

Multi-Scaling of the n -Point Density Function for Coalescing Brownian Motions

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Abstract: This paper gives a derivation for the large time asymptotics of the n -point density function of a system of coalescing Brownian motions on \mathbf{R} .

1. Introduction and Statement of the Main Result

The single species reaction-diffusion systems $A + A \rightarrow A$ (coalescence) and $A + A \rightarrow 0$ (annihilation) have been studied extensively in recent times [6–13]. A common interest in these papers is the departure from mean field behaviour and the computation of exact long-term asymptotics for the particle density.

Recently, it was predicted in [14] that the large time asymptotics for the probability $P_t(n, \Delta V)$ of finding n particles in a fixed volume ΔV :

$$P_t(n, \Delta V) \sim \begin{cases} t^{-\frac{n}{2} - \frac{n(n-1)}{4}} & d = 1 \\ \left(\frac{\ln t}{t}\right)^n (\ln t)^{-\frac{n(n-1)}{2}} & d = 2 \\ t^{-n} & d > 2 \end{cases} \quad (1.1)$$

Note that the predictions for $d > 2$ agree with mean field behaviour. The second part of the exponent in $d = 1$ reflects multi-scaling, or deviation from linear scaling. In $d = 2$ the multi-scaling is manifested in the second logarithmic term. This type of scaling is indicative of particles being *anti-correlated* [14]. These predictions were obtained by use of the dynamical perturbative renormalization group methods in a field theoretic setting [6]. The setting here is for finite rate reactions, leading to annihilations of randomly walking particles on a fixed lattice. After renormalization the large time limit of the reaction rate tends to infinity, corresponding to an instantaneous reaction [6, 13]. Moreover the predictions carry over to the coalescing case, since they have the same effective field theory [1, 8].

The aim of this paper is to examine the conjectures of [14] in dimension $d = 1$, where detailed probabilistic tools are available. We consider a system of coalescing Brownian particles on the real line. Each particle evolves independently until it collides with another particle, at which time the two colliding particles instantaneously coalesce into one. The n -point density function is defined, for distinct y_1, \dots, y_n , by

$$P[\text{there exist particles in } dy_1, \dots, dy_n \text{ at time } t] = \rho_n(y_1, \dots, y_n; t) dy_1 \dots dy_n.$$

Note that ρ_n depends on the initial particle distribution. The existence of such a density is discussed in the Appendix.

Our main result can be stated precisely as the following asymptotic:

$$\rho_n(y_1, \dots, y_n; t) \sim t^{-\alpha(n)} \prod_{1 \leq i < j \leq n} |y_i - y_j|, \quad (1.2)$$

where $\alpha(n) = n/2 + n(n-1)/4$. This asymptotic has the meaning that the left-hand side is bounded above and below by constant multiples C_1, C_2 of the right-hand side. The upper bound is not actually an asymptotic, in that it holds simultaneously for all countable initial conditions, for all $t > 0$ and for all $|y_i| \leq Lt^{1/2}$, with a constant $C_2(L, n) < \infty$ depending only on n and L . The lower bound holds for all $t \geq t_0$ and all $|y_i| \leq Lt^{1/2}$, with a constant $C_1(n, L, t_0) > 0$ that depends on n, L, t_0 and also on the initial condition. For the lower bound, the initial condition must satisfy a mild non-degeneracy condition; in particular it holds for deterministic initial conditions provided the gap between successive particles is bounded, but also if we assume the set of initial positions of particles $\{X_0^i : i \geq 1\}$ is non-empty, translationally invariant and spatially ergodic, in the sense that the distribution of $\{X_0^i : i \geq 1\} \cap (-\infty, 0]$ and $\{X_0^i : i \geq 1\} \cap [L, \infty)$ become independent as $L \rightarrow \infty$. When both the upper and lower bounds hold, we may integrate the density ρ_n over a fixed volume ΔV to obtain the decay rate predicted by the case $d = 1$ of (1.1).

Let us stress that (1.2) is the small distance result: it is valid only in the limit when separation between points is much smaller than correlation length \sqrt{t} . For $n = 2$ it can be obtained as the $\Delta(x)/\sqrt{t} \rightarrow 0$ limit of a well known answer for a pair correlation function for ‘vicious’ walkers, [13, 9, 8, 19]. We also expect (1.2) to hold for the system of annihilating Brownian motions albeit with different amplitudes, see [8].

The key tool is the Karlin-McGregor formula for the non-coincidence probabilities for Brownian motions. Useful upper and lower bounds on this transition density, which already display the key anomalous scaling term $t^{-\alpha(n)}$, are developed in Sect. 3, by exploiting a representation known as the Harish-Chandra-Itzykson-Zuber formula (developed for random matrix problems).

The empty interval method, and its generalizations, have been used to derive expressions for higher order correlation functions in [7–9] for one-dimensional systems with instantaneous reactions. Large time asymptotics for the n -point correlation function density for the coalescing case with Poissonian initial conditions are found in [9], while the corresponding results for the annihilating case are given in [8]. It was shown in this special case that the n -point density correlation function for the two systems are the same apart from the amplitude. This set of exact results was used to test the predictions (1.1) in [14] for $n = 1, 2, 3, 4$. For general n , the derivation of (1.2) from the exact expressions of [8, 9] is not obvious as it results from the cancellation between $(2n-1)!!$ terms with alternating signs.

The problem of deriving rigorously the logarithmic corrections (1.1) to the mean field answers in dimension $d = 2$ remains open. It would also be interesting to find out if

there is a natural multi-fractal interpretation of the multi-scaling. Another simple system for which RG calculations predict multi-scaling in the stationary state is the system of aggregating massive point clusters with a stationary source of light particles. This system is relevant to turbulence, see [15]. It would be interesting to generalize the methods of the present paper to prove multi-scaling for such cluster-cluster aggregation.

2. Proof of the Main Result

The proof is based on the following two lemmas. The first is a bound on the Karlin-McGregor formula for the transition density for non-intersecting Brownian motions [5, 3]. Fix $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ and let $(X^{x_i} : i = 1, \dots, n)$ be independent Brownian motions with $X_0^{x_i} = x_i$. For $y \in \mathbf{R}^n$, denote by $G_t^{KM}(x, y)$ the density of the probability measure

$$P \left[X_t^{x_i} \in dy_i \text{ and the paths } (X_s^{x_i} : s \in [0, t]) \text{ are non-intersecting for } i = 1, \dots, n \right].$$

Then the Karlin-McGregor formula is

$$G_t^{KM}(x, y) = \begin{vmatrix} G_t(x_1, y_1) & \dots & G_t(x_1, y_n) \\ \vdots & & \vdots \\ G_t(x_n, y_1) & \dots & G_t(x_n, y_n) \end{vmatrix}, \quad (2.1)$$

where $G_t(a, b) = (2\pi t)^{-1/2} e^{-|a-b|^2/2t}$ is the one dimensional Brownian transition density. Note that the function $G_t^{KM}(x, y)$ is the transition density for an n -dimensional Brownian motion killed on the set $\cup_{i \neq j} \{y_i = y_j\}$. One can check directly that the determinant expression (2.1) satisfies the heat equation with zero Dirichlet boundary conditions on this boundary, and that $\lim_{t \downarrow 0} G_t^{KM}(x, y) = \delta_{x=y}$.

The following lemma, proved in Sect. 3, gives usable bounds on $G_t^{KM}(x, y)$.

Lemma 1. *For all $x_1 < \dots < x_n$, $y_1 < \dots < y_n$ and $t > 0$,*

$$c_n \prod_{i=1}^n G_t(x_i, y_{n-i+1}) \leq \frac{G_t^{KM}(x, y)}{\Delta(xt^{-1/2}) \Delta(yt^{-1/2})} \leq c_n \prod_{i=1}^n G_t(x_i, y_i), \quad (2.2)$$

where $c_n^{-1} = \prod_{i=1}^n i!$ and $\Delta(x)$ is the Vandermonde determinant defined by

$$\Delta(x) = \Delta(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

The second lemma, proved in Sect. 4, gives a simple upper bound on the n -point correlation function, which reflects the intuition that particles should be anti-correlated, in that the presence of a particle in dx decreases the likelihood that there is a particle at another point dy .

Lemma 2. *For any initial distribution of particles, the n -point density function satisfies*

$$\rho_n(y_1, \dots, y_n; t) \leq (\pi t)^{-n/2} \quad \text{for all } y_i \in \mathbf{R} \text{ and } t > 0.$$

The lower bound in the asymptotic (1.2) follows quickly from the lower bound on the Karlin-McGregor formula (2.2). Indeed, list the set of initial positions of particles as $\{X_0^i : i \geq 1\}$ and let $\Omega_0(t)$ be the event that there exist particles $X_0^{i_1}, \dots, X_0^{i_n}$ satisfying

$$X_0^{i_k} \in [2kt^{1/2}, (2k+1)t^{1/2}] \quad \text{for } k = 1, \dots, n. \quad (2.3)$$

Then, applying the Karlin-McGregor formula between the points $X_0^{i_1}, \dots, X_0^{i_n}$ and y_1, \dots, y_n we find, for $t \geq 1$,

$$\begin{aligned} & P \left[\text{there exist particles at } dy_1, \dots, dy_n \text{ at time } t \right] \\ & \geq E \left[\mathbf{I}(\Omega_0(t)) G_t^{KM} \left((X_0^{i_1}, \dots, X_0^{i_n}), (y_1, \dots, y_n) \right) \right] \\ & \geq c_n E \left[\mathbf{I}(\Omega_0(t)) \Delta(X_0^{i_1} t^{-1/2}, \dots, X_0^{i_n} t^{-1/2}) \prod_{k=1}^n G_t(X_0^{i_k}, y_{n-k+1}) \right] \\ & \quad \times \Delta(y t^{-1/2}) dy_1 \cdots dy_n \\ & \geq C(n, L) t^{-n/2} P[\Omega_0(t)] \Delta(y t^{-1/2}) dy_1 \cdots dy_n, \end{aligned}$$

where we have used (2.3) and $|y_j| \leq L t^{1/2}$ in the final inequality. We have also used $C(n, L, \dots)$ to denote a finite non-zero quantity, depending only on the quantities listed, but whose exact value is unimportant and may change from line to line. It remains only to bound $P[\Omega_0(t)]$ from below, independently of $t \geq t_0$. This clearly holds under the two sets of assumptions described in Sect. 1, and in particular for Poissonian initial conditions.

For the upper bound in the asymptotic (1.2) we estimate the probability that there exist particles at dy_1, \dots, dy_n at time $2t$ by conditioning on the countable set $\{X_t^i : i \in \mathbf{N}\}$ of positions of the particles at time t . For the desired particles to exist at time $2t$, one of the events

$$\Omega_{i_1, \dots, i_n}(t) = \left\{ X_{2t}^{i_k} \in dy_k \text{ and the paths } (X_s^{i_k} : s \in [t, 2t]) \text{ do not intersect for } k = 1, \dots, n \right\},$$

for some $i_1 < \dots < i_n$, must occur. Applying the Markov property at time t and the upper bounds in Lemma 1 we find, for all $t > 0$,

$$\begin{aligned} & P \left[\text{there exist particles at } dy_1, \dots, dy_n \text{ at time } 2t \right] \\ & = P \left[\bigcup_{i_1 < \dots < i_n} \Omega_{i_1, \dots, i_n}(t) \right] \\ & \leq \sum_{i_1 < \dots < i_n} P \left[\Omega_{i_1, \dots, i_n}(t) \right] \\ & \leq c_n \sum_{i_1 < \dots < i_n} E \left[\Delta(X_t^{i_1} t^{-1/2}, \dots, X_t^{i_n} t^{-1/2}) \prod_{k=1}^n G_t(X_t^{i_k}, y_k) \right] \Delta(y t^{-1/2}) dy_1 \cdots dy_n \\ & \leq C(n) \left(\int_{\mathbf{R}^n} G_t(x, y) \Delta(x t^{-1/2}) \rho_n(x; t) dx \right) \Delta(y t^{-1/2}) dy_1 \cdots dy_n \quad (\text{see (B.1)}) \\ & \leq C(n, L) t^{-n/2} \Delta(y t^{-1/2}) dy_1 \cdots dy_n. \end{aligned}$$

The final inequality uses the substitution $x \rightarrow x t^{1/2}$ and the bound from Lemma 2.

3. Proof of Lemma 1, the Karlin-Macgregor Bounds

First note that Brownian scaling implies that $G_t^{KM}(x, y) = t^{-n/2} G_1^{KM}(xt^{-1/2}, yt^{-1/2})$, and we need only prove the lemma for $t = 1$. Factoring out common terms from the rows in (2.1) we find that

$$G_1^{KM}(x, y) = (2\pi)^{-n/2} \det(E(x, y)) \prod_{i=1}^n e^{-(x_i^2 + y_i^2)/2},$$

where $E(x, y)$ is the matrix with entries $\exp(x_i y_j)$. We now use the Harish-Chandra-Itzykson-Zuber formula [2], which is widely used in random matrix theory. This states that

$$\det(E(x, y)) = c_n \Delta(x) \Delta(y) \int_{\mathbf{U}(n)} \exp\left(\operatorname{tr}(UXU^\dagger Y)\right) \mu(dU),$$

where $\operatorname{tr}(A)$ is the trace of a matrix A , $\mu(dU)$ is the (normalized) Haar measure on the unitary group $\mathbf{U}(n)$ and X, Y are the diagonal matrices with entries x_1, \dots, x_n and y_1, \dots, y_n . A short proof of this formula is found in [4] (Appendix A.5).

The fact that U is unitary ensures that the function $F(U) = \operatorname{tr}(UXU^\dagger Y)$ is real valued (recall that U^\dagger is the conjugate transpose of U so that $UU^\dagger = U^\dagger U = I$). By the compactness of $\mathbf{U}(n)$, $F(U)$ achieves its maximum and minimum values and we claim that

$$\min_{U \in \mathbf{U}(n)} F(U) = \sum_{i=1}^n x_{n-i+1} y_i, \quad \max_{U \in \mathbf{U}(n)} F(U) = \sum_{i=1}^n x_i y_i. \quad (3.1)$$

Bounding the integral in the Harish-Chandra-Itzykson-Zuber formula using these maxima and minima in the integrand leads to the bounds (2.2).

It remains only to prove the claim (3.1), for which we will find all the stationary points of the smooth function F on the manifold $\mathbf{U}(n)$. A matrix V is in the tangent space to $\mathbf{U}(n)$ at the point U precisely when $UV^\dagger + VU^\dagger = U^\dagger V + V^\dagger U = 0$. The first derivative of F in the direction of V in the tangent space is given by

$$\begin{aligned} D_V F(U) &= \operatorname{tr}(UXV^\dagger Y) + \operatorname{tr}(VXU^\dagger Y) \\ &= \operatorname{tr}(V^\dagger YUX) + \operatorname{tr}(VXU^\dagger Y) \quad (\text{since } \operatorname{tr}(AB) = \operatorname{tr}(BA) \text{ for Hermitian } B) \\ &= \operatorname{tr}(UV^\dagger YUXU^\dagger) + \operatorname{tr}(VXU^\dagger Y) \quad (\text{since } \operatorname{tr}(UAU^\dagger) = \operatorname{tr}(A)) \\ &= -\operatorname{tr}(VU^\dagger YUXU^\dagger) + \operatorname{tr}(VU^\dagger UXU^\dagger Y) \quad (\text{using } UV^\dagger = -VU^\dagger) \\ &= \operatorname{tr}\left(VU^\dagger [UXU^\dagger, Y]\right) \end{aligned}$$

(where $[A, B] = AB - BA$ is the commutator). The map $V \rightarrow VU^\dagger$ bijectively maps the tangent space onto the Lie group $\operatorname{Lie}(\mathbf{U}(n))$ defined by $\{W : W^\dagger + W = 0\}$. It is straightforward to check that $\operatorname{tr}(WA) = 0$ for all $W \in \operatorname{Lie}(\mathbf{U}(n))$ implies that $A = 0$. The vanishing of $D_V F(U)$ therefore implies that $[UXU^\dagger, Y] = 0$. This in turn implies, since Y is diagonal with distinct eigenvalues, that UXU^\dagger is a diagonal matrix. The matrix UXU^\dagger has the same distinct eigenvalues as X , so that the entries on the diagonal of UXU^\dagger must be a permutation of x_1, \dots, x_n . Hence the value of $F(U)$ at a stationary point is $\sum_{i=1}^n x_{\pi(i)} y_i$ for some permutation π . We must show that this value is maximized when $\pi = \operatorname{Id}$ the identity permutation and minimized

at the involution $\pi(i) = N + 1 - i$. Argue by contradiction and suppose that $\pi \rightarrow \sum_{i=1}^n x_{\pi(i)} y_i$ is maximized over permutations at some $\pi_0 \neq \text{Id}$. Then there exist $i < j$ for which $\pi_0(i) > \pi_0(j)$. Let $\pi^{(jk)}$ be the transposition permutation, swapping the j^{th} and k^{th} elements. Then an increased value is found by taking the composition $\pi_1 = \pi^{(jk)} \circ \pi_0$ permutation, since

$$\sum_{i=1}^n x_{\pi_1(i)} y_i - \sum_{i=1}^n x_{\pi_0(i)} y_i = (x_{\pi_0(i)} - x_{\pi_0(j)})(y_j - y_i) > 0,$$

using the ordering $x_1 < \dots < x_n, y_1 < \dots < y_n$. The minimum follows from a similar argument.

4. Proof of Lemma 2, the n -Point Density Bound

We start with a simple construction of coalescing particles. Fix distinct x_1, \dots, x_m . Take I.I.D. Brownian motions ($B^i : i \geq 1$) and set $\hat{X}_t^{x_i} = x_i + B_t^i$, for $t \geq 0$, the non-interacting Brownian paths. We now coalesce the paths inductively in i as follows. Let $X_t^{x_1} = \hat{X}_t^{x_1}$ for all $t \geq 0$. Supposing the paths $t \rightarrow X_t^{x_j}$ have been defined for $j < i$, we define

$$\tau_i = \inf \left\{ t : \hat{X}_t^{x_i} \in \{X_t^{x_1}, \dots, X_t^{x_{i-1}}\} \right\}.$$

Set $X_t^{x_i} = \hat{X}_t^{x_i}$ for $t \leq \tau_i$. If $\tau_i < \infty$ we may choose $j \in \{1, \dots, i-1\}$ so that $\hat{X}_{\tau_i}^{x_i} = X_{\tau_i}^{x_j}$ and we set $X_t^{x_i} = X_t^{x_j}$ for $t > \tau_i$. In this way we construct, for any $t > 0$, the set $\mathcal{S}_t = \{X_t^{x_i} : 1 \leq i \leq m\}$ of positions of the coalescing particles at time t . The distribution of \mathcal{S}_t depends only on the set of initial positions $\{x_1, \dots, x_m\}$ and not on the order in which we have applied the above coalescence. Moreover, \mathcal{S}_t increases monotonically as the set of initial positions increases. This allows us to define the distribution of the set \mathcal{S}_t starting from a countable initial set of particles as the limit of the distributions from increasing finite subsets. We will write $P_{x_1, x_2, \dots}$ to indicate the starting positions.

The fact that $\rho_1(y) \leq (\pi t)^{-1/2}$, for any initial particle configuration, is well known. For example, in Sect. 9 of [16] it states that the time reversal duality formula for coalescing Brownian motions for $a < b$ is given by

$$P_{x_1, \dots, x_m} [\{X_t^{x_i} : i \leq m\} \cap [a, b] \neq \emptyset] = P_{a, b} \left[[X_t^a, X_t^b] \cap \{x_1, \dots, x_m\} \neq \emptyset \right].$$

We can take the supremum of the above duality over all finite starting configurations to obtain

$$\sup_{\{x_i\}} P_{x_1, \dots, \infty} [\{X_t^{x_i} : i \geq 1\} \cap [a, b] \neq \emptyset] = P_{a, b} [X_t^a < X_t^b],$$

which can be explicitly calculated and is bounded by $(\pi t)^{-1/2}(b-a)$. We give an elementary approach to this bound that avoids duality in Appendix A.

For the extension to ρ_n given in Lemma 2 we fix $a_1 < b_1 < a_2 < \dots < a_n < b_n$. Define

$$\Omega_j = \{\text{there exists a particle in } [a_j, b_j] \text{ at time } t\}.$$

We shall show by induction that

$$P_{x_1, \dots, x_m} [\Omega_1 \cap \dots \cap \Omega_n] \leq \prod_{i=1}^n \frac{(b_i - a_i)}{\pi t^{1/2}}. \quad (4.1)$$

We may suppose that $x_1 < x_2 < \dots < x_m$. Decompose Ω_1 as the disjoint union $\Omega_1 = \cup_k \Omega_{1,k}$, where

$$\Omega_{1,k} = \{X_t^{x_i} < a_1 \text{ for } i = 1, \dots, k-1 \text{ and } X_t^{x_k} \in [a_1, b_1]\}.$$

The construction of the coalescing paths given above, taken in the order x_1, x_2, \dots, x_m , shows that conditional on $\Omega_{1,k}$ the processes $(X^{x_i} : i \geq k+1)$ evolve as coalescing Brownian motions with an extra lower absorbing boundary along the path $t \rightarrow X_t^{x_k}$. Repeating the construction, but using the order $x_m, x_{m-1}, \dots, x_{k+1}$, shows that this extra absorbing path only lowers the probability that any of the positions $(X_t^{x_i} : i \geq k+1)$ lie in the intervals $[a_2, b_2], \dots, [a_n, b_n]$. Thus, applying Bayes formula,

$$\begin{aligned} & P_{x_1, \dots, x_m} [\Omega_1 \cap \dots \cap \Omega_n] \\ &= \sum_{k=1}^m P_{x_1, \dots, x_m} [\Omega_2 \cap \dots \cap \Omega_n \mid \Omega_{1,k}] P_{x_1, \dots, x_m} [\Omega_{1,k}] \\ &\leq \sum_{k=1}^m P_{x_{k+1}, \dots, x_m} [\Omega_2 \cap \dots \cap \Omega_n] P_{x_1, \dots, x_m} [\Omega_{1,k}] \\ &\leq \sum_{k=1}^m P_{x_1, \dots, x_m} [\Omega_2 \cap \dots \cap \Omega_n] P_{x_1, \dots, x_m} [\Omega_{1,k}] \quad (\text{by stochastic monotonicity}) \\ &= P_{x_1, \dots, x_m} [\Omega_2 \cap \dots \cap \Omega_n] P_{x_1, \dots, x_m} [\Omega_1]. \end{aligned}$$

The result (4.1), and hence Lemma 2, follows by induction.

A. An Elementary Approach to the One Point Density

This elementary approach may be of some interest for systems that do not have a duality relation. Fix $a \in [0, 1]$ and let $p(x_1, \dots, x_n)$ be the probability that, starting some coalescing Brownian motions x_1, \dots, x_n , there is a particle at time t in the interval $[0, a]$. We aim to show, by induction on n , that

$$p(x_1, \dots, x_n) \leq Ct^{-1/2}a \quad (\text{A.1})$$

for some constant $C < \infty$. We suppose $x_1 < \dots < x_{n+1}$. Condition on the path $t \rightarrow X_t^{(n+1)}$ started at x_{n+1} . We can consider three cases: if $X_t^{(n+1)} < 0$ then we are sunk since no other path can ever enter $[0, a]$ due to coalescence; if $X_t^{(n+1)} \in [0, a]$ then we are done; finally if $X_t^{(n+1)} > a$ then we still need some of the particles from x_1, \dots, x_n to enter $[0, a]$, but there is an absorbing boundary along the path of $X_t^{(n+1)}$. This absorbing boundary lowers the chance of getting particles where we want. This splitting of possibilities, together with the induction hypothesis (A.1), leads to

$$p(x_1, \dots, x_{n+1}) \leq Ct^{-1/2}aP[X^{(n+1)} > a] + P[X^{(n+1)} \in [0, a]]. \quad (\text{A.2})$$

It is straightforward, by examining ratios of the transition density, to find $c_0 < \infty$ so that for a Brownian motion (X_t) , and for $L \geq 1$, $t > 0$,

$$\frac{P[X_t \in [0, a] | X_0 = x]}{P[X_t < a | X_0 = x]} \leq c_0 L a t^{-1/2} \quad \text{whenever } |x| \leq L t^{-1/2}.$$

Suppose now that $x_i \in [-L t^{1/2}, L t^{1/2}]$ for all i . Using this bound the induction argument works with the choice $C = c_0 L$. In particular we have shown that $p(x_1, \dots, x_n) \leq c_0 L t^{-1/2} a$ whenever $|x_i| \leq L t^{1/2}$ for all i . Applying this with the choice $L = O(\sqrt{\log(t)})$, and using a simple bound that any particle starting outside $[-L t^{1/2}, L t^{1/2}]$ can reach the intervals $[0, a]$ (we need only control particles starting exactly at $\pm L t^{1/2}$ by coalescence) we find there exists $C < \infty$ such that

$$\sup_{\{x_i\}} p(x_1, \dots, x_n) \leq C t^{-1/2} (\log(t))^{1/2} a,$$

where the supremum is over all possible initial configurations. One can remove the unwanted logarithm term by a blocking argument, but this is already longer than the elegant duality argument, which has the extra advantage of achieving the optimal constant.

B. A Note on the Existence of ρ_n

We will show there exists a measure for which $\rho_n(x; t)$ acts as a density, and show its connection to the n -point correlation function. List the (disjoint) positions of the particles at time t as $\{X_t^i : i \geq 1\}$. Define a random measure μ_t^n on \mathbf{R}^n by

$$\mu_t^n = \sum_{i_1} \sum_{i_2 \neq i_1} \cdots \sum_{i_n \neq i_1, \dots, i_{n-1}} \delta(X_t^{i_1}, \dots, X_t^{i_n}),$$

where $\delta(x)$ is a point mass at x . Let $C_n(\delta)$ denote the cube $[0, \delta]^n$. The cubes $k + C_n(2^{-m})$, for $k \in (2^{-m}\mathbf{Z})^n$, partition \mathbf{R}^n . Take $A \subseteq \mathbf{R}^n$ open. For an initial condition with only finitely many particles, the measure μ_t^n has only finitely many atoms which do not lie on $\cup_{i \neq j} \{x_i = x_j\}$. Therefore we have the increasing limit

$$\lim_{m \rightarrow \infty} \sum_{k \in (2^{-m}\mathbf{Z})^n \cap A} \mathbf{I}(\mu_t^n(k + C_n(2^{-m})) > 0) = \mu_t^n(A),$$

where the sum over k is restricted to those terms for which k_1, \dots, k_n are distinct. The event $\mu_t^n(k + C_n(2^{-m})) > 0$ can be rewritten as $[k_j, k_j + 2^{-m}] \cap \{X_t^i : i \geq 1\} \neq \emptyset$ for $j = 1, \dots, n$. Taking expectations and applying (4.1) we find that $E[f d\mu_t^n] \leq (2\pi t)^{-n/2} \int f dx$ when $f = \mathbf{I}(A)$. By an approximation argument this also holds for all measurable f and by monotonicity it holds for any initial distribution of particles. We may then define $\rho_n(x; t) = \rho(x_1, \dots, x_n; t)$ as the density of the measure $E[\mu_t^n]$, that is the Radon Nicodym derivative with respect to Lebesgue measure, so that

$$\int_{\mathbf{R}^n} f(x) \rho_n(x; t) dx = E \left[\sum_{i_1} \sum_{i_2 \neq i_1} \cdots \sum_{i_n \neq i_1, \dots, i_{n-1}} f(X_t^{i_1}, \dots, X_t^{i_n}) \right]. \quad (\text{B.1})$$

The Lebesgue differentiation theorem implies that $\rho(x; t) = \lim_{\delta \rightarrow 0} \delta^{-n} E[\mu_t^n(x + C_n(\delta))]$ for almost all x . Choosing x_1, \dots, x_n disjoint we obtain

$$\rho(x_1, \dots, x_n; t) = \lim_{\delta \rightarrow 0} \delta^{-n} E [N_t([x_1, x_1 + \delta]) \dots N_t([x_n, x_n + \delta])],$$

where $N_t(A)$ is the number of particles inside A at time t . However we claim we may replace $N_t([x_1, x_1 + \delta])$ by the indicator $\mathbf{I}([x_1, x_1 + \delta] \text{ contains a particle})$ in this limit. The error when doing this occurs if there are two or more particles in $[x_1, x_1 + \delta]$ and is dominated by $E[\mu_t^{n+1}((x_1, x_1, x_2, \dots, x_n) + C_{n+1}(\delta))]$. This error is therefore of order δ^{n+1} , so that replacing $N_t([x_i, x_i + \delta])$ in this way for each $i = 1, \dots, n$ we reach the usual definition of $\rho_n(x; t)$ as, for distinct x_1, \dots, x_n ,

$$\rho(x_1, \dots, x_n; t) = \lim_{\delta \rightarrow 0} \delta^{-n} P[\text{all the intervals } [x_i, x_i + \delta] \text{ for } i = 1, \dots, n \text{ are non-empty}].$$

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