A Study of Harmonic Overtones Produced in Indian Drums

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ABSTRACT

This is an investigation of the sounds produced by certain Indian drums. The unique feature of these drums, reported by Raman in 1920, is that their overtones are in integer ratios, as are the overtones of a vibrating string. This is because their membranes are not uniform, but are carefully 'loaded' with a heavy paste at the centre. As a result their sounds have a definite pitch and are perceived by the ear as 'musical'. In contrast, most drums have large numbers of harmonically unrelated overtones, which create a noisy effect. This paper reports the main results of our study of the sounds of Indian drums, and presents a method of calculating how to load a membrane to produce harmonic overtones.

1. Introduction

Any casual listener to music will notice major differences in the sounds of stringed instruments and the sounds of drums. In particular, the sounds of drums do not, in general, produce a sensation of *pitch* in the human ear — they don't correspond to any note in the musical scale. A sound has a definite pitch if it is *periodic*, that is, if its waveform repeats itself after a definite time period. Of course, most sounds decay with time, and hence are not strictly periodic, but nevertheless, if the decay is slow enough the ear does detect a pitch. One reason why drums don't seem to have a pitch is that their sounds decay so fast. But there is a more fundamental reason why, even if a drum does give a sustained sound for a second or so (ample for the ear to detect the pitch), no pitch is apparent.

Both strings and two-dimensional drum membranes, when struck or plucked, can vibrate freely in highly arbitrary manners. However, in both cases (when damping is neglected) there exist special modes of vibration in which the motion of each portion of the string or membrane is the simplest periodic motion possible — a simple harmonic or sinusoidal motion. These are the "normal modes" of vibration (Figure 1), and with each normal mode is associated a sine wave of a particular frequency. Any arbitrary vibration can be decomposed into a superposition of normal modes, and the resulting sound will be a superposition of the sine waves associated with each normal mode. The

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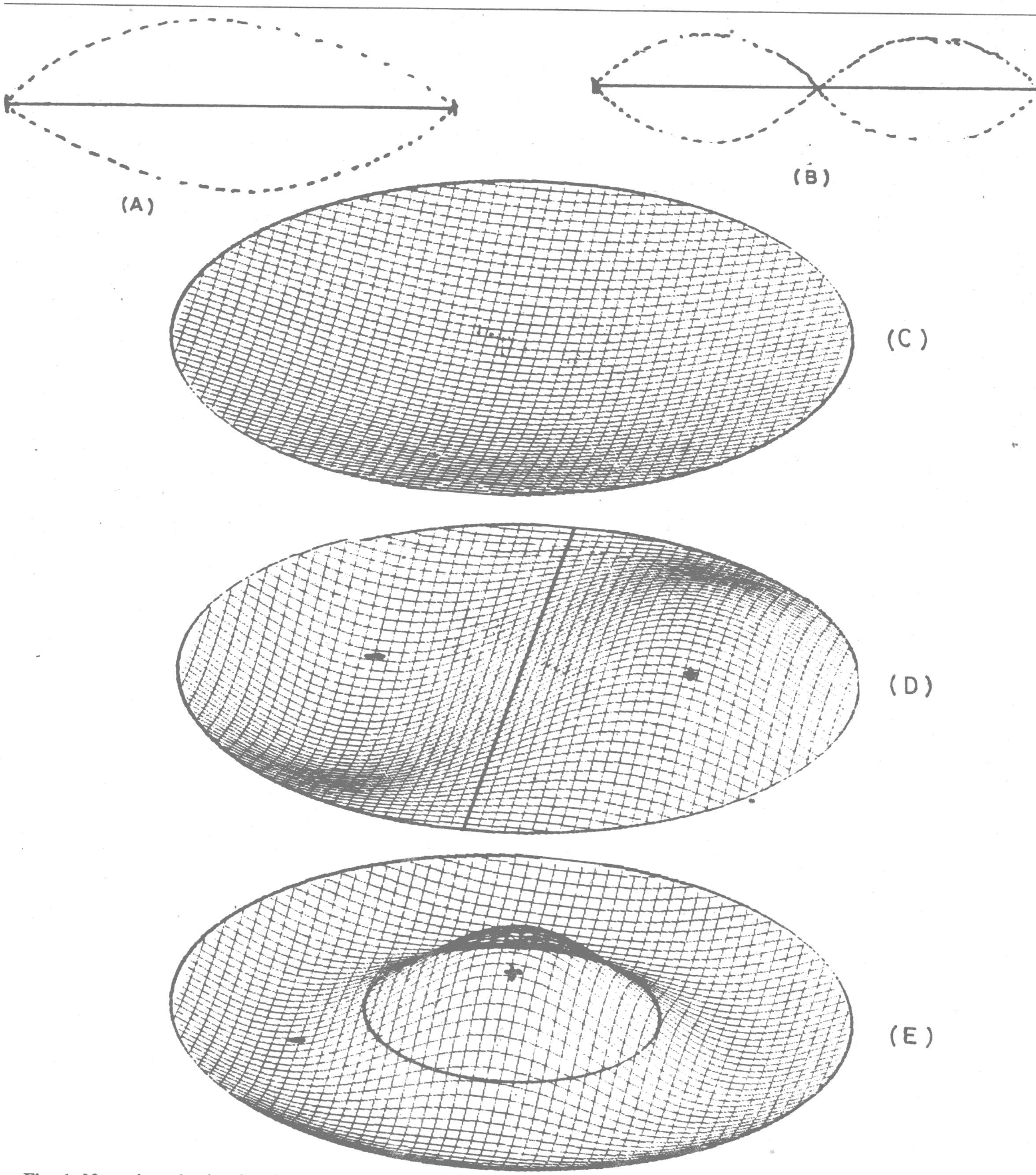

Fig. 1. Normal modes in vibrating strings and circular membranes: (A) The fundamental in a vibrating string; (B) The second harmonic in a vibrating string, with one node (stationary point) other than the endpoints and a frequency twice that of the fundamental; (C) The fundamental (0, 1 mode) in a vibrating circular membrane, with no nodes other than the outer boundary; (D) The (1, 1) overtone in a vibrating membrane, with a nodal diameter; (E) The (0, 2) mode in a vibrating membrane, with two nodal circles including the outer boundary. The nodal (stationary) lines separate, at each instant, regions with positive displacement from regions with negative displacement.

lowest normal mode frequency is called the *fundamental*, and the higher frequencies are called *overtones*.

In a string, all the overtones are integer multiples of the fundamental (when this happens the overtones are called *harmonics*). It is easy to see (Figure 2) that the superposition of sine waves in integer ratios will always give periodic waveforms, so that the sound emitted by a string is periodic with the periodicity of the fundamental. In a uniform circular membrane, the overtones are highly anharmonic (the ratios are not rational numbers at all) which gives an impression of indefinite pitch. (If the frequency of the fundamental is 1, the first few overtones have frequencies 2.405, 3.832, 5.136, 5.520, 6.380, 7.016, ...)

Yet Indian drums, such as the tabla and the mridangam (specifically the treble halves of these instruments) certainly give the impression of a definite pitch or *ṣruti* and are tuned to the tonic, *sā*, of the scale in performance. It was reported by Raman[1,2] that the first nine normal modes of these drums do, in fact, have frequencies in integer ratios, so that the resulting sound is nearly periodic. This is achieved by carefully "loading" the membranes at the centre with a heavy paste to modify the modes of vibration and cause the eight anharmonic overtones to converge to four harmonics.

We decided to verify Raman's observations, and to try and theoretically calculate exactly how a circular membrane should be loaded so as to produce a specified number of harmonic overtones. The method of calculation is given below, but our major findings were: (1) Using our methods, it was very difficult to find a method of loading that would produce harmonic overtones at all – our calculations could produce no more than two, whereas Raman reported eight; and (2) when we ourselves measured the frequencies of several drums, we found that the overtones were, in fact, not harmonic, though they were in integer ratios to each other. Typically, the frequencies were in the ratio 1.07 : 2 : 3 : 4 ... . The overtones can thus be regarded as harmonics of a missing fundamental, the actual fundamental being too high. The effect is in fact quite noticeable when these instruments are played: most notes excite higher harmonics and are tuned to the tonic *sā*, but one particular note, generally called *dhin* and played by striking the membrane sharply at the centre, excites primarily the fundamental and sounds noticeably higher than the tonic. This note is played only occasionally, for effect. Neither Raman nor later authors (e.g. Rossing[3]) appear to have noticed this.

When we repeated the calculations, this time requiring only that the overtones be in integer ratios and not

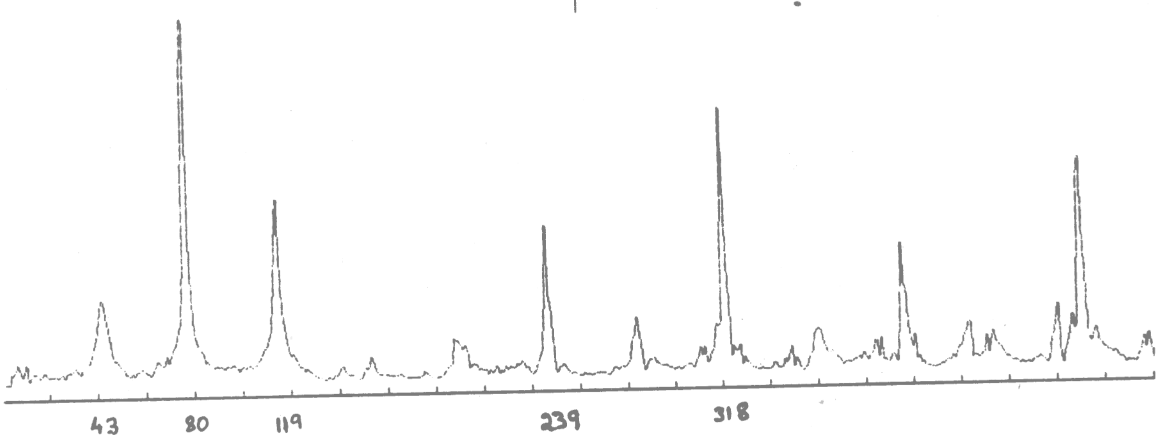

Fig. 2. A graph of the FFT of a section (1024 point, corresponding to 0.2 seconds of the sound) of the waveform of a note on the mridangam called "chapu". Only the first 512 points of the FFT are shown; the next 512 points duplicate the same information. Note that the fundamental dies down quickly, so that only the overtones—notably the first, second, fifth and seventh—are prominent. (Some of the peaks towards the right could be aliases for frequencies above 2500 Hz.) Other notes, such as "Da", "Nam", "Dhin" excite other modes.

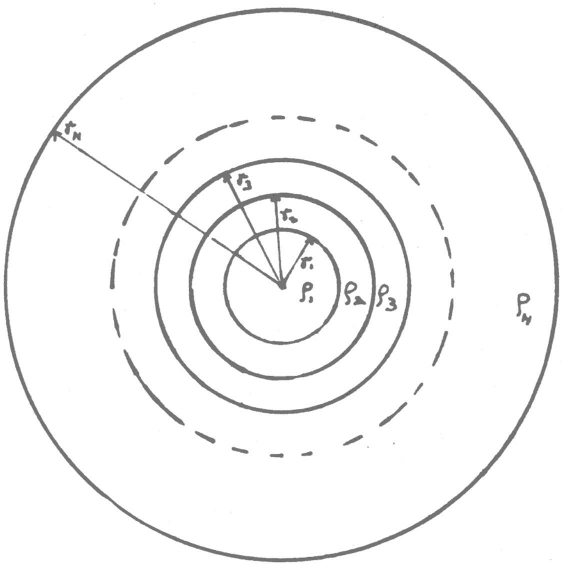

Fig. 3 How to load the membrane. We divide it into N concentric annular regions (except the central region which is a disc), each of which has a uniform density. Then the equations of motion for each region can be solved exactly, and by applying boundary conditions we can solve for the whole membrane.

that they be multiple of the fundamental, the calculations converged readily enough for up to five overtones, while the fundamental was higher by roughly the same amount as in the actual drums.

2. Frequencies Produced by Indian Drums

The sounds of several drums were recorded, fed into a computer, and analysed using a fast Fourier transform (FFT) program. (The Fourier transform is a mathematical procedure for separating a signal into its component frequencies. For the theory of the discrete Fourier transform, of which the FFT is an implementation, see any book on spectral applications, e.g. Newland[4].) The sampling rate of the analog-to-digital converter used was 200 μs, which would have led to the problem of 'aliasing' for frequencies higher than 2500 Hz (that is, frequencies 2500 + δ show up in the transform as spurious peaks at 2500 – δ); however, the frequencies of the first few modes were much lower than this, while higher overtones were much weaker, so this was not a difficulty.

We studied the sounds of two mridangams, a tabla and a pakhawaj. The results are given in Table 1, while a sample graph of a FFT from a mridangam is shown in Figure 3. It is seen that in each case the overtones are in integer ratios (very nearly) while the fundamental is distinctly higher than it should ideally be. This consistent observation leads one to believe that there is some fundamental difficulty in producing overtones harmonic with the fundamental.

Table 1: Component Frequencies in the Sounds of Some Indian Drums, Obtained from Fast Fourier Transforms (1024 points, sampling interval 200 μs) of Recorded Sounds

Drum	n (location of peak in FFT)	Ratios of Frequencies
Mridangam 1	43, 80, 120, 161, 203, 239, 279, 318, 398	1.075 : 2 : 3 : 4.025 : 5.075 : 5.975 : 6.975 : 7.95 : 99.5
Mridangam 2	30, 54, 81, 108, 132, 189, 217	1.11 : 2 : 3 : 4 : 4.89 : 7 : 8.037
Tabla	48, 87, 130, 175, 217, 261	1.103 : 2 : 2.99 : 4.02 : 4.99 : 6
Pakhawaj	29, 54, 82, 108, 133, 160	1.074 : 2 : 3.037 : 4 : 4.93 : 5.93

Note: The FFT transforms a time sequence of N points into a new sequence, numbered from 0 to $N-1$. The frequency corresponding to the nth point in the discrete Fourier transform is $n(N\Delta)$, where Δ is the sampling interval. Thus the numbers above can be converted to frequencies in Hertz by multiplying by 4.8828.

3. Theory of Loading the Membrane

A uniform, flexible vibrating membrane satisfies the two-dimensional wave equation of motion

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

where u is the vertical displacement of the membrane and $v = (\tau/\rho)^{1/2}$ is the velocity of waves on the membrane, τ is the tension of the membrane, and ρ the surface mass density.

In polar coordinates, under the assumption of separation of variables where $u(r, \theta, t)$ is written as $R(r)\,\Theta(\theta)\,T(t)$, the equation can be separated into

$$r^2R'' + rR' + (k^2r^2 - m^2)R = 0, \quad (2)$$

$$\Theta'' + m^2\Theta = 0, \quad (3)$$

$$T'' + k^2v^2T = 0, \quad (4)$$

k and m being constants of separation.

Eq. (2) is the Bessel equation with solution $AJ_m(kr) + BY_m(kr)$, where J_m and Y_m are the Bessel function and the Neumann function respectively of order m; however, since the latter is unbounded at $r = 0$, we must have $B = 0$.

Eqs. (3) and (4) are readily solved to give

$$\Theta = \Theta_0 \cos(m\theta), \quad (5)$$

$$T = T_0 \cos(\omega t), \quad (6)$$

where $\omega \equiv kv = k(\tau/\rho)^{1/2}$, the angular frequency; m must be an integer because $\Theta(\theta + 2\pi) = \Theta(\theta)$. These equations represent a normal mode. More general vibrations are superpositions of solutions for different m and k.

These equations are valid for a uniform membrane. We choose to load our membrane by dividing it into N concentric annular regions, each of constant mass density (Figure 3). Then in each region the equations of motion (2-4) are still valid, though the separation constants may vary from region to region (we then differentiate them with subscripts $n = 1$ to N). However, from the boundary conditions for each region we find that m and ω must be constant for the entire membrane. This means that the k_n are of the form $\alpha\sqrt{\rho_n}$, where α is a constant for the membrane and proportional to the frequency of vibration:

$$\alpha = \omega\tau^{-1/2}, \quad (7)$$

We can now write down the solution of the radial equation for each region as

$$R_n = A_n J_m(k_n r) + B_n Y_m(k_n r) = A_n J_m(\alpha r\sqrt{\rho_n}) + B_n Y_m(\alpha r\sqrt{\rho_n}), \quad (8)$$

with Eqs. (5-6) for Θ and T and with $B_1 = 0$ and m an integer.

The radial solution is subject to certain boundary conditions, namely:

(1) The solutions must be continuous at the boundaries,

(2) The first derivatives of the solutions must be continuous at the boundaries. (This is because the membranes we are considering consist of single, continuous base skins, appropriately loaded. A discontinuous derivative would imply a 'fold' in the membrane, which would not be expected to happen.)

(3) The displacement at the outermost boundary should be zero, since the boundary is fixed. Here for simplicity we may take the radius $R = 1$.

These conditions give rise to $2N - 1$ equations of constraint (since the boundary conditions (1) and (2) each yields $N - 1$ equations, one for each of the inner boundaries). We have the following equations.

Continuity of solution:

$$A_n J_m(\alpha r_n\sqrt{\rho_n}) + B_n Y_m(\alpha r_n\sqrt{\rho_n}) - A_{n+1} J_m(\alpha r_n\sqrt{\rho_{n+1}}) - B_{n+1} Y_m(\alpha r_n\sqrt{\rho_{n+1}}) = 0. \quad (9)$$

Continuity of derivative:

$$A_n\sqrt{\rho_n}\, J'_m(\alpha r_n\sqrt{\rho_n}) + B_n\sqrt{\rho_n}\, Y'_m(\alpha r_n\sqrt{\rho_n}) - A_{n+1}\sqrt{\rho_{n+1}}\, J'_m(\alpha r_n\sqrt{\rho_{n+1}}) - B_{n+1}\sqrt{\rho_{n+1}}\, Y'_m(\alpha r_n\sqrt{\rho_{n+1}}) = 0. \quad (10)$$

Stationary boundary (i.e. solution vanishes at R=1):

$$A_N J_m(\alpha\sqrt{\rho_N}) + B_N Y_m(\alpha\sqrt{\rho_N}) = 0. \quad (11)$$

Here we have $2N$–1 equations, which contain $2N$–1 quantities A_n and B_n (since we have already noted that $B_1 = 0$) in which they are homogeneous. In general, the only solution of such a system is $A_n = B_n = 0$, or the membrane is at rest. Other solutions can only exist if a special condition is satisfied, that is the determinant of the coefficients of A_n and B_n vanishes:

$$\begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 & \cdots & 0 \\ C_{21} & C_{22} & C_{23} & C_{24} & 0 & 0 & \cdots & 0 \\ 0 & 0 & C_{33} & C_{34} & C_{35} & C_{36} & \cdots & 0 \\ 0 & 0 & C_{43} & C_{44} & C_{45} & C_{46} & \cdots & 0 \\ \vdots & & & & & & & \vdots \end{vmatrix} = 0,$$

where $C_{11} = J_m(\alpha r_1 \sqrt{\rho_1})$, $C_{12} = Y_m(\alpha r_1 \sqrt{\rho_1})$, $C_{21} = \sqrt{\rho_1}\, J'_m(\alpha r_1 \sqrt{\rho_1})$. etc. [The coefficients for the first $2N-2$ rows can be read off Eqs. (9) and (10), and for the last row from Eq. (11).]

This determinant is a function of the quantities r_n, ρ_n, α and m. We write the determinantal equation concisely as

$$\Delta(r_n, \rho_n; m; \alpha) = 0. \qquad (12)$$

If we already knew the r_n and ρ_n for every m the determinant would vanish for certain values of α, and we could thus solve for the frequencies corresponding to each m. However, our situation is different: we know the required values of α for each m (or, at least, we know their ratios) and want to *find* the appropriate ρ_n (and possibly r_n as well).

To do this, we assign, to each possible m, appropriate values of α (most conveniently, simple integers). We substitute each pair (m, α) into Eq. (12) to get a new equation, thus forming a system of equations which can be solved numerically for r_n and ρ_n.

First, however, we need to assign a value of N, the number of regions. Then there are two choices: either we can vary the ρ_n and the r_n, which gives us $2N-1$ variables (since $r_N = 1$ and is not varied), or we can fix the r_n and vary only the ρ_n, which gives us N variables. Since the number of variables must equal the number of equations, each of which corresponds to one harmonic overtone, it may seem that the first approach is more economical, as we get $2N-1$ harmonic overtones for just N different regions; however, it turned out that for $N>2$, solutions were hard to find. The second approach was much more workable.

These equations were solved on a computer for values of N up to 5, using the Newton-Raphson method. This method, being based on the Taylor series, converges only if an initial guess close to the actual answer is supplied. To ensure convergence, the program did not directly solve for the ρ_n using the desired α, but instead slowly changed the α values from the flat membrane values to the desired values. This also ensured that the α's were associated with the appropriate modes of vibration (i.e. that the lowest α was the fundamental and not an overtone with the same value of m, and so on). Nevertheless, the calculations often diverged, indicating that solutions did not exist for the given conditions. (Some nonsensical solutions such as negative ρ_n or $r_{n-1} > r_n$ also had to be discarded.)

4. Results

The calculations converged readily for N=1 and N=3, but not for higher N when overtones which were integral multiples of the fundamental were desired. However, when the value of the fundamental was not assigned, convergence was obtained even for $N = 5$ (with r_n appropriately chosen). Details are given in Table 2. It is seen that, when the fundamental is not required to be in the integer series, its value works out higher by roughly the same factor as in the actual drums.

The results were difficult to confirm experimentally, but we tried to load a Western drum (a tom-tom) with rubber sheets in close conformity with the results of the calculations. The overtones produced were roughly as expected, but because of the weight of the loading, the frequencies were too low to be measured very accurately using an FFT.

5. Conclusion

The major conclusion of this study is that Raman and later workers (who are very few in number) appear to have been mistaken in thinking that the higher overtones in the mridangam and tabla are harmonics of the fundamental. Not only are they not so in actual drums, but it seems very difficult even theoretically to produce harmonic overtones in the strict sence. However, since nearly all the overtones are in integer ratios to each other, it seems appropriate not to call the overtones anharmonic, but to regard the fundamental as being a little too high.

The failure of our calculations to produce more than two strictly harmonic overtones should not be taken as proof of the impossibility of such loading, because the calculations were rather crude in nature. It is easy to calculate frequencies corresponding to a given mass

Table 2: Calculation of How to Load a Drum Membrane: Radii and Surface Mass Densities of Annular Regions

N	r_n	ρ_n	Frequencies (m, l mode) (in units of the "ideal" fundamental or half the first overtone)
With harmonic fundamental			
2	1/3, 1	6.2559, 1	1 (0, 1); 2 (1, 1)
2*	0.34261, 1	7.1288, 1	1 (0, 1); 2 (1, 1); 3 (0, 2)
3	1/3, 2/3, 1	11.167, 1, 3.7625	1 (0, 1); 2 (1, 1); 2.6 (0, 2); 3 (2, 1)
Without harmonic fundamental			
2*	0.41623, 1	4.191, 1	1.07 (0, 1); 2 (1, 1); 3 (0.2); 3 (2, 1)
5	0.1, 0.2, 0.3, 0.4, 1	4.989, 8.320, 3.424, 8.132, 1	1.054 (0, 1); 2 (1, 1); 3 (0, 2); 3 (2, 1); 4 (1, 2); 4 (3, 1); 4.607 (2, 2); 4.992 (0, 3)

**In these, both ρ and r were calculated. In the other calculations, r were assigned (by trial and error) and only ρ were calculated.*

Every normal mode can be classified by a pair of numbers, m and l. In the (m, l) mode, m is the number of nodal diameters (each diameter corresponding to a zero of the angular function), as follows from Eq. (5), while l is the number of nodal circles (each circle corresponding to a zero of the radial function).

The exact values of ρ and r are unimportant, and so are their units, only the ratios must be maintained.

distribution on a membrane, but the reverse problem is not so simple, It is likely, however, that if such a loading is possible, it would be rather complicated and of doubtful practical value, particularly as the 'wrong' fundamental does not appear to trouble either musicians or audiences.

Acknowledgements

We wish to thank the Centre for Science Education and Communication, University of Delhi, for providing facilities: Prof. P.K. Srivastava, Department of Physics, University of Delhi, and Director of the Centre, for help and useful discussions; Dr. H. Jacob, Department of Physics, St. Stephen's College, for comments and suggestions; and Mr. M. Padmanabhan and Mr. Sitaram of the Faculty of Music, University of Delhi, for demonstration of and helpful information on the mridangam, tabla and pakhawaj. This work was funded by an award from the Avinder S. Brar Foundation, St. Stephen's College, and fuller details are given in a report presented to the Foundation.

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Super Glue to Plug Micro Holes in Eyes

A Super glue that can effectively plug microscopic holes in the human eye caused by ulcers in the cornea has been developed by scientists at the Indian Institute of Chemical Technology (II CT) in Hyderabad.

Derived from butyl cyanoacrylate, the superglue called cyanoacrylic surgical glue has been made in three-ml bottles, by IICT for opthalmic use.

A drop of 0.5 ml of the glue applied in the cornea can effectively seal the punctures or small holes and provide a painless alternative to sutures.

During clinical trials eye doctors used the glue in 50 patients and reported excellent results with no side-effects like allergy or sepsis.

Butyl cyanoacrylates bind tissue within seconds, have very low toxicity and are ideal for medical applications.

IICT is considering transferring the technology to a company to produce the glue on a larger scale and help reduce imports.

The IICT's superglue is of good quality and costs around Rs. 50 per drop of 0.54 ml against Rs. 300 for that imported from Germany.