POINCARÉ CONJECTURE

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Abstract. Henri Poincaré asked, in the year 1904, whether every simply-connected closed three dimensional manifold is homeomorphic to the sphere $\mathbb{S}^3$. The assertion that it is so is known as Poincaré conjecture. It was finally solved a hundred years later by G. Perelman who was awarded the Fields Medal in the Madrid ICM-2006. In these notes I shall explain the conjecture and a generalization known as 'Thurston's geometrization conjecture', briefly mention Richard Hamilton's result and the core idea of Perelman's work.

1. Basic notions

Let us begin by recalling the classification of (compact) surfaces. Recall that any compact connected surface (=two-dimensional manifold) S without boundary is determined upto homeomorphism by the its Euler characteristic and whether or not it is orientable. Indeed, if the surface S is orientable then its Euler characteristic $\chi(S)$ is an even number not exceeding 2. Writing $\chi(S) = 2 - 2g$, we see that the integer g is non-negative, called the *genus* of S. The surface S is homeomorphic to the 'sphere with g handles'; the case $g = 0$ corresponds to the sphere $\mathbb{S}^2 = \{v \in \mathbb{R}^3 \mid ||v|| = 1\}$. When $g = 1$, the surface S is just the torus $T := \mathbb{S}^1 \times \mathbb{S}^1$. One can describe the higher genus surfaces, that is, surfaces with genus $g \geq 2$, as a connected sum $T\# \cdots \#T$ of g copies of T. Thus we have a list $\mathbb{S}^2, T, T\#T, \cdots$, in which no two surfaces in the list are homeomorphic and any compact oriented surface is homeomorphic exactly one in this list. There is a similar list in the case of compact connected non-orientable surfaces: $\mathbb{P}, \mathbb{P}\#\mathbb{P}, \ldots, \mathbb{P}\# \ldots \#\mathbb{P}, \ldots$, where $\mathbb{P}$ is the projective plane obtained from $\mathbb{S}^2$ by identifying antipodal points $v, -v$ for each $v \in \mathbb{S}^2$. Thus we have a complete classification of surfaces.

One would like to have a result of this nature for higher dimensional manifolds. More precisely, topologists would like to have a collection $\{M_\alpha\}$ of (compact) n-dimensional manifolds upto homeomorphism such that (i) M_α is not homeomorphic M_β for $\alpha \neq \beta$, (ii) any n-dimensional manifold M is homeomorphic to exactly one of the M_α, and, (iii) given manifold M a way to decide (that is an algorithm that will decide) the M_α to which M is homeomorphic. It has been known as a consequence of some very remarkable results established in the 1950s concerning 'word problems' in the theory of group presentations that there cannot be any such classification theorem for manifolds of dimension four and higher. One might view Poincaré conjecture as a classification theorem for simply-connected 3-manifolds:

Poincaré Conjecture: If M is a compact simply connected 3-manifold, then M is homeomorphic to $\mathbb{S}^3$, the three-dimensional sphere.

It is known that any simply connected 3-manifold has the same *homotopy type* as that of $\mathbb{S}^3$. The question is whether such a manifold has the same topological type as $\mathbb{S}^3$. The following is a generalization of the above conjecture:

Generalized Poincaré conjecture: If M is any compact n-manifold having the same homotopy type as the n-sphere $\mathbb{S}^n$, then M is homeomorphic to $\mathbb{S}^n$.

It should be remarked that there are 3-manifolds L_1 and L_2 such that L_1 and L_2 are homotopically equivalent but are not homeomorphic. One can take L_1, L_2 to be certain lens spaces.

Interestingly, the generalized Poincaré conjecture was established first for higher dimensions: For dimensions five and above it was established in 1959 by Stephen Smale who was awarded the Fields Medal in the year 1966 at the Moscow ICM . Affirmative solution in dimension 4 had to wait till 1982; it was proved by Michael Freedman who was awarded the Fields Medal in the year 1986 at ICM, Berkeley.

The original conjecture of Poincaré which was for dimension three was the hardest and had to wait for a hundred years to be settled completely!

If M and N are two manifolds which are homeomorphic, are they necessarily diffeomorphic? This is know to be true for surfaces. It was generally thought that this might be so until John Milnor proved in 1957 that the 7-sphere has exactly 28 distinct inequivalent differentiable structures. For this remarkable achievement he was awarded the Fields Medal in 1962. Milnor and Kervaire obtained examples of topological manifolds which do not admit any differentiable structures.

The situation in dimension three is, however, very special. It is known that any topological 3-manifold admits a unique differentiable structure. Thus two smooth 3-manifolds are homeomorphic if and only if they are diffeomorphic. It is still unknown whether $\mathbb{S}^4$ admits a unique differentiable structure. It should be mentioned here that $\mathbb{R}^n$, $n \neq 4$, is known to admit a unique differentiable structure. It was discovered by Simon Donaldson, who received the Fields Medal in 1986, that $\mathbb{R}^4$ admits 'exotic' differentiable structures.

We give here a brief exposition of some important developments arising out of efforts to classify three-dimensional manifolds. Some of the technical terms used in these notes are explained in the appendix.

2. Topology of 3-manifolds

Recall that the Euler characteristic of a compact connected surface is a very powerful invariant. Just this number and the additional information concerning its orientability determines the topological type. It turns out that in the case of compact 3-manifolds—indeed any odd-dimensional manifolds—the Euler characteristic is always zero. Thus the Euler characteristic reveals nothing about the topology of the 3-manifold. However, the fundamental group turns out to be a very strong invariant for a compact connected 3-manifold M. For instance, when M is orientable it determines its *homology groups* completely. Indeed, $H_0(M) \cong H_3(M) \cong \mathbb{Z}$ for any (connected) oriented 3-manifold. Now

$H_1(M)$ is isomorphic to the abelianised fundamental group $\pi_1(M)/[\pi_1(M), \pi_1(M)]$ by Hurewicz theorem. Now, by Poincaré duality, $H_2(M) \cong H^1(M) = Hom(\pi_1(M); \mathbb{Z})$. In particular, if $\pi_1(M)$ is trivial, then $H_1(M) = 0 = H_2(M)$ and so the manifold M has the same homology groups as $\mathbb{S}^3$. The converse however is not true. Poincaré himself constructed an orientable 3-manifold M which has the same homology groups as that of $\mathbb{S}^3$ but $\pi_1(M)$ is a certain finite subgroup J of $SU(2) \cong \mathbb{S}^3$ which is perfect. Indeed the manifold M can be described as the quotient $\mathbb{S}^3/J$. The group J has centre $Z(J)$ a group of order 2 and $J/Z(J) \subset SO(3)$ can be identified with the group of symmetries of the regular icosahedron. In fact, Poincaré asserted in 1901 that any closed 3-manifold having the same homology groups as the 3-sphere must be homeomorphic to the 3-sphere. He corrected himself by constructing the above example a few years later and asked the question whose affirmative answer is known as the Poincaré conjecture.

We shall now recall a few well-known statements which are known to imply or be equivalent to the Poincaré conjecture.

Suppose that M is a homotopy 3-sphere and S an imbedded 2-sphere in M cutting M along S we get two 3-manifolds-with-boundary, M_1, M_2 each of whose boundary is S. It can be shown that each M_i is contractible. If both M_1, M_2 are homeomorphic to the disk $D := \{x \in \mathbb{R}^3 \mid ||x|| \leq 1\}$, then it can be shown that M is *homeomorphic* to the 3-sphere.

A contractible 3-manifold-with-boundary which is *not* homeomorphic to the 3-disk D is called a fake 3-cell. Thus, Poincaré conjecture is equivalent to the following assertion: *There does not exist any fake* 3*-cell.*

J. H. C. Whitehead gave an example of an *open* (i.e., connected, non-compact) 3-manifold M which is contractible but not homeomorphic to the Euclidean space $\mathbb{R}^3$. (In dimension 2 every contractible open 2-manifold is homeomorphic to $\mathbb{R}^2$.) In the course of its construction, he made essential use of what is now known as the Whitehead link. It consists of two copies imbedded circles in $\mathbb{R}^3$ which have linking number zero but yet they cannot be pulled apart—thus they are 'linked'. Whitehead constructed this example when he found a gap in his own 'proof' of the Poincaré conjecture.

We shall now recall a group theoretic reformulation of the Poincaré conjecture. View $\mathbb{S}^3$ as the one-point compactification $\mathbb{R}^3 \cup \{\infty\}$. Consider the 'standard' imbedding of the (orientable) genus g surface Σ, namely the 2-sphere with g-handles, in $\mathbb{R}^3 \subset \mathbb{S}^3$. The complement of Σ in $\mathbb{S}^3$ consists of two connected components whose closures M_1 and M_2 are compact 3-manifolds-with-boundary with boundary being Σ. In fact M_1 and M_2 are homeomorphic and the manifold $\mathbb{S}^3$ is obtained from M_1 and M_2 by identifying their common boundary Σ. M_1, M_2 are known as 'handle-bodies'. It is known that *any* 3-manifold admits handle-body decomposition $M = M_1 \cup_h M_2$—called Heegaard splitting—for suitable boundary identification h, which is a homeomorphism of the common boundary surface $h : \partial M_1 \longrightarrow \partial M_2$. The genus of the surface $\partial M_i =: \Sigma$ is called the genus of the Heegaard splitting. The homeomorphism type of the resulting 3-manifold depends not only on the genus g of the surface ∂M_1 but also on the homeomorphism h which prescribes the boundary identification $x \sim h(x), x \in \partial M_1$.

There is a notion of equivalence of Heegaard splittings of a given 3-manifold. Waldhausen has shown that any two Heegaard splittings of $\mathbb{S}^3$ over a surface of same genus are equivalent.

Suppose that $M_1 \cup_h M_2 = \mathbb{S}^3$ is a Heegaard splitting of $\mathbb{S}^3$ of genus g. By considering the induced map in fundamental groups of the inclusions $\Sigma \hookrightarrow M_i$ we obtain homomorphisms $\phi_i \colon \pi_1(\Sigma) \longrightarrow \pi_1(M_i)$, $i = 1, 2$ where Σ is the common boundary of $M_i \subset M$. These homomorphisms yield a homomorphism $\phi = (\phi_1, \phi_2)$, that is, $\phi \colon \pi_1(\Sigma) \longrightarrow F_g \times F_g, \phi(g) = (\phi_1(g), \phi_2(g))$ upon identifying $\pi_1(M_i)$ with the free group F_g of rank g. The homomorphism ϕ is called *the splitting homomorphism* associated to the Heegaard splitting.

Using Seifert-Van Kampen theorem, we see that M is simply connected if and only if ϕ is onto. In the case of the standard Heegaard splitting of $\mathbb{S}^3$, the homomorphisms $\phi_i, i = 1, 2$, have a simple and explicit description.

Jaco proved that starting with any surjective homomorphism $\phi = (\phi_1, \phi_2)$ where $\phi_i \colon \pi_1(\Sigma) \longrightarrow F_g$ where Σ_g is a closed orientable surface of genus g, one obtains a Heegaard splitting of a 3-manifold which is simply connected whose associated splitting homomorphism is ϕ.

We say that two homomorphisms $\phi, \psi : G \longrightarrow F_g \times F_g$ of groups are equivalent if there exist automorphisms $\alpha \colon G \longrightarrow G$ and $\beta_i \colon F_g \longrightarrow F_g$ such that $(\beta_1 \times \beta_2) \circ \psi = \phi \circ \alpha$, that is the diagram below is commutative:

$$\begin{array}{ccc} G & \stackrel{\psi}{\longrightarrow} & F_g \times F_g \\ \alpha \downarrow & & \downarrow \beta_1 \times \beta_2 \\ G & \stackrel{\phi}{\longrightarrow} & F_g \times F_g \end{array}$$

The following reformulation of Poincaré conjecture is well-known from the work of Waldhausen and the result of Jaco stated above. See [4] for further details.

Poincaré conjecture holds if and only if for any genus $g \geq 1$, any surjective homomorphism $\psi \colon \pi_1(\Sigma_g) \longrightarrow F_g \times F_g$ is equivalent to the splitting homomorphism ϕ associated to the standard Heegaard splitting of $\mathbb{S}^3$.

A 3-manifold M is called *prime* if $M = M_1 \# M_2$ implies that one of M_1, M_2 is the 3-sphere. M is said to be *irreducible* if every imbedded 2-sphere in M bounds a 3-cell. Thus M is irreducible if it is prime. If M is prime but not irreducible, then M is known to be a 2-sphere bundle over the circle. If $M = M_1 \# M_2$, then using Seifert-Van Kampen theorem one shows that the fundamental group of M is isomorphic to the free product of the fundamental groups of M_1 and M_2. In symbols, $\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2)$. In particular, M is prime if its fundamental group cannot be decomposed as a non-trivial free product.

A basic result in the study of 3-manifolds is the 'prime factorization':
Every closed connected orientable 3-manifold M can be expressed as a connected sum $M_1 \# \cdots \# M_k$ where each M_i is prime. This result was proved by H. Kneser in 1929. In

general, the prime factorization is not unique: There exist closed non-orientable manifolds N such that $N \# N \cong N \# (\mathbb{S}^2 \times \mathbb{S}^1)$. However, Milnor showed that, for orientable 3-manifolds, prime factorization is unique (upto order of the summands and orientation preserving homeomorphisms). In view of prime factorization, if M is a homotopy 3-sphere (equivalently a simply connected closed 3-manifold) then all its prime factors are also homotopy 3-spheres. Many questions concerning 3-manifolds can be reduced to the case of prime manifolds.

Suppose that M is a closed connected 3-manifold and V is an imbedded closed surface of non-positive Euler characteristic. (It is not assumed that V is orientable.) We say that V is *incompressible* in M if the inclusion map $V \subset M$ induces a monomorphism of fundamental groups.

The basic idea of William Thurston in his formulation of *Geometrization Conjecture* is that any 3-manifold can be cut along spheres and incompressible tori or Klein bottles so that each of the resulting connected component has a nice geometric structure. We explain this more precisely in the next section.

Perelman's work claims to prove the geometrization conjecture, of which the Poincaré conjecture is a special case. The mathematical community is awaiting final, decisive word of the experts who are still engaged in the verification of the proof of the geometrization conjecture.

3. Thurston's Geometrization Conjecture

Recall that in two dimensions, there are three Riemannian manifolds, $\mathbb{S}^2, \mathbb{R}^2$, and the Poincaré upper half-space $\mathcal{H}^2$, which are simply-connected, complete and have constant curvature $1, 0$ and -1 respectively. Here $\mathcal{H}^2$ is the Poincaré upper half-space $\{z = x + \sqrt{-1}y \mid y > 0\} \subset \mathbb{C}$ with the Poincaré metric $ds^2 = (1/y^2)(dx^2 + dy^2)$. If S is any compact surface, it admits a Riemannian metric with constant curvature $1, 0$, or -1, depending on whether its Euler characteristic is positive, zero, or, negative. In each case the universal cover $\widetilde{S}$ of S is $\mathbb{S}^2, \mathbb{R}^2$, or $\mathcal{H}^2$ respectively. The covering projection $\widetilde{S} \longrightarrow S$ is a local isometry. Observe also that each of the spaces $\mathbb{S}^2, \mathbb{R}^2$ and $\mathcal{H}^2$ is homogeneous as a metric space: the group SO(3) of proper rotations of $\mathbb{R}^3$ acts transitively on $\mathbb{S}^2$, the group $\mathbb{R}^2$ acts on transitively itself by translations, and the group $\mathrm{PSL}(2, \mathbb{R}) := \mathrm{SL}(2, \mathbb{R})/\{\pm I\}$ acts transitively on $\mathcal{H}^2$ via Möbius transformations. ($\mathrm{SL}(2, \mathbb{R})$ denotes the group of 2×2 matrices over $\mathbb{R}$ with determinant 1.) These are the only *models* of geometries in 2-dimensions: the elliptic, the Euclidean or flat, and the hyperbolic, respectively. Observe that although $\mathbb{R}^2$ and $\mathcal{H}^2$ are *homeomorphic* (even diffeomorphic), they are very different as metric spaces.

In dimension 3, there are eight distinct model geometries. (It turns out that there are a few more geometries possible, but they do not give rise to compact quotients. See [14].) That is, there are eight simply-connected 3-dimensional Riemannian manifolds of whose group of isometries act transitively and which admit compact (or *finite volume*) quotients. They are $\mathbb{S}^3, \mathbb{R}^3, \mathcal{H}^3, \mathcal{H}^2 \times \mathbb{R}, \mathbb{S}^2 \times \mathbb{R}^1, Sol, Nil, \widetilde{\mathrm{SL}}(2,\mathbb{R})$. Of these, the first three in the list have constant sectional curvature $1, 0, -1$ respectively. Here Sol denotes the 2×2-upper triangular matrices over $\mathbb{R}$ with positive diagonal entries, Nil denotes the space of 3×3 unipotent upper triangular matrices, and, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is the universal cover of $\mathrm{SL}(2,\mathbb{R})$. *Note that* $\mathbb{S}^3$ *is the only one in the list which is compact.*

A connected, complete Riemannian manifold 3-manifold M is called locally homogeneous if the universal cover $\widetilde{M}$ of M is homogeneous. Thus $M = \widetilde{M}/\Gamma$ where $\widetilde{M}$ is one of the eight 3-manifolds listed above and $\Gamma = \pi_1(M)$ is a discrete subgroup of the group of isometries of $\widetilde{M}$.

Thurston's geometrization conjecture: *Let M be a closed, oriented, prime 3-manifold. Then there exists an imbedding of a disjoint union of* incompressible *2-tori and Klein bottles in M such that every component of the complement admits a Riemannian metric with respect to which it has finite volume and is locally homogeneous.*

The geometrization conjecture implies the Poincaré conjecture. To see this, let M be any prime 3-manifold with *finite* fundamental group. Then M has no incompressible tori or Klein bottles. By the geometrization conjecture, M has a Riemannian metric with respect to which it is locally homogeneous. Since $\pi_1(M)$ is finite, $\widetilde{M}$ is compact, and hence, remarked earlier, $\widetilde{M}$ has to be the 3-sphere.

The 3-manifolds which are quotients of $\mathbb{S}^3$ were classified by H. Hopf.

4. Ricci flow

Let M be a connected smooth manifold of positive dimension. The set of all Riemannian metrics on M has the structure of a convex cone of an infinite dimensional vector space. Indeed, if g_0 and g_1 are any Riemannian metrics on M and if λ_0 and λ_1 are smooth positive functions on M then $g = \lambda_0 g_0 + \lambda_1 g_1$ is also a Riemannian metric.

Let $Ric(g)$ (or, simply, Ric) denote the Ricci curvature tensor determined by a Riemannian metric g. Recall that Ric is, like the Riemannian metric, a contravariant symmetric 2-tensor on the manifold M. The Ricci flow equation is an equation on the space of all Riemannian metrics on M. The *Ricci flow equation*on a Riemannian manifold (M, g), introduced by Richard Hamilton is:

$$\frac{\partial g(t)}{\partial t} = -2Ric(g(t)) \qquad (*)$$

with the initial condition $g(0) = g$.

A solution to the above equation is a curve $t \mapsto g(t)$ in the space of Riemannian metrics on M starting at $g = g(0)$ such that at any time t, the metric $g(t)$ flows or 'evolves' in the direction of $-2Ric(g(t))$. Suppose that g is an Einstein metric, i.e., the Ricci curvature tensor $Ric(g)$ is a multiple ag of g for some constant g. Then the solution to equation $(*)$ is a rescaling of g. If the multiple $a > 0$, the solution exists for a finite time T, after which the curvature becomes infinite and the manifold becomes extinct. In case $a < 0$

the solution exists for all times. It turns out that in dimension 3, if g is an Einstein metric on M, then the sectional curvature is constant and so M is locally homogeneous. Hamilton's idea is the following. Start with an arbitrary Riemannian metric $g = g(0)$ on M and allow the metric to vary so that the Ricci flow equation ($*$) holds. Then, examine the limiting metric if it is (up to scaling) an Einstein metric. If it is then M is locally homogeneous. There are several problems to overcome. First, Hamilton showed that, if M is compact and $g = g(0)$ is arbitrary, then a solution to the Ricci flow equation always exists and is unique for $0 \leq t < \epsilon$ for some $\epsilon > 0$. If $T < \infty$ is the largest such that the solution exists for the time interval $[0, T)$, then there exists a point p on M at which the Ricci curvature tensor of the metric $g(t)$ becomes unbounded as $t \to T$. We say that 'a singularity develops at p'. Even in the simplest cases, the evolved

A very rough plan of Perelman's proof.

I can do no better than to indicate the major an very broad steps involved in the Perelman's proof of Poincaré conjecture. The complete proofs can be found in [8].

- *Classification of types of singularities.* Singularities at finite time occurs in two types of regions. The first type of regions are those where the curvature becomes unbounded at all points in finite time (as in the case of the 3-sphere with its standard metric). The second type of regions are long thin tube (diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$). Perelman shows that at most *finitely many* singularities can develop at any finite time. Thus the singularities are all isolated at any given time.
- *How to continue the Ricci flow beyond the time when singularity develops?* Perelman performs surgeries in a controlled manner, so that, after the surgery (which in certain cases corresponds to connected sum decomposition), one starts all over again with the Ricci flow with the resulting new Riemannian manifold, which may not be connected. This way Perelman is able to proceed with the Ricci flow. This process proceeds either indefinitely (i.e., for all time $t < \infty$) or until *the manifold becomes extinct*!
- *Finiteness of extinction time.* In case one starts with a simply connected 3-manifold (or a manifold with finite fundamental group), after finitely many surgeries, each of which is along an embedded $\mathbb{S}^2$, one is left with possibly more than one connected component, each of which becomes *extinct* after a finite time. (Recall the case of the standard 3-sphere.)
- *Poincaré conjecture from finite extinction time.* If a simply connected 3-manifold becomes extinct in finite time in the evolution of Ricci flow with surgery, then Perelman shows that it has to a 3-sphere. Since the surgery at each stage was along an imbedded $\mathbb{S}^2$, the the original manifold one started with must have been a connected sum of $\mathbb{S}^3$. Thus the original manifold itself must be diffeomorphic to $\mathbb{S}^3$. This establishes Poincaré conjecture.

Perelman's papers also deal with the more general geometrization conjecture. Here the Perelman surgery corresponds to cutting up the manifold into locally homogeneous pieces.

Note added in 2008: Recently Morgan and Tian [9] have given complete and detailed proof of the geometrization conjecture following Perelman's ideas.

5. Appendix

We recall here some basic notions concerning topology of manifolds.

An n-dimensional *topological* manifold M is a Hausdorff topological space which is *locally Euclidean*, that is, there exists an open covering $\{U\}$ such that each U is homeomorphic to an open subset of $\mathbb{R}^n$. We call a pair (U, ϕ) a *chart* if ϕ is such a homeomorphism. A collection of of charts $\{(U, \phi)\}$ is called an atlas. We shall further assume that M is metrizable (equivalently paracompact). Basic examples or manifolds are open subsets of the Euclidean space $\mathbb{R}^n$ and the unit sphere.

Let M be an n-dimensional manifold and let $\mathfrak{A} = \{(U, \phi)\}$ be an atlas. We say that $\mathfrak{A}$ is a *smooth structure* on the manifold M if, (i) for any two $(U, \phi), (V, \psi) \in \mathfrak{A}$, the map $\psi \circ \phi^{-1} \colon \phi(U \cap V) \longrightarrow \psi(U \cap V)$ is smooth, i.e., all mixed partial derivatives exist and are continuous, and (ii) $\mathfrak{A}$ is a maximal such collection. Starting with any atlas $\mathfrak{A}$ satisfying condition (i), one can always expand it to make it maximal in a unique way. For this reason, condition (ii) is not needed upon in order to define smooth structure on M.

A continuous map $f \colon M \longrightarrow N$ between two smooth manifolds $(M, \mathfrak{A}), (N, \mathfrak{B})$ is said to be smooth if, for any charts $(U, \phi) \in \mathfrak{A}, (V, \psi) \in \mathfrak{B}$ such that $f(U) \subset V$, the map $\psi \circ f \circ \phi^{-1}$ is smooth. Thanks to chain rule for differentiation, this definition does not depend on the choice of the charts $(U, \phi), (V, \psi)$.

If $f \colon M \longrightarrow N$ is smooth, one-to-one and onto, whose inverse is also smooth, then we say that f is a *diffeomorphism.* This is the basic notion of 'equivalence' in the study of smooth manifolds; thus we identify two manifolds if they are diffeomorphic to each other.

Suppose that $f \colon U \longrightarrow \mathbb{R}^n$, $x \mapsto (f_1(x), \cdots, f_n(x))$ is a smooth map where U is an open subset of $\mathbb{R}^{m+n}$. If q is a regular value (that is, the Jacobian matrix $J(f) = (\partial f_i / \partial x_j)$ of f has maximum rank at all points of $f^{-1}(q)$), then $M := f^{-1}(q)$ is in a natural way a smooth manifold of dimension m. Thus the unit sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid ||x||^2 = 1\}$ is a smooth manifold of dimension n. Likewise one sees that the group $\mathrm{SL}(n, \mathbb{R})$ of $n \times n$ real matrices with determinant equal to 1 is a smooth manifold of dimension $n^2 - 1$.

A celebrated theorem of H. Whitney says that any smooth manifold M of dimension n can be imbedded as a submanifold of the Euclidean space $\mathbb{R}^{2n}$, that is M can be regarded as a subspace of $\mathbb{R}^{2n}$ where the inclusion map $M \hookrightarrow \mathbb{R}^{2n}$ is smooth.

The tangent space Let M be a smooth manifold of dimension n and let $p \in M$. A *tangent vector* to M at p is an operator which assigns to each smooth function $f \colon M \longrightarrow \mathbb{R}$ a real number $v(f)$ which satisfies the following properties:
(i) $v(f_1) = v(f_2)$ if f_1 and f_2 agree in some neighbourhood of p
(ii) $v(af + g) = av(f) + v(g)$, and,
(iii) *Leibnitz rule:* $v(fg) = g(p)v(f) + f(p)v(g)$ for any smooth functions f, g and $a \in \mathbb{R}$.

The set of all tangent vectors to M at p is a vector space denoted T_pM and is called the *tangent space* to M at p. The number $v(f)$ is the *directional derivative* of f along v.

The set $TM := \coprod_{p\in M} T_pM$ has the structure of a smooth manifold whose dimension is twice that of M such that the obvious projection map $\pi : TM \longrightarrow M$ is smooth. TM is called *the tangent bundle* of M. A smooth map $v : M \longrightarrow TM$ is called a vector field if $\pi \circ v = id_M$.

When M is an open subset of $\mathbb{R}^n$, any tangent vector at T_pM is the vector space $\{\sum_{1\leq i\leq n} a_i\partial_i|_p \mid a_i \in \mathbb{R}\}$ where $\partial_i|_p(f) = \frac{\partial f}{\partial x_i}(p)$. The tangent bundle TM is just $M \times \mathbb{R}^n$ where $\sum a_i\partial_i|_p$ is identified with $(p; a_1, \cdots, a_n) \in M \times \mathbb{R}^n$.

Suppose that M is imbedded as a smooth submanifold in $\mathbb{R}^d$. If $\sigma\colon (-\epsilon, \epsilon) \longrightarrow M$ is a smooth curve such that $\sigma(0) = p$, then the velocity vector $\sigma'(0) := \frac{d\sigma}{dt}|_{t=0} \in \mathbb{R}^d$ is the tangent vector to M at p defined as follows: if f is a smooth function on M, then $\sigma'(0)(f) = \frac{d(f\circ\sigma)}{dt}|_{t=0} \in \mathbb{R}$. Any tangent vector to M can be realised as a velocity vector of a suitable smooth curve and so that the space of all velocity vectors at p can be identified with the tangent space T_pM; in particular $T_pM \subset \mathbb{R}^d$.

If $M = f^{-1}(q)$ where q is the regular value of a smooth function $f\colon U \longrightarrow \mathbb{R}^m$, U open in $\mathbb{R}^{m+n}$, then the tangent space T_pM can be identified with the kernel of the linear map $\mathbb{R}^{m+n} \longrightarrow \mathbb{R}^m$ defined by the Jacobian at p, $J_p(f)$. Thus TM is identified with the subspace $\{(p, \sum_i a_i\partial_i|_p) \in \mathbb{R}^n \times \mathbb{R}^n = T\mathbb{R}^n \mid p \in M, J_p(f)(\sum_i a_i\partial_i|_p) = 0\} \subset M \times \mathbb{R}^n$.

For example, if $\mathbb{S}^n = f^{-1}(1)$ where $f\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ is the smooth function $f(x) = x_0^2 + \cdots + x_n^2$. Observe that 1 is a regular value for f. Indeed the Jacobian of f is the column matrix $(2x_0, \cdots, 2x_n)^t$. The corresponding linear map $\mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ is $u \mapsto 2u_0x_0 + 2u_1x_1 + \cdots + 2u_nx_n = 2u.x$. The kernel of this linear map at $x \in \mathbb{S}^n$ is $\{u \in \mathbb{R}^{n+1} \mid u\cdot x = 0\}$ which is just the plane orthogonal to the vector $x \in \mathbb{R}^{n+1}$ (with respect to the standard innerproduct on $\mathbb{R}^{n+1}$.

Note that when M is an imbedded submanifold of a Euclidean space $\mathbb{R}^d$, the tangent space T_pM inherits in a natural way an innerproduct since T_pM is a vector subspace of the Euclidean space $T_p\mathbb{R}^d$ which is canonically identified with $\mathbb{R}^d$. As the point p varies on M the innerproduct varies 'smoothly'. This leads to the notion of a Riemannian metric on M.

A *Riemannian metric* on M is the choice of an innerproduct $\langle\cdot,\cdot\rangle_p$ on the tangent space T_pM for each $p \in M$ such that the innerproducts vary smoothly with respect to p. Smoothness with respect to p is the requirement that if $X, Y : M \longrightarrow TM$ are smooth vector fields on M, then $p \mapsto \langle X_p, Y_p\rangle_p$ be a smooth function on M.

An *affine connection* on a manifold M is an $\mathbb{R}$-linear operator ∇_X, associated to each smooth vector field X on M, on the vector space of all smooth vector fields of M satisfying the following axioms:

(1) $\nabla_X(aY + bZ) = a\nabla_X Y + b\nabla_X Z$,

(2) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$, and,

(3) $\nabla_{fX+gY} = f\nabla_X + g\nabla_Y$,

for all smooth functions f, g on M, vector fields X, Y, Z on M and all $a, b \in \mathbb{R}$. In view of (3) above, it follows easily that $(\nabla_X Y)_p = (\nabla_{X'} Y)_p \in T_pM$ if $X_p = X'_p$. Thus it is meaningful to define, for $v \in T_pM$, $\nabla_v Y \in T_pM$ to be $(\nabla_X Y)_p$ for any vector field X such that $X_p = v$.

If M is a Riemannian manifold and ∇ is an affine connection on M, we say that ∇ is *compatible with the Riemannian metric* if, for any three smooth vector fields X, Y, Z on M, one has

$$Z(\langle X, Y\rangle) = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle.$$

A connection ∇ is said to be *symmetric* if $\nabla_X Y - \nabla_Y X = \nabla_{[X,Y]}$. On a given Riemannian manifold, there exists a unique symmetric connection ∇ which is compatible with the metric. It is called the *Levi-Civita* connection.

Let ∇ be the Levi-Civita connection on M. Define the *curvature tensor* R of ∇ as follows: $R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$.

Let X and Y be smooth vector fields on M and for each $p \in M$ we define an endomorphism $T_pM \longrightarrow T_pM$ where $v \mapsto R(X_p, v)(Y)$. Taking trace of this endomorphism gives a smooth function $p \mapsto Trace(R(X_p, v)(Y)$. $Ric(X, Y)$ is symmetric and bilinear in X, Y and so $(X, Y) \mapsto Ric(X, Y)$ is a $(0, 2)$-tensor and is called the Ricci tensor.

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