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Is there an energy cascade in strong wave turbulence?

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Abstract. *Constant fluxes of conserved quantities in stationary wave turbulence exactly determine the inertial range scalings of appropriate flux-carrying correlation functions just as energy conservation determines the scaling of the third order structure function in hydrodynamic turbulence. This constraint on the flux-carrying correlation function, which we refer to as a* constant flux relation *(CFR) requires no assumption of weak nonlinearity. It thus provides a natural departure point for the study of strong wave turbulence. In this paper we state the theoretical results and illustrate the ideas using a finite dimensional toy model. We predict that the energy cascade in strong wave turbulence, provided that a local cascade is possible, would be of a different character to most familiar cascade mechanisms and must involve a non-trivial conversion between linear wave energy and nonlinear wave self-interaction energy.*
Keywords*:* strong wave turbulence, higher order correlation functions.

Wave turbulence [1] is modelled as a Hamiltonian equation for the complex wave amplitudes, $a_{\mathbf{k}}$, coupled to a source and sink of energy which are widely separated in $\mathbf{k}$ (wave-vector) space :

$$\partial_t a_{\mathbf{k}} = i\frac{\delta H}{\delta \bar{a}_{\mathbf{k}}} + f_{\mathbf{k}} - \gamma_{\mathbf{k}} a_{\mathbf{k}}. \tag{1}$$

H has a linear and nonlinear part, $H = T + gU$. The latter induces interactions between waves enabling transfer of energy among normal modes resulting in cascades in $\mathbf{k}$-space. The linear part of the energy describes a collection of linear oscillators: $T = \int \omega_{\mathbf{k}} a_{\mathbf{k}} \bar{a}_{\mathbf{k}} d\mathbf{k}$. For 3-wave interactions, the nonlinear part of the energy is $U = \int V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} \left(a_{\mathbf{k}} a_{\mathbf{k}_1} \bar{a}_{\mathbf{k}_2} + \bar{a}_{\mathbf{k}} \bar{a}_{\mathbf{k}_1} a_{\mathbf{k}_1}\right) \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\, d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2$. We take the dispersion relation, $\omega_{\mathbf{k}}$, and the triad interaction coefficient, $V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ to be homogeneous functions of their arguments having degree α and γ respectively. Almost everything is known [1] about the stationary statistics of Eq. (1)

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in the limit of weak nonlinearity when the interactions between waves can be treated perturbatively. In particular, T, is conserved to leading order and cascades to small scales in an analytically tractable way. Practically nothing is known about the strong nonlinearity limit when both T and U cascade.

In the stationary state [2], conservation of energy, $T+U$, requires that

$$\int \prod_{i=1}^{2} (dk_i k_i^{d-1}) \left[T_{k;k_1,k_2} \Pi_{0;1,2} - T_{k_1;k,k_2} \Pi_{1;0,2} \right] = 0 \tag{2}$$

in the inertial range. In this equation

$$\Pi_{0;1,2} = \int \prod_{i=0}^{2} d\Omega_i \langle Re(a_{\vec{k}} \partial_t \bar{a}_{\vec{k}_1} \bar{a}_{\vec{k}_2}) \rangle \tag{3}$$

is the energy flux correlation function. Note the unusual structure of this correlation function: it involves time derivatives of amplitudes and is thus a composite quantity. Assuming a locality condition requiring convergence of a certain integral involving an unknown scaling function, one may show that this is only possible if $\Pi_{0;1,2}$ has homogeneity degree $-3d-\gamma$. This result, analogous to Kolmogorov's $4/5$-Law, holds regardless of the strength of the underlying wave turbulence. This is the only general exact result known to us for strong wave turbulence. In the weak limit, the assumption of locality can be checked a-posteriori. This is not the case in general. Locality can be non-trivial [3]. In general, we require numerical simulation to study locality of a cascade.

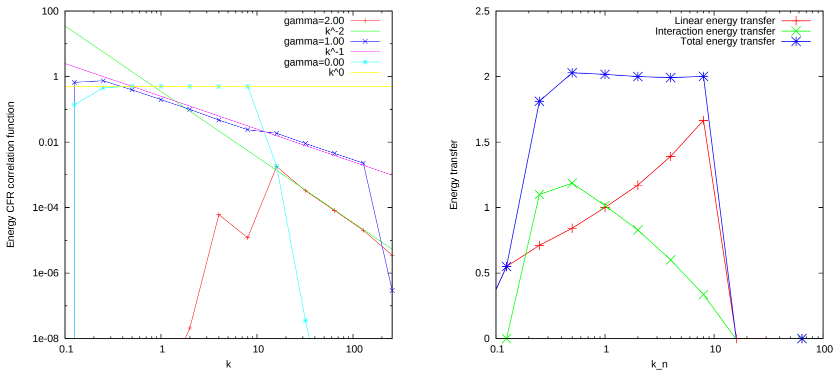

Figure 1: Left panel: Verification of CFR scaling, Eq. (6), for the toy model. Right panel: Stationary fluxes of linear, nonlinear and total energy for $\gamma = 0.0$.

To separate complications coming from the unusual structure of $\Pi_{0;1,2}$ from those coming from locality, we studied a toy model of strong 3-wave interactions. The model is a discrete chain of oscillators with wavenumbers, $k_n = 2^n$

and frequencies $\omega_n = k_n^\alpha$. Energy is injected by forcing oscillator 0 and removed by damping at both ends. The Hamiltonian is $H = \sum_{n=-N}^{n=N} T_n + U_n$ where

$$T_n = \omega_n a_n \overline{a}_n \qquad U_n = k_{n-1}^\gamma \left(\overline{a}_n a_{n-1}^2 + a_n \overline{a}_{n-1}^2 \right). \tag{4}$$

The chain reaches a statistically stationary state in which energy is injected at the centre, cascades through the chain and is removed at the ends. The flows of linear and nonlinear energies are:

$$\frac{dT_n}{dt} = Q_n^{(T)} - Q_{n-1}^{(T)} - R_n \qquad \frac{dU_n}{dt} = Q_n^{(U)} - Q_{n-1}^{(U)} + R_n \tag{5}$$

where $Q_n^{(T)} = -4g\omega_n k_n^\gamma \mathrm{Im}[a_{n+1}\overline{a}_n^2]$, $Q_n^{(U)} = -4g^2(k_n k_{n-1})^\gamma \mathrm{Im}[\overline{a}_{n+1} a_n a_{n-1}^2]$ and $R_n = \left(1 - \frac{\omega_n}{2\omega_{n-1}}\right) Q_n^{(T)}$ Taking averages in the stationary state yields for the average transfer of total energy: $\langle Q_n^{(H)} \rangle - \langle Q_{n-1}^{(H)} \rangle = 0$ where $Q_n^{(H)} = Q_n^{(T)} + Q_n^{(U)}$ is the flux of total energy. This has the simple solution

$$\mathrm{Re}\langle \overline{a}_{n+1}\, a_n\, d_t a_n \rangle = -\frac{Q_0}{4g} k_n^{-\gamma}, \tag{6}$$

where Q_0 is the rate of dissipation of total energy. This is the analogue of the constant flux relation for our toy model. Its numerical verification for several values of γ is shown in the left panel of Fig. 1. The right panel of Fig. 1 demonstrates the constancy of the flux of total energy for the case $\gamma = 0$. The results indicate that the flux carrying correlation function can indeed exhibit scaling behaviour despite its composite nature. The cascade of total energy in the toy model is clearly of a different character to a more familiar cascade of a quadratic quantity. The cascade involves the transmutation of linear energy into nonlinear energy in order to maintain a constant flux of the sum of the two.

We have not yet studied the extent to which these lessons carry over to the original wave turbulence problem. It is clear, however, that there is some potential here for some interesting new results on strong wave turbulence.

References

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