
- Media-induced behavioral changes *can* perturb the stable endemic equilibrium.

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Is homogenous/random mixing adequate to reflect the reality ?

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NETWORK MODELS

- Scale-free
- Modular
- Multiplex

Thank You

The existence of equilibrium E_0 is trivial. We prove the existence of E^* in detail. In equilibrium $E^*(Y^*, X_a^*, N^*, M^*)$, the values of Y^*, X_a^*, N^* and M^* are obtained by solving the following algebraic equations(for Y $\neq 0$):

$$\beta(N - Y - X_a) - (\nu + \alpha + d) = 0, \tag{1}$$

$$\lambda(N - Y - X_a)M - \lambda_0 X_a - dX_a = 0, \tag{2}$$

$$A - dN - \alpha Y = 0, \tag{3}$$

$$\phi Y - \phi_0 M = 0. \tag{4}$$

Using equations (1) and (4) in equation (2) , we get,

$$X_a = \frac{\lambda\phi(\nu + \alpha + d)Y}{\beta\phi_0(\lambda_0 + d)}. \tag{5}$$

Further, using equations (3), (4) and (5) in equation (1), we get

$$\frac{A-\alpha Y}{d} - Y - \frac{\lambda\phi(\nu+\alpha+d)Y}{\beta\phi_0(\lambda_0+d)} = \frac{(\nu+\alpha+d)}{\beta}. \tag{6}$$

This yields the value of Y as

$$\begin{aligned} Y &= \frac{(\beta A - d(\nu+\alpha+d))}{d\beta(1+\frac{\alpha}{d}+\frac{\phi\lambda(\nu+\alpha+d)}{\phi_0\beta(\lambda_0+d)})} \\ &= Y^*(say), \end{aligned} \tag{7}$$

which is positive provided $R_0 > 1$. Finally using this value of $Y = Y^*$ in equations (3), (4) and (5), we get positive values of N, M and X_a respectively.

Back

The Jacobian matrix 'J' for the model system (1.2) is as follows:

$$J = \begin{bmatrix} a_{11} & -\beta Y & \beta Y & 0 \\ -\lambda M & -(\lambda M + \lambda_0 + d) & \lambda M & \lambda(N - Y - X_a) \\ -\alpha & 0 & -d & 0 \\ \phi & 0 & 0 & -\phi_0 \end{bmatrix}.$$

where, $a_{11} = \beta(N - 2Y - X_a) - (\nu + \alpha + d)$

Now the Jacobian matrix 'J', evaluated at the equilibrium E_0 is given by

$$J_{E_0} = \begin{bmatrix} \beta(A/d) - (\nu + \alpha + d) & 0 & 0 & 0 \\ 0 & -(d + \lambda_0) & 0 & \lambda(A/d) \\ -\alpha & 0 & -d & 0 \\ \phi & 0 & 0 & -\phi_0 \end{bmatrix}.$$

- Eigenvalues of matrix J_{E_0} are $(\beta(A/d) - (\nu + \alpha + d))$, $-(\lambda_0 + d)$, $-d$ and $-\phi_0$.
- One eigenvalue of this matrix i.e., $(\beta(A/d) - (\nu + \alpha + d))$, is positive whenever E^* exits (i.e., $R_0 > 1$).
- Thus if E^* exists, then E_0 is a saddle point with stable manifold locally in the $X_a - N - M$ space and with unstable manifold locally in the Y-direction.
- Thus E_0 is unstable whenever E^* exists.

Furthermore, to establish the local stability of endemic equilibrium E^*, we evaluate the Jacobian matrix 'J' at equilibrium E^* as

$$J_{E^*} = \begin{bmatrix} \beta Y^* & -\beta Y^* & \beta Y^* & 0 \\ -\lambda M^* & -(\lambda M^* + \lambda_0 + d) & \lambda M^* & \lambda(N^* - Y^* - X_a^*) \\ -\alpha & 0 & -d & 0 \\ \phi & 0 & 0 & -\phi_0 \end{bmatrix}.$$

The characteristic equation for matrix J_{E^*} is given by,

$$\psi^4 + A_1\psi^3 + A_2\psi^2 + A_3\psi + A_4 = 0, \tag{8}$$

where,

$$\begin{aligned}
A_1 =& \beta Y^* + \lambda M^* + \lambda_0 + \phi_0 + 2d, \\
A_2 =& \beta Y^*(\lambda_0 + \phi_0 + \alpha + 2d) + (\lambda M^* + \lambda_0 + d)(\phi_0 + d) + \phi_0 d, \\
A_3 =& \beta Y^* \phi_0(\alpha + d) + \beta Y^*(\phi_0 + \alpha + d)(\lambda_0 + d) + \beta Y^* \lambda(N^* - Y^* - X_a^*)\phi \\
& + (\lambda M^* + \lambda_0 + d)\phi_0 d, \\
A_4 =& \beta Y^* \phi_0(\lambda_0 + d)(\phi_0 + d) + \beta Y^* \lambda(N^* - Y^* - X_a^*)\phi d.
\end{aligned}$$

Now, it is apparent from here that all the A_i's for $i = 1, 2, 3, 4$ are positive. Thus, it follows from Routh-Hurwitz criterion that all the roots of equation (8) are either be negative or with negative real part provided, $A_1A_2A_3 - A_3^2 - A_1^2A_4 > 0$.

Consider the following positive definite function,

$$V = (Y - Y^* - Y^* \ln \frac{Y}{Y^*}) + \frac{m_1}{2}(X_a - X_a^*)^2 + \frac{m_2}{2}(N - N^*)^2) + \frac{m_3}{2}(M - M^*)^2 \tag{9}$$

where m_1, m_2 and m_3 are some positive constants to be chosen appropriately later on. On differentiating 'V' with respect to 't' we get,

$$\dot{V} = (Y - Y^*)\frac{\dot{Y}}{Y} + m_1(X_a - X_a^*)\dot{X}_a + m_2(N - N^*)\dot{N} + m_3(M - M^*)\dot{M}, \tag{10}$$

where $\dot{}$ represents differentiation w.r.t. time. Now the value of $\dot{V}$ along the solutions of model system (1.2) is computed as,

$$\dot{V} = -\beta(Y - Y^*)^2 - m_1(\lambda M + \lambda_0 + d)(X_a - X_a^*)^2 - m_2 d(N - N^*)^2$$

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$$\begin{aligned}\dot{V} \quad = \quad & -\beta(Y - Y^*)^2 - m_1(\lambda M + \lambda_0 + d)(X_a - X_a^*)^2 - m_2 d(N - N^*)^2 \\ & - m_3\phi_0(M - M^*)^2 - (\beta + m_1\lambda M)(Y - Y^*)(X_a - X_a^*) \\ & +(-m_2\alpha + \beta)(Y - Y^*)(N - N^*) + m_3\phi(Y - Y^*)(M - M^*) \\ & + m_1\lambda M(X_a - X_a^*)(N - N^*) + m_1\lambda(N^* - X_a^* - M^*)(X_a - X_a^*)(M - M^*).\end{aligned}$$

On choosing $m_2 = \beta/\alpha$ and making some simple algebraic manipulations we get,

$$\begin{aligned}\dot{V} &= -m_1\lambda M(X_a - X_a^*)^2 - \beta(Y - Y^*)^2 - m_1(\lambda_0 + d)(X_a - X_a^*)^2 \\ &\quad -\frac{\beta d}{\alpha}(N - N^*)^2 - m_3\phi_0(M - M^*)^2 - (\beta + m_1\lambda M)(Y - Y^*)(X_a - X_a^*) \\ &\quad + m_3\phi(Y - Y^*)(M - M^*) + m_1\lambda M(X_a - X_a^*)(N - N^*) \\ &\quad + m_1\lambda(N^* - X_a^* - M^*)(X_a - X_a^*)(M - M^*).\end{aligned}$$

Now $\dot{V}$ will be negative definite inside the region of attraction Ω, provided

$$\beta < \frac{m_1(\lambda_0 + d)}{3} \tag{11}$$

$$m_1\lambda^2 M_R^2 < \frac{\beta(\lambda_0 + d)}{3} \tag{12}$$

$$m_3\phi^2 < \frac{2\beta\phi_0}{3} \tag{13}$$

$$m_1\lambda^2 M_R^2 < \frac{\beta d(\lambda_0 + d)}{\alpha} \tag{14}$$

$$m_1\lambda^2(N^* - X_a^* - M^*)^2 < \frac{m_3\phi_0(\lambda_0 + d)}{2} \tag{15}$$

From inequality (13), we may choose a positive value of m_3 as $m_3 = \dfrac{4\beta\phi_0}{9\phi^2}$.
Thereafter, from rest of the inequalities, we may choose positive m_1 if the following inequality holds.

$$\frac{3\lambda^2}{(\lambda_0 + d)^2} < \min\left\{\frac{1}{3M_R^2},\ \frac{d}{\alpha M_R^2},\ \frac{2\phi_0^2}{9(N^* - Y^* - X_a^*)^2}\right\}. \tag{16}$$

Finally using the fact that $M_R = \dfrac{\phi A}{\phi_0 d}$ (*see region of attraction*), the above inequality (16) reduces to inequality (1.6). Back

To obtain the equilibria of model system (2.2), we solve the following algebraic equations, which are obtained by setting growth rate of all the variables equal to zero.

$$\begin{aligned}
\beta_{HV}(N_H - Y_H - A_H)Y_V - (\nu + \alpha + d_H)Y_H &= 0, && (1)\\
\lambda(N_H - Y_H - A_H)M - (\lambda_0 + d_H)A_H &= 0, && (2)\\
\Lambda - \alpha Y_H - d_H N_H &= 0, && (3)\\
\beta_{VH}(N_V - Y_V)Y_H - (\theta A_H + d_V)Y_V &= 0, && (4)\\
rN_V\left(1 - \frac{N_V}{K}\right) - \theta A_H N_V &= 0, && (5)\\
\phi Y_H - \phi_0(M - M_0) &= 0. && (6)
\end{aligned}$$

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\Lambda - \alpha Y_H - d_H N_H &= 0, \quad (3)\\
\beta_{VH}(N_V - Y_V)Y_H - (\theta A_H + d_V)Y_V &= 0, \quad (4)\\
rN_V\left(1 - \frac{N_V}{K}\right) - \theta A_H N_V &= 0, \quad (5)\\
\phi Y_H - \phi_0(M - M_0) &= 0. \quad (6)
\end{aligned}$$

The existence of E_0 and E_1 is trivial. So, here we discuss the existence of E^* only. For this purpose, using equations (2), (3) and (6), we obtain

$$A_H = \frac{\lambda(\Lambda - (\alpha + d_H)Y_H)(\phi Y_H + \phi_0 M_0)}{d_H(\lambda(\phi Y_H + \phi_0 M_0) + \phi_0(\lambda_0 + d_H))} = g(Y_H)(\text{say}). \tag{7}$$

Further, for $N_V \neq 0$, equation (5) yields

$$N_V = \frac{K}{r}(r - \theta g(Y_H)). \tag{8}$$

From equations (4) and (8), we obtain

$$Y_V = \frac{\beta_{VH} K(r - \theta g(Y_H))Y_H}{r(\beta_{VH} Y_H + \theta g(Y_H) + d_V)}. \tag{9}$$

Finally using all these values in equation (1) for $Y_H \neq 0$, we get a function $f(Y_H)$ as

$$\begin{aligned} f(Y_H) &= \beta_{HV}\beta_{VH}\left(\frac{\Lambda - \alpha Y_H}{d_H} - Y_H - g(Y_H)\right)\frac{K(r-\theta g(Y_H))}{r(\beta_{VH}Y_H + \theta g(Y_H) + d_V)} \\ &\quad -(\nu + \alpha + d_H) = 0. \end{aligned} \tag{10}$$

The investigation of function $f(Y_H)$ leads to following observations,

(i) $f(0) = \beta_{HV}\beta_{VH}\dfrac{(1-p)\Lambda K(r - \theta p(\Lambda/d_H))}{r d_H(\theta p(\Lambda/d_H) + d_V)} - (\nu + \alpha + d_H)$, which is positive provided

$$\frac{\beta_{HV}\beta_{VH}(1-p)\Lambda K(r-\theta p(\Lambda/d_H))}{r d_H(\theta p(\Lambda/d_H) + d_V)(\nu + \alpha + d_H)} > 1. \tag{11}$$

(ii) $f(\frac{\Lambda}{\alpha + d_H}) = -(\nu + \alpha + d_H)$, which is negative.

Therefore, we may obtain a unique positive value of Y_H in the interval $(0, \Lambda/(\alpha + d_H))$ provided $f'(Y_H) < 0$. Let us denote this positive value of Y_H as Y_H^*. Furthermore, the substitution of this value in equations (3), (6), and (7), yields the equilibrium values of N_H, M, A_H as N_H^*, M^*, A_H^*. Finally, using value of Y_H^* in (8) and (9), we get a positive value of N_V and Y_V as N_V^* and Y_V^* provided $r > \theta g(Y_H^*)$ i.e., $r > \theta A_H^*$. Back

The Jacobian matrix 'J' for model system (2.2) is given by:

$$J=\begin{bmatrix} -a_{11} & -\beta_{HV}Y_V & \beta_{HV}Y_V & a_{14} & 0 & 0 \\ -\lambda M & -a_{22} & \lambda M & 0 & 0 & a_{26} \\ -\alpha & 0 & -d_H & 0 & 0 & 0 \\ a_{41} & -\theta Y_V & 0 & -a_{44} & \beta_{VH}Y_H & 0 \\ 0 & -\theta N_V & 0 & 0 & r-2r\frac{N_V}{K}-\theta A_H & 0 \\ \phi & 0 & 0 & 0 & 0 & -\phi_0 \end{bmatrix}$$

where,

$a_{11} = \beta_{HV}Y_V + \nu + \alpha + d_H$, $a_{14} = \beta_{HV}(N_H - Y_H - A_H)$, $a_{22} = \lambda M + \lambda_0 + d_H$,

$a_{26} = \lambda(N_H - Y_H - A_H)$, $a_{41} = \beta_{VH}(N_V - Y_V)$, $a_{44} = \beta_{VH}Y_H + \theta A_H + d_V$.

The Jacobian matrix 'J' for model system (2.2) is given by:

$$
\mathrm{J}=\begin{bmatrix}
-a_{11} & -\beta_{HV}Y_V & \beta_{HV}Y_V & a_{14} & 0 & 0 \\
-\lambda M & -a_{22} & \lambda M & 0 & 0 & a_{26} \\
-\alpha & 0 & -d_H & 0 & 0 & 0 \\
a_{41} & -\theta Y_V & 0 & -a_{44} & \beta_{VH}Y_H & 0 \\
0 & -\theta N_V & 0 & 0 & r-2r\frac{N_V}{K}-\theta A_H & 0 \\
\phi & 0 & 0 & 0 & 0 & -\phi_0
\end{bmatrix}
$$

where,

$a_{11}=\beta_{HV}Y_V+\nu+\alpha+d_H$, $a_{14}=\beta_{HV}(N_H-Y_H-A_H)$, $a_{22}=\lambda M+\lambda_0+d_H$,

$a_{26}=\lambda(N_H-Y_H-A_H)$, $a_{41}=\beta_{VH}(N_V-Y_V)$, $a_{44}=\beta_{VH}Y_H+\theta A_H+d_V$.

The Jacobian matrix 'J', evaluated at the equilibrium E_0 is given by

$$J_{E_0} = \begin{bmatrix} -(\nu+\alpha+d_H) & 0 & 0 & \beta_{HV}\frac{(1-p)\Lambda}{d_H} & 0 & 0 \\ -\lambda M_0 & -a_{22} & \lambda M_0 & 0 & 0 & a_{26} \\ -\alpha & 0 & -d_H & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\theta\frac{p\Lambda}{d_H}+d_V) & 0 & 0 \\ 0 & 0 & 0 & 0 & r-\theta\frac{p\Lambda}{d_H} & 0 \\ \phi & 0 & 0 & 0 & 0 & -\phi_0 \end{bmatrix}$$

where,

$a_{22} = \lambda M_0 + \lambda_0 + d_H$, $a_{26} = \lambda\frac{(1-p)\Lambda}{d_H}$.

It is apparent that the eigenvalues of J_{E_0} are $-(\nu+\alpha+d_H)$, $-(\lambda M_0+\lambda_0+d_H)$, $-d_H$, $-(\theta\frac{p\Lambda}{d_H}+d_V)$, $r-\theta\frac{p\Lambda}{d_H}$ and $-\phi_0$. Therefore, all eigenvalues of J_{E_0} are negative provided $r < \theta p(\Lambda/d_H)$. Hence, J_{E_0} is stable until $r < \theta p(\Lambda/d_H)$ and it becomes unstable if $r > \theta p(\Lambda/d_H)$ i.e., E_1 exists. Thus, if E_1 exists, then E_0 is a saddle point with stable manifold locally in the $Y_H - A_H - N_H - Y_V - M$ space and unstable manifold locally in the N_V-direction. Thus E_0 is unstable whenever E_1 exists.

Further, the Jacobian matrix 'J', evaluated at equilibrium E_1 is,

$$J_{E_1} = \begin{bmatrix} -(\nu+\alpha+d_H) & 0 & 0 & \beta_{HV}\frac{(1-p)\Lambda}{d_H} & 0 \\ -\lambda M_0 & -a_{221} & \lambda M_0 & 0 & 0 & \lambda\frac{(1}{} \\ -\alpha & 0 & -d_H & 0 & 0 \\ \beta_{VH} & 0 & 0 & -(\theta\frac{p\Lambda}{d_H}+d_V) & 0 \\ 0 & -(\theta K(r-\theta\frac{p\Lambda}{d_H}))/r & 0 & 0 & -(r-\theta\frac{p\Lambda}{d_H}) \\ \phi & 0 & 0 & 0 & 0 & - \end{bmatrix}$$

where, $a_{221} = (\lambda M_0 + \lambda_0 + d_H)$

Form J_{E_1} it is found that four eigenvalues i.e., $-(\lambda M_0 + \lambda_0 + d_H)$, $-d_H$, $-(r - \theta p(\Lambda/d_H))$, $-\phi_0$ are negative.

The rest two eigenvalues are roots of following quadratic equation

$$\xi^2 + q_1\xi + q_2 = 0, \tag{12}$$

where,
$q_1 = \nu + \alpha + d_H + \theta p(\Lambda/d_H) + d_V,$
$q_2 = -\beta_{HV}\beta_{VH}\dfrac{(1-p)\Lambda K(r - \theta p(\Lambda/d_H))}{rd_H(\theta(\Lambda/d_H)p + d_V)} + (\nu + \alpha + d_H).$
It is noted that if $q_2 > 0$, then the roots of equation (12) are either negative or with negative real part. On the contrary if $q_2 < 0$, then one root of equation (12) is positive. In this case, E_1 has an unstable manifold locally either in Y_H-direction or in Y_V-direction and stable manifold locally in $A_H - N_H - N_V - M$ space. It is interesting to note here that q_2 becomes negative if $R_0 > 1$, which implies the existence of E^*. So we infer that E_1 becomes unstable whenever E^* exits. Back

Consider a Liapunov's function as,

$$V = \frac{1}{2}y_1^2 + \frac{k_1}{2}a_1^2 + \frac{k_2}{2}n_1^2 + \frac{k_3}{2}y_2^2 + \frac{k_4}{2N_V^*}n_2^2 + \frac{k_5}{2}m^2, \qquad (13)$$

where, k_1, k_2, k_3, k_4 and k_5 are positive constants to be chosen appropriately. Here $y_1, a_1, n_1, y_2, n_2,$ and m are small perturbations in Y_H, A_H, N_H, Y_V, N_V and M around the equilibrium E^*, respectively. Now differentiating 'V' with respect to 't', we get

$$\dot{V} = y_1\dot{y}_1 + k_1 a_1\dot{a}_1 + k_2 n_1\dot{n}_1 + k_3 y_2\dot{y}_2 + \frac{k_4}{N_V^*}n_2\dot{n}_2 + k_5 m\dot{m}, \qquad (14)$$

where $\dot{}$ represents differentiation w.r.t. time.

Consider a Liapunov's function as,

$$V = \frac{1}{2}y_1^2 + \frac{k_1}{2}a_1^2 + \frac{k_2}{2}n_1^2 + \frac{k_3}{2}y_2^2 + \frac{k_4}{2N_V^*}n_2^2 + \frac{k_5}{2}m^2, \tag{13}$$

where, k_1, k_2, k_3, k_4 and k_5 are positive constants to be chosen appropriately. Here $y_1, a_1, n_1, y_2, n_2,$ and m are small perturbations in Y_H, A_H, N_H, Y_V, N_V and M around the equilibrium E^*, respectively. Now differentiating 'V' with respect to 't', we get

$$\dot{V} = y_1\dot{y}_1 + k_1 a_1\dot{a}_1 + k_2 n_1\dot{n}_1 + k_3 y_2\dot{y}_2 + \frac{k_4}{N_V^*}n_2\dot{n}_2 + k_5 m\dot{m}, \tag{14}$$

where $\dot{}$ represents differentiation w.r.t. time.

Using the linearized system of model system (2.2) corresponding to E^*, we get

$$
\begin{aligned}
\dot{V} \quad = \quad & y_1[-a_{11}^* y_1 - \beta_{HV} Y^* a_1 + \beta_{HV} Y^* n_1 + a_{14}^* y_2] \\
& + k_1 a_1[-\lambda M^* y_1 - a_{22}^* a_1 + \lambda M^* n_1 + a_{26}^* m] \\
& + k_2 n_1[-\alpha y_1 - d_H n_1] \\
& + k_3 y_2[a_{41}^* y_1 - \theta Y_V^* a_1 - a_{44}^* y_2 + \beta_{VH} Y_H^* n_2] \\
& + k_4 n_2[-\theta a_1 - (r/K) n_2] \\
& + k_5 m[\phi y_1 - \phi_0 m].
\end{aligned}
$$

Here, a_{ij}^* denotes the values of a_{ij} in Jacobian matrix J_{E^*} evaluated at E^*.

After choosing $k_2 = \frac{\beta_{HV} Y_V^*}{\alpha}$, a little algebraic manipulation yields,

$$\begin{aligned}\dot{V} \quad = \quad & -a_{11}^* y_1^2 - k_1 a_{22}^* a_1^2 - k_2 d_H n_1^2 - k_3 a_{44}^* y_2^2 - k_4 (r/K) n_2^2 - k_5 \phi_0 m^2 \\ & + y_1 a_1 [-\beta_{HV} Y_V^* - k_1 \lambda M^*] + y_1 y_2 [a_{14}^* + k_3 a_{41}^*] + y_1 m [k_5 \phi] \\ & + a_1 n_1 [k_1 \lambda M^*] + a_1 y_2 [-k_3 \theta Y_V^*] + a_1 n_2 [-k_4 \theta] + a_1 m [k_1 a_{26}^*] \\ & + y_2 n_2 [k_3 \beta_{VH} Y_H^*].\end{aligned}$$

Now, $\dot{V}$ will be negative definite provided the following inequalities are satisfied,

$$\beta_{HV}^2 {Y_V^*}^2 < \frac{2}{15} k_1 a_{11}^* a_{22}^*, \tag{15}$$

$$k_1 \lambda^2 {M^*}^2 < \frac{2}{15} a_{11}^* a_{22}^*, \tag{16}$$

$${a_{14}^*}^2 < \frac{1}{5} k_3 a_{11}^* a_{44}^*, \tag{17}$$

$$k_3 {a_{41}^*}^2 < \frac{1}{5} a_{11}^* a_{44}^*, \tag{18}$$

$$k_5 \phi^2 < \frac{2}{5} a_{11}^* \phi_0, \tag{19}$$

$$k_1 \lambda^2 {M^*}^2 < \frac{2}{3} \frac{\beta_{HV} Y_V^* d_H a_{22}^*}{\alpha}, \tag{20}$$

$$k_3 \theta^2 {Y_V^*}^2 < \frac{1}{6} k_1 a_{22}^* a_{44}^*, \tag{21}$$

$$k_4 \theta^2 < \frac{1}{3} k_1 \frac{r}{K} a_{22}^*, \tag{22}$$

$$k_1 {a_{26}^*}^2 < \frac{1}{3} k_5 a_{22}^* \phi_0, \tag{23}$$

From inequality (19), we can choose $k_5 = \dfrac{2a_{11}^*\phi_0}{6\phi^2}$. Now using this value of k_5, we can choose a positive value of k_1 from inequalities (15), (16), (20) and (23), as

$$\frac{15}{2}\frac{\beta_{HV}^2 {Y_V^*}^2}{a_{11}^* a_{22}^*} < k_1 < \min\left\{\frac{2}{15}\frac{a_{11}^* a_{22}^*}{\lambda^2 {M^*}^2}, \frac{2}{3}\frac{\beta_{HV} Y_V^* d_H a_{22}^*}{\lambda^2 {M^*}^2 \alpha}, \frac{1}{3}\frac{k_5 \phi_0 a_{22}^*}{{a_{26}^*}^2}\right\}. \qquad (25)$$

Using inequalities (17), (18) and (21), a positive value of k_3 can be chosen as

$$\frac{5{a_{14}^*}^2}{a_{11}^* a_{44}^*} < k_3 < \min\left\{\frac{a_{11}^* a_{44}^*}{5{a_{41}^*}^2}, \frac{a_{22}^* a_{44}^* k_1}{6\theta^2 {Y_V^*}^2}\right\}. \tag{26}$$

Finally using values of k_1 and k_3 as chosen above, in inequalities (22) and (24) we may choose a positive value of k_4 provided following inequality holds,

$$\frac{\beta_{VH}^2 {Y_H^*}^2}{a_{44}^*} k_3 < \frac{r^2 a_{22}^*}{6K^2\theta^2} k_1. \tag{27}$$

Hence the proof. Back

Consider the following positive definite function,

$$W = \frac{1}{2}(Y_H - Y_H^*)^2 \;+\; \frac{p_1}{2}(A_H - A_H^*)^2 \;+\; \frac{p_2}{2}(N_H - N_H^*)^2 \;+\; \frac{p_3}{2}(Y_V -$$
$$+\; p_4(N_V - N_V^* - N_V^* \ln \frac{N_V}{N_V^*}) \;+\; \frac{p_5}{2}(M - M^*)^2$$

where the coefficients p_1, p_2, p_3, p_4 and p_5 are positive constants to be chosen suitably later on. Differentiating (28) with respect to 't' we get,

$$\dot{W} = (Y_H - Y_H^*)\dot{Y}_H \;+\; p_1(A_H - A_H^*)\dot{A}_H \;+\; p_2(N_H - N_H^*)\dot{N}_H \;+\; p_3(Y$$
$$+\; p_4(N_V - N_V^*)\frac{\dot{N}_V}{N_V} \;+\; p_5(M - M^*)\dot{M},$$

where $\dot{}$ represents differentiation w.r.t. time.

Consider the following positive definite function,

$$\begin{aligned} W = \frac{1}{2}(Y_H - Y_H^*)^2 \quad &+ \quad \frac{p_1}{2}(A_H - A_H^*)^2 + \frac{p_2}{2}(N_H - N_H^*)^2 + \frac{p_3}{2}(Y_V - Y_V^*)^2 \\ &+ \quad p_4(N_V - N_V^* - N_V^* \ln \frac{N_V}{N_V^*}) + \frac{p_5}{2}(M - M^*)^2 \qquad (28) \end{aligned}$$

where the coefficients p_1, p_2, p_3, p_4 and p_5 are positive constants to be chosen suitably later on. Differentiating (28) with respect to 't' we get,

$$\begin{aligned} \dot{W} = (Y_H - Y_H^*)\dot{Y}_H \quad &+ \quad p_1(A_H - A_H^*)\dot{A}_H + p_2(N_H - N_H^*)\dot{N}_H + p_3(Y_V - Y_V^*)\dot{Y}_V \\ &+ \quad p_4(N_V - N_V^*)\frac{\dot{N}_V}{N_V} + p_5(M - M^*)\dot{M}, \qquad (29) \end{aligned}$$

where $\dot{}$ represents differentiation w.r.t. time.

Evaluating $\dot{W}$ along the solutions of model system (2.2), we get

$$\begin{aligned}
\dot{W} \quad = \quad & (Y_H - Y_H^*)[\beta_{HV}\{(N_H - Y_H - A_H)Y_V - (N_H^* - Y_H^* - A_H^*)Y_V^*\} - (\nu + \alpha + d_H) \\
& + p_1(A_H - A_H^*)[\lambda\{(N_H - Y_H - A_H)M - (N_H^* - Y_H^* - A_H^*)M^*\} \\
& -(\lambda_0 + d_H)(A_H - A_H^*)] + p_2(N_H - N_H^*)[-d_H(N_H - N_H^*) - \alpha(Y_H - Y_H^*)] \\
& + p_3(Y_V - Y_V^*)[\beta_{VH}\{Y_H(N_V - Y_V) - Y_H^*(N_V^* - Y_V^*)\} \\
& -\theta(A_H Y_V - A_H^* Y_V^*) - d_V(Y_V - Y_V^*)] \\
& + p_4(N_V - N_V^*)[-\frac{r}{K}(N_V - N_V^*) - \theta(A_H - A_H^*)] \\
& + p_5(M - M^*)[\phi(Y_H - Y_H^*) - \phi_0(M - M^*)].
\end{aligned}$$

On rearranging the terms and setting $p_2 = \dfrac{\beta_{HV} Y_V^*}{\alpha}$, $\dot{W}$ reduces to,

$$\begin{aligned}
\dot{W} \quad = \quad & -(\beta_{HV} Y_V^* + \nu + \alpha + d_H)(Y_H - Y_H^*)^2 - p_1(\lambda M + \lambda_0 + d_H)(A_H - A_H^*)^2 \\
& -\frac{\beta_{HV} d_H Y_V^*}{\alpha}(N_H - N_H^*)^2 - p_3(\beta_{VH} Y_H + \theta A_H + d_V)(Y_V - Y_V^*)^2 \\
& -p_4 \frac{r}{K}(N_V - N_V^*)^2 - p_5 \phi_0 (M - M^*)^2 \\
& +[\beta_{HV}(N_H - Y_H - A_H) + p_3 \beta_{VH}(N_V^* - Y_V^*)](Y_H - Y_H^*)(Y_V - Y_V^*) \\
& -[\beta_{HV} Y_V^* + p_1 \lambda M](Y_H - Y_H^*)(A_H - A_H^*) + [p_5 \phi](Y_H - Y_H^*)(M - M^*) \\
& +[p_1 \lambda M](A_H - A_H^*)(N_H - N_H^*) - [p_3 \theta Y_V^*](A_H - A_H^*)(Y_V - Y_V^*) \\
& -[p_4 \theta](A_H - A_H^*)(N_V - N_V^*) + [p_1 \lambda (N_H^* - Y_H^* - A_H^*)](A_H - A_H^*)(M - M^*) \\
& +[p_3 \beta_{VH} Y_H](Y_V - Y_V^*)(N_V - N_V^*).
\end{aligned}$$

Using region of attraction Ω, we infer that $\dot{W}$ will be a negative definite provided the following inequalities hold:

$$
\begin{aligned}
\beta_{HV}^2 {Y_V^*}^2 &< \frac{2}{15} p_1(\beta_{HV} Y_V^* + \nu + \alpha + d_H)(\lambda_0 + d_H), && (31)\\
p_1 \lambda^2 M_R^2 &< \frac{2}{15}(\beta_{HV} Y_V^* + \nu + \alpha + d_H)(\lambda_0 + d_H), && (32)\\
\frac{\beta_{HV}^2 \Lambda^2}{d_H^2} &< \frac{1}{5} p_3(\beta_{HV} Y_V^* + \nu + \alpha + d_H) d_V, && (33)\\
p_3 \beta_{VH}^2 (N_V^* - Y_V^*)^2 &< \frac{1}{5}(\beta_{HV} Y_V^* + \nu + \alpha + d_H) d_V, && (34)\\
p_5 \phi^2 &< \frac{2}{5}(\beta_{HV} Y_V^* + \nu + \alpha + d_H)\phi_0, && (35)
\end{aligned}
$$

$$
\begin{aligned}
p_1\lambda^2 M_R^2 &< \frac{2}{3}\frac{\beta_{HV} Y_V^* d_H(\lambda_0 + d_H)}{\alpha}, && (36)\\
p_3\theta^2 {Y_V^*}^2 &< \frac{1}{6}p_1(\lambda_0 + d_H)d_V, && (37)\\
p_4\theta^2 &< \frac{1}{3}p_1\frac{r(\lambda_0 + d_H)}{K}, && (38)\\
p_1\lambda^2(N_H^* - Y_H^* - A_H^*)^2 &< \frac{1}{3}p_5\phi_0(\lambda_0 + d_H), && (39)\\
p_3\frac{\Lambda^2\beta_{VH}^2}{d_H^2} &< \frac{1}{2}p_4\frac{rd_V}{K}, && (40)
\end{aligned}
$$

From (35), a positive value of p_5 can be chosen as $p_5 = \dfrac{2(\beta_{HV} Y_V^* + \nu + \alpha + d_H)\phi_0}{6\phi^2}$.

Further, using inequalities (31), (32), (36) and (39), we may choose a positive value of p_1 as follows

$$\frac{15}{2} \frac{\beta_{HV}^2 {Y_V^*}^2}{(\beta_{HV} Y_V^* + \nu + \alpha + d_H)(\lambda_0 + d_H)} < p_1 < \min\left\{ \frac{2(\beta_{HV} Y_V^* + \nu + \alpha + d_H)(\lambda_0 + d_H)}{15\lambda^2 M_R^2}, \frac{2\beta_{HV} Y_V^* (\lambda_0 + d_H) d_H}{3\alpha\lambda^2 M_R^2}, \frac{\phi_0(\lambda_0 + d_H) p_5}{3\lambda^2 (N_H^* - Y_H^* - A_H^*)^2} \right\}. \tag{41}$$

Further, from inequalities (33), (34) and (37), we can choose a positive value of p_3 as,

$$\frac{5\beta_{HV}^2 \Lambda^2}{(\beta_{HV} Y_V^* + \nu + \alpha + d_H) d_V d_H^2} < p_3 < \min\left\{ \frac{(\beta_{HV} Y_V^* + \nu + \alpha + d_H) d_V}{\beta_{VH}^2 (N_V^* - Y_V^*)^2}, \frac{p_1(\lambda_0 + d_H) d_V}{6\theta^2 {Y_V^*}^2} \right\}. \tag{42}$$

Using positive values of p_1 and p_3 as obtained from (41) and (42) in inequalities (38) and (40) respectively, we may choose a positive value of p_4 if the following inequality is satisfied,

$$\frac{\beta_{VH}^2\Lambda^2}{d_H^2 d_V} p_3 < \frac{(\lambda_0 + d_H)r^2}{6K^2\theta^2} p_1. \tag{43}$$

Hence, we made the assertion that W is a Liapunov's function for model system (2.2), provided conditions (2.6), (2.7) and (2.8) hold. Back