Prof. R.P. Agarwal

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KSR: Professor Agarwal you are the person who is considered as the Patron of the Society for Special Functions and their Applications in India. And ever since you finished your second Ph.D. with Professor W.N. Bailey and returned to India, I think you have been the leader to this group of people. You have been also working on Ramanujan and Ramanujan's Note Books. You have a long record and your recent three volume publication on Resonance of Ramanujan's Mathematics clearly shows that you have made a break through about the mock theta functions of order 3, 5 and 7, which were mentioned by Ramanujan in his last letter to Hardy, written in January of 1920. I would like you to first tell us how you got interested in Ramanujan's work and when?

RPA: Well, in fact, I got interested in Ramanujan's work right when I went to England. Prof. Bailey was already interested in Ramanujan's work. Probably, you will be interested to know how Bailey knew Ramanujan. Bailey was an undergraduate student at Cambridge when Ramanujan went there. And they became rather friends and Watson was a post graduate student there. Watson was senior to Bailey by two years. And Bailey was interested in Ramanujan's work. When I went to work with him at London he asked me whether I would like to work in self reciprocal functions and Hankel transforms, which was the work I did for my first Ph.D. in India, or I would like to work on generalized hypergeometric series. Well, naturally, I gave him a reply that "with you I would like to work with on generalized hypergeometric series". And he gave me certain papers of Ramanujan and particularly, those which were connected with elliptic functions and partitions. I quickly recognized them and I said that one side is, of course, elliptic functions and the other side partition functions. And there I got interested. In fact, the first paper that I did at London was on partial sums of hypergeometric series. And that was one of the results that Ramanujan had given connecting an infinite series with a finite series and of course that was a particular case of a certain hypergeometric transformation and I gave the most general transformation of that type which is known up to this day. It has not been further generalized b anybody this is how I got interested in Ramanujan's work. Then, I did some other work also along with Ramanujan's work. When I came back to India, then, I got really interested. But my real interest in Ramanujan's work and hypergeometric series began in 1967, when I went to States in the Pennsylvania State University as a Fulbright professor. There some work of Andrews was published and I saw that work.

And I quickly saw that it has lot of potentiality in basic hypergeometric series and I gave the most general transformation of the basic hypergeometric series which was connected with that. I sent a copy of that to George Andrews and it was then that we became friends – that was my first acquaintance with George Andrews. And then began our journey through Ramanujan's work with him. We exchanged certain letters and he said that: "yes indeed defuse and write Ramanujan's work in the most general hypergeometric series setting. Then it can throw better light on the intrinsic value of that result and one could find more results of that type". Then again, well I came back from there and one of my students wanted to work on Ramanjan's work and it was then that I got more interested in Ramanujna's work. We got a copy of the first and second Notebook. We began to obtain certain hypergeometric series results and then again, in 1979, when Ramanujan's 'lost' Notebook – the so called 'lost' Notebook – was found by George Andrews, when he went to Cambridge as a Fulbright fellow, I saw one of his papers, or, rather, Andrews showed me one of his papers on partial theta functions. Well, you know, lady luck played an important role in this subject and I saw that paper. It was a long paper published in the Advances of Mathematics, in 1981, I told Andrews that this is a three line result and a known result given by Sears, in 1951. And quick was his response that how come it can be obtained from that and we did obtain those results. And, in fact, the attempt of Sears through the Sears transformation made me tell him that we could get 110 more results of this type. One has to sit and find it out and see how many of them are different from each other; how many agree with each other, and, in fact, when I came back then one of my students submitted a Ph.D. thesis in which he found out all the 110 reults and showed that five or six of them are new and the others are just paraphrasing of one or the other of results. There was another result on partial theta functions, which Andrews gave in that paper and I said that, that is not a straight forward result, that is a mixture of two results, at least: $a = b, b = c$ and we have to find b. And indeed one of my students, Arun Varma who is a Professor of Mathematics in the University of Roorkee now, he was able to find out the solution in a few months. And there it was. Now the hypergeometric approach, rather, the basic hypergeometric approach, then proved to me to be more effective tool rather than finding these results of Ramanujan by means of scattered results or giving certain remarks and then finding it out. We began applying basic hypergeometric series to find out those results. This is how we got interested in basic hypergeometric series and Ramanujan's work.

KSR: I would like to ask you now about your recent work in the years 1996 to 1999. You have published a three part work on "Resonance of Ramanujan's Mathematics". This is mostly about the work of Ramanujan contained in the 'lost' Notebook, or, does it also include the work contained in other Notebooks ?

RPA: No. It contains the work in other Notebooks also. But the work of Ramanujan on the mock theta functions which was found in the 'lost' Notebook forms the part 3 of that volume. The other volumes contain other results also which were given in his first and second Notebooks. In fact, you know that the second Notebook is a better form of the first Notebook and therefore, we only talk of the second Notebook which is written in a more orderly form. So, these 3 books that I have published are some work of the second Notebook and other work of the 'lost' Notebook, particularly on mock theta functions and continued fractions.

KSR: One question which I would like to ask you because you have been in this country for so many years and you have been a teacher *par excellence* and you rose to become the Vice Chancellor of the Luckow University and also the Vice Chancellor of the Rajasthan University. If there is a revision which is to be made in syllabi, at what level can we include some of the work of Ramanujan ? And do you think it is something which should be done ? Why is it that it has not be done, if you think it should have been done ?

RPA: Well, I think we could include Ramanujan's work at the under graduate level itself, through the solutions of differential equations. We have to raise our standards in teaching differential equations, particularly solution in series, and then through those results in the post graduate classes we could introduce more of q-series and through q-series then obtain results say of the type Bailey obtained in his book, or, Slater has obtained in her book, or, some of these you find also in my monograph, published by Asia Publishing House and once you do that,then our post graduate student would be ready to take up and understand any one of the research papers that are published on Ramanujan's work, or, for that matter, on generalized hypergeometric series also.

KSR: In your work on mock theta functions, which is the last work which Ramanujan left as a legacy for mathematicians, you pointed out, you have obtained, some very beautiful results which relate, which are regarding the order of the mock theta functions and, in fact, it is also known, that Ramanujan discovered several of them of order 3, 5 and 7 and in more recent years, in the 1990's, something like about 70 years after he had done this work, people have discovered mock theta functions of even order like order 6 or order 10. Would you like to tell us about this contribution ?

RPA: Well, in fact, when Ramanujan in his last letter reported these 17 functions, some of order 3, 5 and 7, he made certain statements. He gave 7 functions of order 3, and 10 functions of order 5, and 3 functions of order 7, and he said that in the 5th order functions there are two groups containing 5 each and they are not related to each other. This is what he said. Now, as early as 1924 or 23, Selberg also tried to a give definition for order. Then, later on, much later, Dragnet and Rademacher tried to give a meaning to order for the function. Watson, of course, disposed of by saying that it is easy to see that they are of order this and added 3 more functions of order 3. So there were 20 known to us. And relations between them were given by Ramanujan. But then the question arose why after all they are called of order 3, 5 or 7 ? Ramanujan made one more statement that there is no general theory for the 5th order transformation theory, for the 5th order functions and the 7th order functions. Why ? Well, no reply ? Now, then, I thought in the 90's that there should be a plausible solution to this and this problem should not vex youngsters and we, analysts and number theorists, should once for all decide why should these be called of order 3, 5 and 7. And also, try to give an answer to the discussion why there is no transformation theory. Now the seeds of this where sown earlier by Watson himself in 1935 and 36, when he proved these identies given by Ramanujan through hypergeometric means, basic hypergeometric, I mean, and then later on N. J. Fine also showed that the three third order functions are all representable by a ${}_2\phi_1$. Now, this was enough hint to try whether 5th order ones can be representented by ${}_3\phi_2$s and the 7th order ones can be represented by ${}_4\phi_3$s. And indeed to my surprise the 5th order ones were all limiting case of ${}_3\phi_2$s and not of a lower degree function and 7th order ones were representable as limiting cases of ${}_4\phi_3$s. Now this was enough for me to conclude that if there are basic hypergeometric functions of order $2r+1, 2r$ and then take a limit of these functions, then the mock theta function is of order $4r+1$. Now the question arose whether the conjecture of Ramanujan that there is no transformation theory for the 5th and 7th order functions was also explained by this, because we have got a full transformation theory known for a ${}_2\phi_1$ with a general argument, the Sears' work. But, there is no general theory for the transformations between two ${}_3\phi_2$s with a general argument, or between ${}_4\phi_3$s with a general argument. So, that gave a complete answer to the querries raised by Ramanujan, or, conjectures raised by Ramanujan. But when came the 6th order functions, there was difficulty. And just trying to explain the 6th order functions by means of our conjecture, that this should be the definition, we found that they are not of order of 6 but they are of order 3, or, of order 5. Again on the same standards that they were limiting cases of either of a ${}_2\phi_1$, or, limiting cases of irreducible ${}_3\phi_2$s. Of course, the limit could be taken of a number of these. So, this is how I gave these definitions. Now, recently, very recently, in 1991 came the paper of George Andrews and the person who gave the 6th order functions. Very lately, in 1999, came a paper by Chou, a Korean and he described some 10th order functions. Now this raised our eye-brows again! When you go on describing functions when you know that the mock theta functions do exist or not, we still do not know, and we go on describing and enunciating that this function is of order 10, or, order 6. Why should we do that? There must be some logic behind it and we found in some private correspondence with them that they have named them, or, classified them as of order 6, or 10, on some combinatorial basis. But, remember Ramanujan never said that and these identities are found in the 'lost' notebook themselves. If Ramanujan had given these identities, he would have certainly given much earlier than prior to his death! Because, his last letter only enunciates only 3rd, 5th and 7th order functions. So, those identities must have been given earlier. If Ramanujan had any intention of calling them as of 6th order, or, of 10th order, he should have mentioned it in the letter to Hardy. But, he did not. I recently saw on the internet that somebody has defined functions, mock theta functions, of order 8. Well, this is making, to my mind, a mockery of the entire thing. We should come out and try to tell the posterity whether we have to use the combinatorial basis or the hypergeometric basis. This must be decided by the analysts and the number theorists once for all. So that, each one of us who works knows what he is doing. Recently, one of my students has tried the 6th order functions. He has shown, to my utter surprise, that the 6th order functions are nothing but combinations of ordinary ${}_3\phi_2$s and ${}_2\phi_1$s. Not even with quadratic terms. The ordinary ${}_3\phi_2$s and ${}_2\phi_1$s. So, if we tried find sum up ${}_3\phi_2$s and ${}_2\phi_1$s, the combinations of them and then prove the result, probably it may be a more direct method and a more logical method of doing that. Then, this will also confirm that Ramanujan had no intention of calling them as of order 6, or, order 10, or, for that purpose, as order 8!