Relevance of Srinivasa Ramanujan at the dawn of the new millennium

K. Srinivasa Rao
The Institute of Mathematical Sciences, Chennai- 600 113.
(e-mail: rao@imsc.ernet.in)

The work of Ramanujan will be appreciated, as long as people do mathematics, opined the Astrophysicist Nobel Laureate Dr. S. Chandrasekhar, at the time of the birth centenary of Ramanujan. Prof. E.H. Neville began a broadcast in Hindustani, in 1941, as follows: *Srinivasa Ramanujan was a mathematician so great that his name transcends jealousies, the one superlatively great mathematician whom India has produced in the last thousand years.* Undoubtedly, Srinivasa Ramanujan (Dec. 22, 1887 – April 26, 1920) is one of the greatest Mathematicians of the twentieth century. For his mathematical abilities and *natural genius* he has been compared, by his contemporaries Professors G.H. Hardy and J.E. Littlewood, with all-time great mathematicians Leonhard Euler, Carl Friedrich Gauss and Karl Gustav Jacobi. Marc Kac said: *An ordinary genius is a fellow that you and I would be just as good as, if we were only many times better. There is no mystery as to how his mind works. Once we understand what he has done, we feel certain that we, too, could have done it. It is different with the magicians ······ the working of their minds is for all intents and purposes incomprehensible. Even after we understand what they have done, the process by which they have done it is completely dark.* Prof. Bruce C. Berndt who has methodically and thoroughly edited every one of the 3254 entries of Ramanujan in his three Notebooks – in *Ramanujan's Notebooks*, Parts I to V, published by Springer-Verlag (1985 - 1997) – states that though there are a few scattered errors in these notebooks, *Ramanujan's accuracy is amazing* and *mystery ··· still surrounds some of his work.* This sums up the kind of mathematical genius that Ramanujan was. The original notebooks of Ramanujan are now with the University of Madras.

Due to the orchestrated efforts of Ramanujan's friends and admirers – Professor V. Ramaswamy Iyer, Dewan Bahadur S. Ramachandra Rao, Mr. S. Narayana Iyer, Prof. P.V. Seshu Iyer, Sir Francis Spring, Prof. G.T. Walker, Prof. E.H. Neville – Ramanujan first got the post of a clerk in the Accountant General's office in Madras, in 1912. A month later, he secured a clerical post (class III, Grade IV) in the Account's section of the Madras Port Trust, before recognition came to him in the form of the *first research scholarship* of the University of Madras in May 1913, even though he did not have the academic degrees considered essential for research. Then due to Ramanujan's initiative – his historic first letter to Prof. Hardy – came the recognition of his talent and the invitation to go to Cambridge from Prof. Hardy, in 1914, with financial support from the University of Madras.

The authorities of the University of Madras have to be commended for their interest in nurturing Ramanujan's mathematical talent by granting him not only the first research scholarship in Mathematics of the University, for two years; but also offering him a scholarship, of £250 per year, to go to Trinity College, Cambridge, along with an adequate allowance (of £100) for his passage by ship and initial outfit; extending his scholarship until his return to India in March 1919 – on the basis of Ramanujan's prolific research output from Cambridge endorsed and strongly recommended by Prof. Hardy; and offering him an allowance, on his return, to enable him to continue his research work, until his untimely demise.

Ramanujan was awarded the B.A. degree by research, in March 1916, for his work on *Highly Composite Numbers*, his longest paper (62 pages) in the Journal of the London Mathematical Society. He was the first Indian Mathematician to be awarded the prestigious Fellowship of the Royal Society, in Feb. 1918. The citation read: *Research Student in Mathematics, Distinguished as a pure mathematician particularly for his investigations in elliptic functions and the theory of numbers.* Ramanujan was elected to a Trinity College Fellowship, in October 1918 – a Prize fellowship worth £250 a year for six years (which he was not destined to enjoy, since he returned to India in March 1919 and died in April 1920).

The publication of the *Collected Papers of Srinivasa Ramanujan* (Edited by P.V. Seshu Iyer, B.M. Wilson and G.H. Hardy, for the Cambridge University Press), in 1927 and Hardy's book entitled: *Ramanujan: Twelve lectures on subjects suggested by his life and work* (1940), created a flurry of publications by contemporary mathematicians – Professors G.N. Watson, W.N. Bailey, C.T. Preece and others.

In 1923, Prof. Hardy spent a few months in editing a chapter in the Notebooks of Ramanujan, on hypergeometric series, and found it so daunting a task that he felt that he would not be able to do any other research work if he continued to look into these Notebooks. He persuaded the University of Madras to take up this task and in 1931, Prof. G.N. Watson agreed to edit the notebooks in collaboration with Prof. B.M. Wilson. Unfortunately, the untimely death of Wilson, in 1935, at the age of 38, put an end to this effort. Prof. Watson, however, lectured on Ramanujan's Notebooks at the London Mathematical Society, in 1931 and his Presidential Address to the Royal Society, in 1935, was on 'mock' theta functions invented by Ramanujan. Twenty-two years later, in 1957, due to the efforts of Professors S.R. Ranganathan and K.S. Krishnan, Homi J. Bhabha and K. Chandrasekharan, the Tata Institute of Fundamental Research brought out a facsimile edition of these notebooks, without any commentary. This resulted in a spurt of activity on Ramanujan's entries in the Notebooks.

The discovery of the so called *'Lost' Notebook of Ramanujan* by Prof. George E. Andrews, in the spring of 1976, in the estate of late Prof. G.N. Watson, contributed to a resurgence of interest in the life and work of Ramanujan. This 'Lost' Notebook contained some 600 theorems on what Ramanujan called as 'mock' theta functions. These are results he noted on about 100 loose sheets of paper, during the last year of his life, after his return to India, in March 1919. Prof. Berndt and Prof. Andrews are at present editing this 'Lost' Notebook. A facsimile edition of the 'Lost' Notebook of Ramanujan was released by the then Prime Minister of India, Mr. Rajiv Gandhi, at an International Conference in Chennai, organized by the Institute of Mathematical Sciences to mark the birth centenary celebrations of Ramanujan, on Dec. 22, 1987. This volume contains an article by Prof. Andrews about the genesis of this 'Lost' Notebook and includes some of the Unpublished papers of Ramanujan.

It is a remarkable fact that hundreds of papers have been inspired by Ramanujan's entries in his Notebooks and his *Collected Papers*. Furthermore, Ramanujan's name has appeared in the titles and abstracts of innumerable research papers and this is continuing unabated at the dawn of the 21st century.

It is also significant to note that today there are three journals named after Srinivasa Ramanujan and these are: **The Hardy - Ramanujan Journal** (since 1975); the **Journal of the Ramanujan Mathematical Society** (since 1985); and **Ramanujan Journal** (since 1997). This is a tribute befitting the greatest.

The following is an assessment of Ramanujan, the mathematician: "Paul Erdös, a renowned Hungarian mathematician, has passed on to us Hardy's personal ratings of mathematicians: Suppose that we rate mathematicians on the basis of pure talent on a scale from 0 to 100, Hardy gave himself a score of 25, Littlewood 30, Hilbert 80, and Ramanujan 100." (Prof. Berndt, 1985).

The *Collected Papers* and Hardy's *Ramanujan*, referred above, are two books which have been reprinted by the American Mathematical Society, in 1999, with editorial comments and notes by Prof. Bruce C. Berndt. The National Board for Higher Mathematics has also released, in March 2000, an Indian edition of the *Collected Papers*, albeit without the two interesting biographical articles which were an integral part of the first edition.

A Ramanujan Museum was created in 1993 in the premises of the Avvai Academy in Royapuram by the dedicated Mathematics School Teacher Mr. P.K. Srinivasan with the help of Mr. A.T.B. Bose. In 1967, Mr. P.K. Srinivasan brought out two volumes from the Muthialpet High School to commemorate Ramanujan's 75th Birth Anniversary. He was also responsible for bringing out three booklets entitled *Creativity of Ramanujan* for Primary, Middle and Higher Secondary Schools, published by the Association for Mathematics Teachers of India. In these some of the simple entries of Ramanujan are reproduced with explanations.

The Taxi cab number anecdote, in which Hardy started a conversation at a nursing home where Ramanujan was with the statement that the number 1729 was rather a dull number, to which Ramanujan immediately reacted by saying *No. It is the smallest number expressible as the sum of two cubes in two different ways*:

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

As a sequel Ramanujan was asked by Hardy what was the smallest number that could be expressed as the sum of two fourth powers in two different ways and Ramanujan after some thought is reported to have told Hardy that it should be a large number. Hardy later recalled that:

$$59^4 + 158^4 = 133^4 + 134^4 = 635318657$$

is the smallest number for the fourth power problem, a solution which was known to Euler. Ramanujan's quick response to the Taxi cab number is attributable to the fact that in three questions he posed to the Journal of Indian Mathematical Society (JIMS) and in two entries in his Notebooks, reference has been made to the Euler equation $X^3 + Y^3 + Z^3 = U^3$ or its solutions, though the number 1729 itself does not find an explicit mention, an entry on p.225, Chapter XVIII, of his second Notebook reads: $1^3 + 12^3 = 9^3 + 10^3$. This could be an intersting way of introducing the subject of Diophantine equations in Higher Secondary Schools and Colleges.

The following are a few examples out of the 59 questions posed by Ramanujan, in the JIMS, which could be part of the curriculum in the appropriate stages of the mathematics education of our country:

- Q. 289, JIMS vol. IV, p.226: Find the values of:

$$(i) \qquad \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + \cdots}}}}}$$

$$(ii) \quad \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + 4\sqrt{9 + 5\sqrt{10 + \cdots}}}}}$$

Persuaded by his friends, Ramanujan provided the answer to this question himself:
(i) Let $f(n) = n(n+2)$, then

$$\begin{aligned} f(n) &= n\sqrt{(n+2)^2} \\ &= n\sqrt{n^2 + 4n + 4} \\ &= n\sqrt{1 + (n+1)(n+3)} \\ &= n\sqrt{1 + f(n+1)}. \end{aligned}$$

By recursive use of this equation, it is straightforward to show:

$$n(n+2) = n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\sqrt{1+(n+4)\sqrt{1+\cdots}}}}}$$

and in this if we set $n = 1$, we get the desired result that (i) in Q. 289 is nothing but 3 represented in the nested root form!
(ii) Let $f(n) = n(n+3)$, then

$$\begin{aligned} f(n) &= n\sqrt{(n+3)^2} \\ &= n\sqrt{n^2+6n+9} \\ &= n\sqrt{(n+5)+(n+1)(n+4)} \\ &= n\sqrt{(n+5)+f(n+1)}. \end{aligned}$$

By recursive use of this equation, it is straightforward to show:

$$f(n) = n\sqrt{(n+5)+(n+1)\sqrt{(n+6)+(n+2)\sqrt{(n+7)+(n+3)\sqrt{(n+8)+\cdots}}}}$$

and in this if we set $n = 1$, we get the desired result that (ii) in Q. 289 is nothing but 4 represented in the nested root form! In this elementary mathematics example, one can see how Ramanujan should have arrived at the question itself:

$$3 = \sqrt{9} = \sqrt{1+8} = \sqrt{1+2*4} = \sqrt{1+2\sqrt{1+15}} = \sqrt{1+2\sqrt{1+3*5}}$$

$$= \sqrt{1+2\sqrt{1+3\sqrt{25}}} = \sqrt{1+2\sqrt{1+3\sqrt{1+24}}} = \sqrt{1+2\sqrt{1+3\sqrt{1+4*6}}}$$

$$= \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{36}}}} = \cdots = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\cdots}}}}}$$

$$4 = \sqrt{16} = \sqrt{6+10} = \sqrt{6+2*5} = \sqrt{6+2\sqrt{25}} = \sqrt{6+2\sqrt{7+3*6}}$$

$$= \sqrt{6+2\sqrt{7+3\sqrt{36}}} = \sqrt{6+2\sqrt{7+3\sqrt{8+28}}} = \sqrt{6+2\sqrt{7+3\sqrt{8+4*7}}}$$

$$= \sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{49}}}} = \cdots = \sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+5\sqrt{10+\cdots}}}}}$$

It is plausible that having arrived at these exotic representations for 3 and 4, like a detective story writer, he posed the question, knowing its answer before hand! In his Notebooks he however noted down the more general result (p.139, Entry 4 of the second Notebook of Ramanujan):

$$x + n + a = \sqrt{(ax + (n+a)^2 + x\sqrt{(a(x+n) + (n+a)^2 + (x+n)\sqrt{etc.}}}$$

from which by putting $x = 2, n = 1$ and $a = 0$ or 1 give the results (i) or (ii), respectively. Finally, it may be conjectured that this entry in his notebook was at about the time when his classmate and friend C.V. Rajagopalachari posed the question: if $\sqrt{x} + y = 7$ and $x + \sqrt{y} = 11$, what are the values of x and y? For which, Ramanujan shot back the answer $x = 9$ and $y = 4$.

• Q. 464: $2^n - 7$ is a perfect square for the values 3,4,5,7,15 of n. Find other values.

That the Diophantine equation

$$x^2 + 7 = 2^n$$

has only 5 solutions was proved by Nagell in 1940. A generalization of this equation:

$$x^2 + D = p^n$$

is called the Ramanujan-Nagell equation. Prof. T.N. Shorey has shown that:

$$x^2 + 7 = 4y^n$$

has no solutions for $x \geq 1,\ y > 2,\ n > 1$.

• Q. 469: 1+n! is a perfect square for values 4,5,7 of n. Find other values.

The Diophantine equation:

$$1 + n! = m^2$$

is called the Brocard-Ramanujan equation in literature today. Prof. H. Gupta (1935) has shown that there are no solutions to this equation for $8 \leq n \leq 63$ and more recently (2000) Prof. Berndt and W. Galway showed that there are no solutions up to $n = 10^9$. It is yet to be proved that the equation has only three solutions!

It will remain a mystery whether Ramanujan had proofs that there were no more solutions to the above two Diophantine equations (in Q. 464 and 469) and he still posed them as questions, in his inimitable style! In the case of the celebrated Rogers-Ramanujan identities, which were also posed by Ramanujan as Q.584 in the JIMS, Ramanujan had no proof but when asked by Prof. Hardy he came up with a proof of the identities.

It is surprising that no serious effort has been made by the various curriculum development authorities for mathematics education in our country to include in the school/college syllabi some aspects of the mathematics of Ramanujan. Perhaps this will happen, at least now, in the new millennium, since Ramanujan's work, his conjectures, his questions in the JIMS and his tantalisingly recorded results in his Notebooks have been a source of inspiration and stimulated the research work of generations of mathematicians the world over. Prof. Atle Selberg, in 1988, at an extempore lecture at the Tata Institute of Fundamental Research observed that *a felicitous but unproved conjecture may be of much more consequence than the proof of many a respectable theorem.* The many conjectures of Ramanujan opened up new ares of research in mathematics.

A πie Pavilion – highlighting briefly the contributions of the Indian mathematicians, Euler, Gauss, Jacobi and the several formulae of Ramanujan including his 17 infinite series representations for π – and a replica of the Ramanujan Museum at Royapuram were designed and created for the Indian Science Congress Exhibition, at Chennai in Jan. 1999, by the author. Subsequently, steps were taken to house the exhibitions with some modifications in the Periyar Science and Technology Center (PSTC), which was opened for the public on Feb. 28, 1999. With additional funds from the Department of Science and Technology, Government of India, a *Ramanujan Photo Gallery* was added in June 1999, in the PSTC.

A Pilot CD-ROM Project on the Life and Work of Srinivasa Ramanujan created by the author for the Indian Science Congress Exhibition in 1999, held in Chennai, has been the precursor to a full length Project proposed by the Institute of Mathematical Sciences and the National Multimedia Research Center of C-DAC, Pune. This has been approved by the Department of Science and Technology, Government of India, and this project is expected to be completed in 2002.

It is heartening to note that through a new initiative of Dr. Pon Kothandaraman, Vice Chancellor, University of Madras, the Ramanujan Institute and the Institute of Mathematical Sciences are planning an appropriate Museum to house the original Notebooks of Ramanujan and a few of the memorabilia connected with his life – such as his passport, the slate used by him, some of the original letters written by him or received by him. This live Museum is expected to stimulate the interest of generations of mathematicians, students and the public to the life and work of Ramanujan.

Freeman J. Dyson, the renowned Physicist concludes his article entitled: *A Walk through Ramanujan's Garden* – in *Ramanujan Revisited*, Proceedings of the Ramanujan Centenary conference, in 1987, at University of Illinois – with the following advice:

> *In conclusion, I would like to urge all of you who are working in the many fields of mathematics which have been enriched by Ra-*

manujan's ideas to go back to the source the collected papers and the notebooks. ... The notebooks ... are now appearing in a splendidly annotated version edited by Bruce Berndt. The "lost" notebook is now accessible to us through the devoted labors of George Andrews. When I started my walk through Ramanujan's garden 47 years ago, only the collected papers were available. A year after I chose Hardy and Wright's "Theory of Numbers" (Oxford, Clarendon Press, 1938) as a school prize, I won another prize. For the second prize I chose Ramanujan's collected papers. The collected papers have traveled with me from England to America and are still as fresh to-day as they were in 1940. Whenever I am angry or depressed, I pull down the collected papers from the shelf and take a quiet stroll in Ramanujan's garden. I recommend this therapy to all of you who suffer from headaches or jangled nerves. And Ramanujan's papers are not only a good therapy for headaches. They also are full of beautiful ideas which may help you to do more interesting mathematics.

Certainly several of us will not find this therapy quite as effective as it was for Dyson! Dyson, Erdös and Selberg were among the very best in their chosen fields and were greatly benefited by an early introduction to the *Collected Papers* of Ramanujan. However, it is not an exaggeration to say that the prolific creative work of the mathematical genius Ramanujan contained in his *Collected Papers*, in his original Notebooks and the 'Lost' Notebook, the study and lectures on Ramanujan's work by Prof. Hardy and the exhaustive editing of Ramanujan's Notebooks by Prof. Berndt, will be be the sources for inspiring many more generations of students of mathematics in the new millennium.

Acknowledgement: I wish to thank Prof. Ashok K. Agarwal for inviting me to write this article, in lieu of the actual contribution I made at the Chandigarh conference on my work on *Multiplicative Diophantine Equations*.