we can prove that solutions of this equation have the foregoing properties, provided

$$\sigma - \sum_{1}^{n-1} \rho_j + n > 0,$$

$$\sigma + 1 + \sum_{n-k}^{n-1} (1 - \rho_j) + m \sum_{1}^{n-1-k} (1 - \rho_j) > 0, \quad (k = 0, 1, 2, \ldots, n-2).$$

These all reduce to the single condition

$$\sigma - \sum_{1}^{n-1} \rho_j + n > 0$$

when $m = 1$. It is clear that all the conditions are satisfied if

$$\sigma + 1 > 0; \quad 1 - \rho_k > 0, \quad (k = 1, 2, \ldots, n-1).$$

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# MODULAR EQUATIONS AND APPROXIMATIONS TO $\pi$.

By S. Ramanujan.

1. IF we suppose that

(1) $(1 + e^{-\pi\sqrt{n}})(1 + e^{-3\pi\sqrt{n}})(1 + e^{-5\pi\sqrt{n}}) \ldots = 2^{\frac{1}{4}} e^{-\pi\sqrt{n}/24} G_n$

and

(2) $(1 - e^{-\pi\sqrt{n}})(1 - e^{-3\pi\sqrt{n}})(1 - e^{-5\pi\sqrt{n}}) \ldots = 2^{\frac{1}{4}} e^{-\pi\sqrt{n}/24} g_n$,

then $G_n$ and $g_n$ can always be expressed as roots of algebraical equations when $n$ is any rational number. For we know that

(3) $(1 + q)(1 + q^3)(1 + q^5) \ldots = 2^{\frac{1}{6}} q^{\frac{1}{24}} (kk')^{-\frac{1}{12}}$

and

(4) $(1 - q)(1 - q^3)(1 - q^5) \ldots = 2^{\frac{1}{6}} q^{\frac{1}{24}} k^{-\frac{1}{12}} k'^{\frac{1}{6}}$.

Now the relation between the moduli $k$ and $l$, which makes

$$n \frac{K'}{K} = \frac{L'}{L},$$

where $n = r/s$, $r$ and $s$ being positive integers, is expressed by the modular equation of the $rs^{\text{th}}$ degree. If we suppose that



$k = l'$, $k' = l$, so that $K = L'$, $K' = L$, then

$$q = e^{-\pi L'/L} = e^{-\pi\sqrt{n}},$$

and the corresponding value of $k$ may be found by the solution of an algebraical equation.

From (1), (2), (3), and (4) it may easily be deduced that

(5) $$g_{4n} = 2^{\frac{1}{4}} g_n G_n,$$

(6) $$G_n = G_{1/n}, \quad 1/g_n = g_{4/n},$$

(7) $$(g_n G_n)^8 (G_n^8 - g_n^8) = \tfrac{1}{4}.$$

I shall consider only integral values of $n$. It follows from (7) that we need only consider one of $G_n$ or $g_n$ for any given value of $n$; and from (5) that we may suppose $n$ not divisible by 4. It is most convenient to consider $g_n$ when $n$ is even, and $G_n$ when $n$ is odd.

2. Suppose then that $n$ is odd. The values of $G_n$ and $g_{2n}$ are got from the same modular equation. For example, let us take the modular equation of the 5[th] degree, viz.,

(8) $$\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3 = 2\left(u^2v^2 - \frac{1}{u^2v^2}\right),$$

where $2^{\frac{1}{4}} q^{\frac{1}{24}} u = (1 + q)(1 + q^3)(1 + q^5) \ldots$

and $2^{\frac{1}{4}} q^{\frac{5}{24}} v = (1 + q^5)(1 + q^{15})(1 + q^{25}) \ldots$.

By changing $q$ to $-q$ the above equation may also be written as

(9) $$\left(\frac{v}{u}\right)^3 - \left(\frac{u}{v}\right)^3 = 2\left(u^2v^2 + \frac{1}{u^2v^2}\right),$$

where $2^{\frac{1}{4}} q^{\frac{1}{24}} u = (1 - q)(1 - q^3)(1 - q^5) \ldots$

and $2^{\frac{1}{4}} q^{\frac{5}{24}} v = (1 - q^5)(1 - q^{15})(1 - q^{25}) \ldots$.

If we put $q = e^{-\pi/\sqrt{5}}$ in (8), so that $u = G_{1/5}$ and $v = G_5$, and hence $u = v$, we see that

$$v^4 - v^{-4} = 1.$$

Hence $v^4 = \dfrac{1 + \sqrt{5}}{2}$, $\quad G_5 = \left(\dfrac{1 + \sqrt{5}}{2}\right)^{\frac{1}{4}}$.

Similarly, by putting $q = e^{-\pi\sqrt{(2/5)}}$, so that $u = g_{2/5}$ and $v = g_{10}$, and hence $u = 1/v$, we see that

$$v^6 - v^{-6} = 4.$$

Hence $$v^2 = \frac{1+\sqrt{5}}{2}, \quad g_{10} = \sqrt{\left(\frac{1+\sqrt{5}}{2}\right)}.$$

Similarly it can be shown that

$$G_9 = \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{\frac13}, \quad g_{18} = (\sqrt{2}+\sqrt{3})^{\frac13},$$

$$G_{17} = \sqrt{\left(\frac{5+\sqrt{17}}{8}\right)} + \sqrt{\left(\frac{\sqrt{17}-3}{8}\right)},$$

$$g_{34} = \sqrt{\left(\frac{7+\sqrt{17}}{8}\right)} + \sqrt{\left(\frac{\sqrt{17}-1}{8}\right)},$$

and so on.

3. In order to obtain approximations for $\pi$ we take logarithms of (1) and (2). Thus

$$(10) \qquad \pi = \frac{24}{\sqrt{n}} \log(2^{\frac14} G_n),$$

$$\pi = \frac{24}{\sqrt{n}} \log(2^{\frac14} g_n),$$

approximately, the error being nearly $\frac{24}{\sqrt{n}} e^{-\pi\sqrt{n}}$ in both cases. These equations may also be written as

$$(11) \qquad e^{\pi\sqrt{n}/24} = 2^{\frac14} G_n, \quad e^{\pi\sqrt{n}/24} = 2^{\frac14} g_n.$$

In those cases in which $G_n^{12}$ and $g_n^{12}$ are simple quadratic surds we may use the forms

$$(G_n^{12} + G_n^{-12})^{\frac{1}{12}}, \quad (g_n^{12} + g_n^{-12})^{\frac{1}{12}},$$

instead of $G_n$ and $g_n$; for we have

$$g_n^{12} = \tfrac18 e^{\frac12\pi\sqrt{n}} - \tfrac32 e^{-\frac12\pi\sqrt{n}},$$

approximately, and so

$$g_n^{12} + g_n^{-12} = \tfrac18 e^{\frac12\pi\sqrt{n}} + \tfrac{13}{2} e^{-\frac12\pi\sqrt{n}},$$



approximately, so that

$$(12) \qquad \pi = \frac{2}{\sqrt{n}} \log\{8(g_n^{12} + g_n^{-12})\},$$

the error being about $\frac{104}{\sqrt{n}} e^{-\pi\sqrt{n}}$, which is of the same order as the error in the formulæ (10). The formula (12) often leads to simpler results. Thus the formula (10) gives

$$e^{\pi\sqrt{18}/24} = 2^{\frac14} g_{18}$$

or $$e^{\frac14\pi\sqrt{18}} = 10\sqrt{2} + 8\sqrt{3}.$$

But if we use the formula (12), or

$$e^{\pi\sqrt{n}/24} = 2^{\frac14}(g_n^{12} + g_n^{-12})^{\frac{1}{12}},$$

we get a simpler form, viz.,

$$e^{\frac18\pi\sqrt{18}} = 2\sqrt{7}.$$

4. The values of $g_{2n}$ and $G_n$ are obtained from the same equation. The approximation by means of $g_{2n}$ is preferable to that by $G_n$ for the following reasons.

(a) It is more accurate. Thus the error when we use $G_{65}$ contains a factor $e^{-\pi\sqrt{65}}$, whereas that when we use $g_{130}$ contains a factor $e^{-\pi\sqrt{130}}$.

(b) For many values of $n$, $g_{2n}$ is simpler in form than $G_n$; thus

$$g_{130} = \sqrt{\left\{(2+\sqrt{5})\left(\frac{3+\sqrt{13}}{2}\right)\right\}},$$

while

$$G_{65} = \left\{\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{13}}{2}\right)\right\}^{1/4} \times \sqrt{\left\{\sqrt{\left(\frac{9+\sqrt{65}}{8}\right)} + \sqrt{\left(\frac{1+\sqrt{65}}{8}\right)}\right\}}.$$

(c) For many values of $n$, $g_{2n}$ involves quadratic surds only, even when $G_n$ is a root of an equation of higher order. Thus $G_{23}$, $G_{29}$, $G_{31}$ are roots of cubic equations, $G_{47}$, $G_{79}$ are those of quintic equations, and $G_{71}$ is that of a septic equation, while $g_{46}$, $g_{58}$, $g_{62}$, $g_{94}$, $g_{142}$, and $g_{158}$ are all expressible by quadratic surds.

5. Since $G_n$ and $g_n$ can be expressed as roots of algebraical equations with rational coefficients, the same is true of $G_n^{24}$ or $g_n^{24}$. So let us suppose that

$$1 = a g_n^{-24} - b g_n^{-48} + \dots,$$

or

$$g_n^{24} = a - b g_n^{-24} + \dots.$$

But we know that

$$64 e^{-\pi\sqrt{n}} g_n^{24} = 1 - 24 e^{-\pi\sqrt{n}} + 276 e^{-2\pi\sqrt{n}} - \dots,$$

$$64 g_n^{24} = e^{\pi\sqrt{n}} - 24 + 276 e^{-\pi\sqrt{n}} - \dots,$$

$$64a - 64 b g_n^{-24} + \dots = e^{\pi\sqrt{n}} - 24 + 276 e^{-\pi\sqrt{n}} - \dots,$$

$$64a - 4096 b e^{-\pi\sqrt{n}} + \dots = e^{\pi\sqrt{n}} - 24 + 276 e^{-\pi\sqrt{n}} - \dots,$$

that is

$$(13) \quad e^{\pi\sqrt{n}} = (64a + 24) - (4096b + 276) e^{-\pi\sqrt{n}} + \dots.$$

Similarly, if

$$1 = a G_n^{-24} - b G_n^{-48} + \dots,$$

then

$$(14) \quad e^{\pi\sqrt{n}} = (64a - 24) - (4096b + 276) e^{-\pi\sqrt{n}} + \dots.$$

From (13) and (14) we can find whether $e^{\pi\sqrt{n}}$ is very nearly an integer for given values of $n$, and ascertain also the number of 9's or 0's in the decimal part. But if $G_n$ and $g_n$ be simple quadratic surds we may work independently as follows. We have for example

$$g_{22} = \sqrt{(1 + \sqrt{2})}.$$

Hence

$$64 g_{22}^{24} = e^{\pi\sqrt{22}} - 24 + 276 e^{-\pi\sqrt{22}} - \dots,$$

$$64 g_{22}^{-24} = 4096 e^{-\pi\sqrt{22}} + \dots.$$

so that

$$64 (g_{22}^{24} + g_{22}^{-24}) = e^{\pi\sqrt{22}} - 24 + 4372 e^{-\pi\sqrt{22}} + \dots$$
$$= 64 \{(1 + \sqrt{2})^{12} + (1 - \sqrt{2})^{12}\}.$$

Hence

$$e^{\pi\sqrt{22}} = 2508951.9982\dots.$$

Again

$$G^{37} = (6 + \sqrt{37})^{\frac{1}{4}},$$

$$64 G_{37}^{24} = e^{\pi\sqrt{37}} + 24 + 276 e^{-\pi\sqrt{37}} + \dots,$$

$$64 G_{37}^{-24} = 4096 e^{-\pi\sqrt{37}} - \dots,$$



so that

$$64 (G_{37}^{24} + G_{37}^{-24}) = e^{\pi\sqrt{37}} + 24 + 4372 e^{-\pi\sqrt{37}} - \dots$$
$$= 64 \{(6 + \sqrt{37})^6 + (6 - \sqrt{37})^6\}.$$

Hence

$$e^{\pi\sqrt{37}} = 199148647.999978\dots.$$

Similarly, from

$$g^{58} = \sqrt{\left(\frac{5 + \sqrt{29}}{2}\right)},$$

we obtain

$$64 (g_{58}^{24} + g_{58}^{-24}) = e^{\pi\sqrt{58}} - 24 + 4372 e^{-\pi\sqrt{58}} + \dots$$
$$= 64 \left\{\left(\frac{5 + \sqrt{29}}{2}\right)^{12} + \left(\frac{5 - \sqrt{29}}{2}\right)^{12}\right\}.$$

Hence

$$e^{\pi\sqrt{58}} = 24591257751.99999982\dots.$$

6. I have calculated the values of $G_n$ and $g_n$ for a large number of values of $n$. Many of these results are equivalent to results given by Weber; for example,

$$G_{13}^4 = \frac{3 + \sqrt{13}}{2}, \qquad G_{25} = \frac{1 + \sqrt{5}}{2},$$

$$g_{30}^6 = (2 + \sqrt{5})(3 + \sqrt{10}), \qquad G_{37}^4 = 6 + \sqrt{37},$$

$$G_{49} = \frac{7^{1/4} + \sqrt{(4 + \sqrt{7})}}{2}, \qquad g_{58}^2 = \frac{5 + \sqrt{29}}{2},$$

$$g_{70}^2 = \frac{(3 + \sqrt{5})(1 + \sqrt{2})}{2},$$

$$G_{73} = \sqrt{\left(\frac{9 + \sqrt{73}}{8}\right)} + \sqrt{\left(\frac{1 + \sqrt{73}}{8}\right)},$$

$$G_{85} = \left(\frac{1 + \sqrt{5}}{2}\right)\left(\frac{9 + \sqrt{85}}{2}\right)^{1/4},$$

$$G_{97} = \sqrt{\left(\frac{13 + \sqrt{97}}{8}\right)} + \sqrt{\left(\frac{5 + \sqrt{97}}{8}\right)},$$

$$g_{190}^2 = (2 + \sqrt{5})(3 + \sqrt{10}),$$

$$G_{385}^2 = \tfrac{1}{8}(3 + \sqrt{11})(\sqrt{5} + \sqrt{7})(\sqrt{7} + \sqrt{11})(3 + \sqrt{5}),$$

and so on. I have also many results not given by Weber. I give a complete table of new results. In Weber's notation, $G_n = 2^{-\frac{1}{4}} f\{\sqrt{(-n)}\}$ and $g_n = 2^{-\frac{1}{4}} f_1\{\sqrt{(-n)}\}$.

TABLE I.

$$g_{62} + \frac{1}{g_{62}} = \tfrac{1}{2}\{\sqrt{(1+\sqrt{2})} + \sqrt{(9+5\sqrt{2})}\},$$

$$G_{65}^2 = \sqrt{\left\{\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{13}}{2}\right)\right\}} \times \left\{\sqrt{\left(\frac{1+\sqrt{65}}{8}\right)} + \sqrt{\left(\frac{9+\sqrt{65}}{8}\right)}\right\},$$

$$g_{66}^2 = \sqrt{(\sqrt{2}+\sqrt{3})}\,(7\sqrt{2}+3\sqrt{11})^{1/6} \times \left\{\sqrt{\left(\frac{7+\sqrt{33}}{8}\right)} + \sqrt{\left(\frac{\sqrt{33}-1}{8}\right)}\right\},$$

$$G_{69}^2 = (3\sqrt{3}+\sqrt{23})^{1/4}\left(\frac{5+\sqrt{23}}{4}\right)^{1/6} \times \left\{\sqrt{\left(\frac{6+3\sqrt{3}}{4}\right)} + \sqrt{\left(\frac{2+3\sqrt{3}}{4}\right)}\right\},$$

$$G_{77}^2 = \{\tfrac{1}{2}(\sqrt{7}+\sqrt{11})\,(8+3\sqrt{7})\}^{1/4} \times \left\{\sqrt{\left(\frac{6+\sqrt{11}}{4}\right)} + \sqrt{\left(\frac{2+\sqrt{11}}{4}\right)}\right\},$$

$$G_{81}^3 = \frac{(2\sqrt{3}+2)^{1/3}+1}{(2\sqrt{3}-2)^{1/3}-1},$$

$$g_{90} = \{(2+\sqrt{5})\,(\sqrt{5}+\sqrt{6})\}^{1/6} \times \left\{\sqrt{\left(\frac{3+\sqrt{6}}{4}\right)} + \sqrt{\left(\frac{\sqrt{6}-1}{4}\right)}\right\},$$

$$g_{94} + \frac{1}{g_{94}} = \tfrac{1}{2}\{\sqrt{(7+\sqrt{2})} + \sqrt{(7+5\sqrt{2})}\},$$

$$g_{98} + \frac{1}{g_{98}} = \tfrac{1}{2}\{\sqrt{2} + \sqrt{(14+4\sqrt{14})}\},$$

$$g_{114}^2 = \sqrt{\{\sqrt{2}+\sqrt{3})}\,(3\sqrt{2}+\sqrt{19})^{1/6} \times \left\{\sqrt{\left(\frac{23+3\sqrt{57}}{8}\right)} + \sqrt{\left(\frac{15+3\sqrt{57}}{8}\right)}\right\},$$

$$G_{117} = \tfrac{1}{2}\left(\frac{3+\sqrt{13}}{2}\right)^{1/4}(2\sqrt{3}+\sqrt{13})^{1/6}\,\{3^{1/4}+\sqrt{(4+\sqrt{3})}\},$$



$$\begin{cases} G_{121} + \dfrac{1}{G_{121}} = \left(\dfrac{11}{2}\right)^{1/6}\left\{\left(3+\dfrac{1}{3\sqrt{3}}\right)^{1/3} + \left(3-\dfrac{1}{3\sqrt{3}}\right)^{1/3}\right\} \\ \dfrac{1}{G_{121}} = \dfrac{1}{3\sqrt{2}}\left[(11-3\sqrt{11})^{1/3}\{(3\sqrt{11}+3\sqrt{3}-4)^{1/3} + (3\sqrt{11}-3\sqrt{3}-4)^{1/3}\} - 2\right], \end{cases}$$

$$g_{126} = \sqrt{\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)}\,(\sqrt{6}+\sqrt{7})^{1/6} \times \left\{\sqrt{\left(\frac{3+\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{\sqrt{2}-1}{4}\right)}\right\}^2,$$

$$g_{138}^2 = \sqrt{\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)}\,(78\sqrt{2}+23\sqrt{23})^{1/6} \times \left\{\sqrt{\left(\frac{5+2\sqrt{6}}{4}\right)} + \sqrt{\left(\frac{1+2\sqrt{6}}{4}\right)}\right\},$$

$$G_{141}^2 = (4\sqrt{3}+\sqrt{47})^{1/4}\left(\frac{7+\sqrt{47}}{\sqrt{2}}\right)^{1/6} \times \left\{\sqrt{\left(\frac{18+9\sqrt{3}}{4}\right)} + \sqrt{\left(\frac{14+9\sqrt{3}}{4}\right)}\right\},$$

$$G_{145}^2 = \sqrt{\left\{\frac{(2+\sqrt{5})\,(5+\sqrt{29})}{2}\right\}} \times \left\{\sqrt{\left(\frac{17+\sqrt{145}}{8}\right)} + \sqrt{\left(\frac{9+\sqrt{145}}{8}\right)}\right\},$$

$$\frac{1}{G_{147}} = 2^{-1/12}\left[\frac{1}{2} + \frac{1}{\sqrt{3}}\left\{\sqrt{\left(\frac{7}{4}\right)} - (28)^{1/6}\right\}\right],$$

$$G_{153} = \left\{\sqrt{\left(\frac{5+\sqrt{17}}{8}\right)} + \sqrt{\left(\frac{\sqrt{17}-3}{8}\right)}\right\}^2 \times \left\{\sqrt{\left(\frac{37+9\sqrt{17}}{4}\right)} + \sqrt{\left(\frac{33+9\sqrt{17}}{4}\right)}\right\}^{1/3},$$

$$g_{154}^2 = \sqrt{\left\{(2\sqrt{2}+\sqrt{7})\left(\frac{\sqrt{7}+\sqrt{11}}{2}\right)\right\}} \times \left\{\sqrt{\left(\frac{13+2\sqrt{22}}{4}\right)} + \sqrt{\left(\frac{9+2\sqrt{22}}{4}\right)}\right\}.$$

$$g_{158} + \frac{1}{g_{158}} = \tfrac{1}{2}\{\sqrt{(9+\sqrt{2})} + \sqrt{(17+13\sqrt{2})}\},$$

$$\left\{\begin{aligned} G_{169} + \frac{1}{G_{169}} &= \left(\frac{13}{4}\right)^{1/6}\left\{\left(1+\frac{1}{3\sqrt{3}}\right)^{1/3} + \left(1-\frac{1}{3\sqrt{3}}\right)^{1/3}\right\}^2, \\ \frac{1}{G_{169}} &= \frac{1}{3}\left[(\sqrt{13}-2) + \left(\frac{13-3\sqrt{13}}{2}\right)^{1/3}\right. \\ &\qquad \left. \times \left\{\left(3\sqrt{3} - \frac{11-\sqrt{13}}{2}\right)^{1/3} - \left(3\sqrt{3}+\frac{11-\sqrt{13}}{2}\right)^{1/3}\right\}\right], \end{aligned}\right.$$

$$g_{198} = \sqrt{(1+\sqrt{2})}\,(4\sqrt{2}+\sqrt{33})^{1/6} \times \left\{\sqrt{\left(\frac{9+\sqrt{33}}{8}\right)} + \sqrt{\left(\frac{1+\sqrt{33}}{8}\right)}\right\},$$

$$G_{205} = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3\sqrt{5}+\sqrt{41}}{2}\right)^{1/4} \times \left\{\sqrt{\left(\frac{7+\sqrt{41}}{8}\right)} + \sqrt{\left(\frac{\sqrt{41}-1}{8}\right)}\right\},$$

$$G^2_{213} = (5\sqrt{3}+\sqrt{71})^{1/4}\left(\frac{59+7\sqrt{71}}{4}\right)^{1/6} \times \left\{\sqrt{\left(\frac{21+12\sqrt{3}}{2}\right)} + \sqrt{\left(\frac{19+12\sqrt{3}}{2}\right)}\right\},$$

$$G^2_{217} = \left\{\sqrt{\left(\frac{9+4\sqrt{7}}{2}\right)} + \sqrt{\left(\frac{11+4\sqrt{7}}{2}\right)}\right\} \times \left\{\sqrt{\left(\frac{12+5\sqrt{7}}{4}\right)} + \sqrt{\left(\frac{16+5\sqrt{7}}{4}\right)}\right\},$$

$$G_{225} = \left(\frac{1+\sqrt{5}}{4}\right)(2+\sqrt{3})^{1/3}\{\sqrt{(4+\sqrt{15})} + 15^{1/4}\},$$

$$g_{238} = \left\{\sqrt{\left(\frac{1+2\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{5+2\sqrt{2}}{4}\right)}\right\} \times \left\{\sqrt{\left(\frac{1+3\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{5+3\sqrt{2}}{4}\right)}\right\},$$

$$G^2_{265} = \sqrt{\left\{(2+\sqrt{5})\left(\frac{7+\sqrt{53}}{2}\right)\right\}} \times \left\{\sqrt{\left(\frac{89+5\sqrt{265}}{8}\right)} + \sqrt{\left(\frac{81+5\sqrt{265}}{8}\right)}\right\},$$

$$G_{289} = \left[\sqrt{\left\{\frac{17+\sqrt{17}+17^{1/4}(5+\sqrt{17})}{16}\right\}} + \sqrt{\left\{\frac{1+\sqrt{17}+17^{1/4}(5+\sqrt{17})}{16}\right\}}\right]^2,$$



$$G^2_{301} = \left\{(8+3\sqrt{7})\left(\frac{23\sqrt{43}+57\sqrt{7}}{2}\right)\right\}^{1/4} \times \left\{\sqrt{\left(\frac{46+7\sqrt{43}}{4}\right)} + \sqrt{\left(\frac{42+7\sqrt{43}}{4}\right)}\right\},$$

$$g_{310} = \left(\frac{1+\sqrt{5}}{2}\right)\sqrt{(1+\sqrt{2})} \times \left\{\sqrt{\left(\frac{7+2\sqrt{10}}{4}\right)} + \sqrt{\left(\frac{3+2\sqrt{10}}{4}\right)}\right\},$$

$$\left\{\begin{aligned} G_{325} &= \left(\frac{3+\sqrt{13}}{2}\right)^{1/4} t, \text{ where} \\ t^3 + t^2\left(\frac{1-\sqrt{13}}{2}\right)^2 &+ t\left(\frac{1+\sqrt{13}}{2}\right)^2 + 1 \\ &= \sqrt{5}\left\{t^3 - t^2\left(\frac{1+\sqrt{13}}{2}\right) + t\left(\frac{1-\sqrt{13}}{2}\right) - 1\right\}, \end{aligned}\right.$$

$$G_{333} = \tfrac{1}{2}(6+\sqrt{37})^{1/4}(7\sqrt{3}+2\sqrt{37})^{1/6} \times \{\sqrt{(7+2\sqrt{3})} + \sqrt{(3+2\sqrt{3})}\},$$

$$\left\{\begin{aligned} G_{363} &= 2^{\frac{5}{12}} t, \text{ where} \\ 2t^3 - t^2\{(4+\sqrt{33}) &+ \sqrt{(11+2\sqrt{33})}\} \\ &- t\{1+\sqrt{(11+2\sqrt{33})}\} - 1 = 0, \end{aligned}\right.$$

$$G^2_{441} = \left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)(2+\sqrt{3})^{1/3} \times \left\{\frac{2+\sqrt{7}+\sqrt{(7+4\sqrt{7})}}{2}\right\}\left\{\frac{\sqrt{(3+\sqrt{7})}+(6\sqrt{7})^{1/4}}{\sqrt{(3+\sqrt{7})}-(6\sqrt{7})^{1/4}}\right\},$$

$$G_{445} = \sqrt{(2+\sqrt{5})}\left(\frac{21+\sqrt{445}}{2}\right)^{1/4} \times \sqrt{\left\{\left(\frac{13+\sqrt{89}}{8}\right)} + \sqrt{\left(\frac{5+\sqrt{89}}{8}\right)\right\}},$$

$$G^2_{465} = \sqrt{\left\{(2+\sqrt{3})\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3\sqrt{3}+\sqrt{31}}{2}\right)\right\}}(5\sqrt{5}+2\sqrt{31})^{1/6} \times \left\{\sqrt{\left(\frac{2+\sqrt{31}}{4}\right)} + \sqrt{\left(\frac{6+\sqrt{31}}{4}\right)}\right\} \times \left\{\sqrt{\left(\frac{11+2\sqrt{31}}{2}\right)} + \sqrt{\left(\frac{13+2\sqrt{31}}{2}\right)}\right\},$$

$$G_{505}^2 = (2+\sqrt{5})\sqrt{\left\{\left(\frac{1+\sqrt{5}}{2}\right)(10+\sqrt{101})\right\}} \times \left\{\left(\frac{5\sqrt{5}+\sqrt{101}}{4}\right)+\sqrt{\left(\frac{105+\sqrt{505}}{8}\right)}\right\},$$

$$g_{522} = \sqrt{\left(\frac{5+\sqrt{29}}{2}\right)}(5\sqrt{29}+11\sqrt{6})^{1/6} \times \left\{\sqrt{\left(\frac{9+3\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{5+3\sqrt{6}}{4}\right)}\right\},$$

$$G_{553}^2 = \left\{\sqrt{\left(\frac{96+11\sqrt{79}}{4}\right)}+\sqrt{\left(\frac{100+11\sqrt{79}}{4}\right)}\right\} \times \left\{\sqrt{\left(\frac{141+16\sqrt{79}}{2}\right)}+\sqrt{\left(\frac{143+16\sqrt{79}}{2}\right)}\right\},$$

$$g_{630} = (\sqrt{14}+\sqrt{15})^{1/6}\sqrt{\left\{(1+\sqrt{2})\left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)\right\}} \times \left\{\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}+2}{4}\right)}+\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}-2}{4}\right)}\right\} \times \left\{\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}+4}{8}\right)}+\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}-4}{8}\right)}\right\},$$

$$G_{765}^2 = \left(\frac{3+\sqrt{5}}{2}\right)(16+\sqrt{255})^{1/6}\sqrt{\left\{(4+\sqrt{15})\left(\frac{9+\sqrt{85}}{2}\right)\right\}} \times \left\{\sqrt{\left(\frac{6+\sqrt{51}}{4}\right)}+\sqrt{\left(\frac{10+\sqrt{51}}{4}\right)}\right\} \times \left\{\sqrt{\left(\frac{18+3\sqrt{51}}{4}\right)}+\sqrt{\left(\frac{22+3\sqrt{51}}{4}\right)}\right\},$$

$$G_{777}^2 = \sqrt{\left\{(2+\sqrt{3})(6+\sqrt{37})\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)\right\}}(246\sqrt{7}+107\sqrt{37})^{1/6} \times \left\{\sqrt{\left(\frac{6+3\sqrt{7}}{4}\right)}+\sqrt{\left(\frac{10+3\sqrt{7}}{4}\right)}\right\} \times \left\{\sqrt{\left(\frac{15+6\sqrt{7}}{2}\right)}+\sqrt{\left(\frac{17+6\sqrt{7}}{2}\right)}\right\},$$



$$G_{1225} = \left(\frac{1+\sqrt{5}}{2}\right)(6+\sqrt{35})^{1/4}\left\{\frac{7^{1/4}+\sqrt{(4+\sqrt{7})}}{2}\right\}^{3/2} \times \left[\sqrt{\left\{\frac{43+15\sqrt{7}+(8+3\sqrt{7})\sqrt{(10\sqrt{7})}}{8}\right\}}+\sqrt{\left\{\frac{35+15\sqrt{7}+(8+3\sqrt{7})\sqrt{(10\sqrt{7})}}{8}\right\}}\right],$$

$$G_{1353}^2 = \sqrt{\left\{(3+\sqrt{11})(5+3\sqrt{3})\left(\frac{11+\sqrt{123}}{2}\right)\right\}} \times \left(\frac{6817+321\sqrt{451}}{4}\right)^{1/6} \times \left\{\sqrt{\left(\frac{17+3\sqrt{33}}{8}\right)}+\sqrt{\left(\frac{25+3\sqrt{33}}{8}\right)}\right\} \times \left\{\sqrt{\left(\frac{561+99\sqrt{33}}{8}\right)}+\sqrt{\left(\frac{569+99\sqrt{33}}{8}\right)}\right\},$$

$$G_{1645}^2 = (2+\sqrt{5})\sqrt{\left\{(3+\sqrt{7})\left(\frac{7+\sqrt{47}}{2}\right)\right\}}\left(\frac{73\sqrt{5}+9\sqrt{329}}{2}\right)^{1/4} \times \left\{\sqrt{\left(\frac{119+7\sqrt{329}}{8}\right)}+\sqrt{\left(\frac{127+7\sqrt{329}}{8}\right)}\right\} \times \left\{\sqrt{\left(\frac{743+41\sqrt{329}}{8}\right)}+\sqrt{\left(\frac{751+41\sqrt{329}}{8}\right)}\right\}.$$

7. Hence we deduce the following approximate formulæ.

TABLE II.

$$e^{\frac{1}{8}\pi\sqrt{18}} = 2\sqrt{7}, \quad e^{\pi\sqrt{22}/12} = 2+\sqrt{2}, \quad e^{\frac{1}{4}\pi\sqrt{30}} = 20\sqrt{3}+16\sqrt{6},$$

$$e^{\frac{1}{4}\pi\sqrt{34}} = 12(4+\sqrt{17}), \quad e^{\frac{1}{2}\pi\sqrt{46}} = 144(147+104\sqrt{2}),$$

$$e^{\frac{1}{4}\pi\sqrt{42}} = 84+32\sqrt{6}, \quad e^{\pi\sqrt{58}/12} = \frac{5+\sqrt{29}}{\sqrt{2}},$$

$$e^{\frac{1}{4}\pi\sqrt{70}} = 60\sqrt{35}+96\sqrt{14}, \quad e^{\frac{1}{4}\pi\sqrt{78}} = 300\sqrt{3}+208\sqrt{6},$$

$$e^{\pi\sqrt{55}/24} = \frac{1+\sqrt{(3+2\sqrt{5})}}{\sqrt{2}}, \quad e^{\frac{1}{4}\pi\sqrt{102}} = 800\sqrt{3}+196\sqrt{51},$$

$$e^{\frac{1}{4}\pi\sqrt{130}} = 12(323+40\sqrt{65}), \quad e^{\pi\sqrt{190}/12} = (2\sqrt{2}+\sqrt{10})(3+\sqrt{10}),$$

$$\pi = \frac{12}{\sqrt{130}}\log\left\{\frac{(2+\sqrt{5})(3+\sqrt{13})}{\sqrt{2}}\right\},$$

$$\pi = \frac{24}{\sqrt{142}} \log\left\{\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right\},$$

$$\pi = \frac{12}{\sqrt{190}} \log\{(2\sqrt{2}+\sqrt{10})(3+\sqrt{10})\}.$$

$$\pi = \frac{12}{\sqrt{310}} \log[\tfrac{1}{4}(3+\sqrt{5})(2+\sqrt{2})\{(5+2\sqrt{10})+\sqrt{(61+20\sqrt{10})}\}].$$

$$\pi = \frac{4}{\sqrt{522}} \log\left[\left(\frac{5+\sqrt{29}}{\sqrt{2}}\right)^3 (5\sqrt{29}+11\sqrt{6}) \times \left\{\sqrt{\left(\frac{9+3\sqrt{6}}{4}\right)} + \sqrt{\left(\frac{5+3\sqrt{6}}{4}\right)}\right\}^6\right].$$

The last five formulæ are correct to 15, 16, 18, 22, and 31 places of decimals respectively.

8. Thus we have seen how to approximate to $\pi$ by means of logarithms of surds. I shall now show how to obtain approximations in terms of surds only. If

$$n\frac{K'}{K} = \frac{L'}{L},$$

we have

$$\frac{n\,dk}{kk'^2K^2} = \frac{dl}{ll'^2L^2}.$$

But, by means of the modular equation connecting $k$ and $l$, we can express $dk/dl$ as an algebraic function of $k$, a function moreover in which all coefficients which occur are algebraic numbers. Again,

$$q = e^{-\pi K'/K}, \quad q^n = e^{-\pi L'/L},$$

and

$$(15) \quad \frac{q^{\frac{1}{12}}(1-q^2)(1-q^4)(1-q^6)\dots}{q^{\frac{1}{12}n}(1-q^{2n})(1-q^{4n})(1-q^{6n})\dots} = \left(\frac{kk'}{ll'}\right)^{\frac{1}{6}} \sqrt{\left(\frac{K}{L}\right)}.$$

Differentiating this equation logarithmically, and using the formula

$$\frac{dq}{dk} = \frac{\pi^2 q}{2kk'^2K^2},$$

we see that

$$(16) \quad n\left\{1-24\left(\frac{q^{2n}}{1-q^{2n}} + \frac{2q^{4n}}{1-q^{4n}} + \dots\right)\right\} - \left\{1-24\left(\frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \dots\right)\right\} = \frac{KL}{\pi^2} A(k),$$



where $A(k)$ denotes an algebraic function of the special class described above. I shall use the letter $A$ generally to denote a function of this type.

Now, if we put $k = l'$ and $k' = l$ in (16), we have

$$(17) \quad n\left\{1-24\left(\frac{1}{e^{2\pi\sqrt{n}}-1} + \frac{2}{e^{4\pi\sqrt{n}}-1} + \dots\right)\right\} - \left\{1-24\left(\frac{1}{e^{2\pi/\sqrt{n}}-1} + \frac{2}{e^{4\pi/\sqrt{n}}-1} + \dots\right)\right\} = \left(\frac{K}{\pi}\right)^2 A(k).$$

The algebraic function $A(k)$ of course assumes a purely numerical form when we substitute the value of $k$ deduced from the modular equation. But by substituting $k = l'$ and $k' = l$ in (15) we have

$$n^{\frac{1}{4}} e^{-\pi\sqrt{n}/12}(1-e^{-2\pi\sqrt{n}})(1-e^{-4\pi\sqrt{n}})(1-e^{-6\pi\sqrt{n}})\dots = e^{-\pi/(12\sqrt{n})}(1-e^{-2\pi/\sqrt{n}})(1-e^{-4\pi/\sqrt{n}})(1-e^{-6\pi/\sqrt{n}})\dots.$$

Differentiating the above equation logarithmically we have

$$(18) \quad n\left\{1-24\left(\frac{1}{e^{2\pi\sqrt{n}}-1} + \frac{2}{e^{4\pi\sqrt{n}}-1} + \dots\right)\right\} + \left\{1-24\left(\frac{1}{e^{2\pi/\sqrt{n}}-1} + \frac{2}{e^{4\pi/\sqrt{n}}-1} + \dots\right)\right\} = \frac{6\sqrt{n}}{\pi}.$$

Now, adding (17) and (18), we have

$$(19) \quad 1 - \frac{3}{\pi\sqrt{n}} - 24\left(\frac{1}{e^{2\pi\sqrt{n}}-1} + \frac{2}{e^{4\pi\sqrt{n}}-1} + \dots\right) = \left(\frac{K}{\pi}\right)^2 A(k).$$

But it is known that

$$1 - 24\left(\frac{q}{1+q} + \frac{3q^3}{1+q^3} + \frac{5q^5}{1+q^5} + \dots\right) = \left(\frac{2K}{\pi}\right)^2 (1-2k^2),$$

so that

$$(20) \quad 1 - 24\left(\frac{1}{e^{\pi\sqrt{n}}+1} + \frac{3}{e^{3\pi\sqrt{n}}+1} + \dots\right) = \left(\frac{K}{\pi}\right)^2 A(k).$$

Hence, dividing (19) by (20), we have

$$(21) \quad \frac{1 - \frac{3}{\pi\sqrt{n}} - 24\left(\frac{1}{e^{2\pi\sqrt{n}}-1} + \frac{2}{e^{4\pi\sqrt{n}}-1} + \dots\right)}{1 - 24\left(\frac{1}{e^{\pi\sqrt{n}}+1} + \frac{3}{e^{3\pi\sqrt{n}}+1} + \dots\right)} = R,$$

where $R$ can always be expressed in radicals if $n$ is any rational number. Hence we have

$$(22) \qquad \pi = \frac{3}{(1-R)\sqrt{n}}$$

nearly, the error being about $8\pi e^{-\pi\sqrt{n}}(\pi\sqrt{n}-3)$.

9. We may get a still closer approximation from the following results.

It is known that

$$1+240\sum_{r=1}^{r=\infty}\frac{r^3q^{2r}}{1-q^{2r}} = \left(\frac{2K}{\pi}\right)^4(1-k^2k'^2),$$

and also that

$$1-504\sum_{r=1}^{r=\infty}\frac{r^5q^{2r}}{1-q^{2r}} = \left(\frac{2K}{\pi}\right)^6(1-2k^2)(1+\tfrac{1}{2}k^2k'^2).$$

Hence, from (19), we see that

$$(23) \qquad \left\{1-\frac{3}{\pi\sqrt{n}}-24\sum_{r=1}^{r=\infty}\frac{r}{e^{2\pi r\sqrt{n}}-1}\right\}\left\{1+240\sum_{r=1}^{r=\infty}\frac{r^3}{e^{2\pi r\sqrt{n}}-1}\right\} = R'\left\{1-504\sum_{r=1}^{r=\infty}\frac{r^5}{e^{2\pi r\sqrt{n}}-1}\right\},$$

where $R'$ can always be expressed in radicals for any rational value of $n$. Hence

$$(24) \qquad \pi = \frac{3}{(1-R')\sqrt{n}},$$

nearly, the error being about $24\pi(10\pi\sqrt{n}-31)e^{-2\pi\sqrt{n}}$.

It will be seen that the error in (24) is much less than that in (22), if $n$ is at all large.

10. In order to find $R$ and $R'$ the series in (16) must be calculated in finite terms. I shall give the final results for a few values of $n$.

### TABLE III.

$$q = e^{-\pi K'/K}, \quad q^n = e^{-\pi L'/L},$$

$$f(q) = n\left(1-24\sum_1^\infty\frac{q^{2mn}}{1-q^{2mn}}\right)-\left(1-24\sum_1^\infty\frac{q^{2m}}{1-q^{2m}}\right),$$



$$f(2) = \frac{4KL}{\pi^2}(k'+l),$$

$$f(3) = \frac{4KL}{\pi^2}(1+kl+k'l'),$$

$$f(4) = \frac{4KL}{\pi^2}(\sqrt{k'}+\sqrt{l})^2,$$

$$f(5) = \frac{4KL}{\pi^2}(3+kl+k'l')\sqrt{\left(\frac{1+kl+k'l'}{2}\right)},$$

$$f(7) = \frac{12KL}{\pi^2}(1+kl+k'l'),$$

$$f(11) = \frac{8KL}{\pi^2}\{2(1+kl+k'l')+\sqrt{(kl)}+\sqrt{(k'l')}-\sqrt{(kk'll')}\},$$

$$f(15) = \frac{4KL}{\pi^2}[\{1+(kl)^{\frac{1}{4}}+(k'l')^{\frac{1}{4}}\}^4-\{1+kl+k'l'\}],$$

$$f(17) = \frac{4KL}{\pi^2}\sqrt{\{44(1+k^2l^2+k'^2l'^2)+168(kl+k'l'-kk'll') - 102(1-kl-k'l')(4kk'll')^{\frac{1}{3}}-192(4kk'll')^{\frac{2}{3}}\}},$$

$$f(19) = \frac{24KL}{\pi^2}\{(1+kl+k'l')+\sqrt{(kl)}+\sqrt{(k'l')}-\sqrt{(kk'll')}\},$$

$$f(23) = \frac{4KL}{\pi^2}[11(1+kl+k'l')-16(4kk'll')^{\frac{1}{6}} \times\{1+\sqrt{(kl)}+\sqrt{(k'l')}\}-20(4kk'll')^{\frac{1}{3}}],$$

$$f(31) = \frac{12KL}{\pi^2}[3(1+kl+k'l')+4\{\sqrt{(kl)}+\sqrt{(k'l'}+\sqrt{(kk'll')}\} - 4(kk'll')^{\frac{1}{4}}\{1+(kl)^{\frac{1}{4}}+(k'l')^{\frac{1}{4}}\}],$$

$$f(35) = \frac{4KL}{\pi^2}[2\{\sqrt{(kl)}+\sqrt{(k'l')}-\sqrt{(kk'll')}\} + (4kk'll')^{-\frac{1}{6}}\{1-\sqrt{(kl)}-\sqrt{(k'l')}\}^3].$$

Thus the sum of the series (19) can be found in finite terms, when $n=2, 3, 4, 5$, etc., from the equations in Table III. We can use the same table to find the sum of (19) when $n=9, 25, 49$, etc.; but then we have also to use the equation

$$\frac{3}{\pi} = 1-24\left(\frac{1}{e^{2\pi}-1}+\frac{2}{e^{4\pi}-1}+\frac{3}{e^{6\pi}-1}+\ldots\right),$$

which is got by putting $k=k'=1/\sqrt{2}$ and $n=1$ in (18).

Similarly we can find the sum of (19) when $n = 21, 33, 57, 93$, etc., by combining the values of $f(3)$ and $f(7)$, $f(3)$ and $f(11)$, and so on, obtained from Table III.

11. The errors in (22) and (24) being about

$$8\pi e^{-\pi\sqrt{n}}(\pi\sqrt{n}-3), \quad 24\pi(10\pi\sqrt{n}-31)e^{-2\pi\sqrt{n}},$$

we cannot expect a high degree of approximation for small values of $n$. Thus, if we put $n = 7, 9, 16$, and $25$ in (24), we get

$$\frac{19}{16}\sqrt{7} = 3.14180\ldots,$$

$$\frac{7}{3}\left(1 + \frac{\sqrt{3}}{5}\right) = 3.14162\ldots,$$

$$\frac{99}{80}\left(\frac{7}{7 - 3\sqrt{2}}\right) = 3.14159274\ldots,$$

$$\frac{63}{25}\left(\frac{17 + 15\sqrt{5}}{7 + 15\sqrt{5}}\right) = 3.14159265380\ldots,$$

while

$$\pi = 3.14159265358\ldots.$$

But if we put $n = 25$ in (22), we get only

$$\frac{9}{5} + \sqrt{\frac{9}{5}} = 3.14164\ldots.$$

12. Another curious approximation to $\pi$ is

$$\left\{9^2 + \frac{19^2}{22}\right\}^{\frac{1}{4}} = 3.14159265262\ldots.$$

This value was obtained empirically, and it has no connection with the preceding theory.

The actual value of $\pi$, which I have used for purposes of calculation, is

$$\frac{355}{113}\left(1 - \frac{.0003}{3533}\right) = 3.1415926535897943\ldots,$$

which is greater than $\pi$ by about $10^{-15}$. This is obtained by simply taking the reciprocal of $1 - (113\pi/355)$.

In this connection it may be interesting to note the following simple geometrical constructions for $\pi$. The first merely gives the ordinary value $355/113$. The second gives the value $(9^2 + 19^2/22)^{1/4}$ mentioned above.



(1) Let $AB$ (fig. 1) be a diameter of a circle whose centre is $O$.

Bisect $AO$ at $M$ and trisect $OB$ at $T$.

Draw $TP$ perpendicular to $AB$ and meeting the circumference at $P$.

Draw a chord $BQ$ equal to $PT$ and join $AQ$.

Draw $OS$ and $TR$ parallel to $BQ$ and meeting $AQ$ at $S$ and $R$ respectively.

Draw a chord $AD$ equal to $AS$ and a tangent $AC = RS$.

Join $BC$, $BD$, and $CD$ and draw $EX$, parallel to $CD$, meeting $BC$ at $X$.

Then the square on $BX$ is very nearly equal to the area of the circle, the error being less than a tenth of an inch when the diameter is 40 miles long.

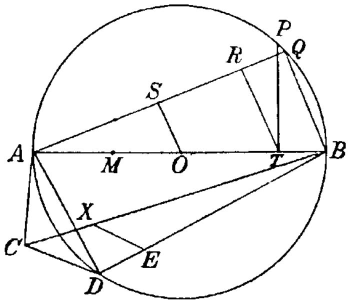


Fig. 1.

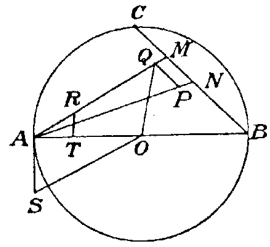


Fig. 2.

(2) Let $AB$ (fig 2) be a diameter of a circle whose centre is $O$.

Bisect the arc $ACB$ at $C$ and trisect $AO$ at $T$.

Join $BC$ and cut off from it $CM$ and $MN$ equal to $AT$.

Join $AM$ and $AN$ and cut off from the latter $AP$ equal to $AM$.

Through $P$ draw $PQ$ parallel to $MN$ and meeting $AM$ at $Q$.

Join $OQ$ and through $T$ draw $TR$, parallel to $OQ$, and meeting $AQ$ at $R$.

Draw $AS$ perpendicular to $AO$ and equal to $AR$, and join $OS$.

Then the mean proportional between $OS$ and $OB$ will be very nearly equal to a sixth of the circumference, the error being less than a twelfth of a inch when the diameter is 8000 miles long.

13. I shall conclude this paper by giving a few series for $1/\pi$.

It is known that, when $k \leq 1/\sqrt{2}$,

$$(25)\qquad \left(\frac{2K}{\pi}\right)^2 = 1 + \left(\frac{1}{2}\right)^3 (2kk')^2 + \left(\frac{1.3}{2.4}\right)^3 (2kk')^4 + \dots .$$

Hence we have

$$(26)\qquad q^{\frac{1}{3}}(1-q^2)^4(1-q^4)^4(1-q^6)^4 \dots$$
$$= \left(\frac{1}{4}kk'\right)^{\frac{2}{3}} \left\{1 + \left(\frac{1}{2}\right)^3 (2kk')^2 + \left(\frac{1.3}{2.4}\right)^3 (2kk')^4 + \dots\right\}.$$

Differentiating both sides in (26) logarithmically with respect to $k$, we can easily show that

$$(27)\qquad 1 - 24\left(\frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \frac{3q^6}{1-q^6} + \dots\right)$$
$$= (1-2k^2)\left\{1 + 4\left(\frac{1}{2}\right)^3 (2kk')^2 + 7\left(\frac{1.3}{2.4}\right)^3 (2kk')^4 + \dots\right\}.$$

But it follows from (19) that, when $q = e^{-\pi\sqrt{n}}$, $n$ being a rational number, the left-hand side of (27) can be expressed in the form

$$A\left(\frac{2K}{\pi}\right)^2 + \frac{B}{\pi},$$

where $A$ and $B$ are algebraic numbers expressible by surds. Combining (25) and (27) in such a way as to eliminate the term $(2K/\pi)^2$, we are left with a series for $1/\pi$. Thus, for example,

$$(28)\qquad \frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \dots,$$
$$(q = e^{-\pi\sqrt{3}},\ 2kk' = \tfrac{1}{2}),$$

$$(29)\qquad \frac{16}{\pi} = 5 + \frac{47}{64}\left(\frac{1}{2}\right)^3 + \frac{89}{64^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{131}{64^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \dots,$$
$$(q = e^{-\pi\sqrt{7}},\ 2kk' = \tfrac{1}{8}),$$

$$(30)\qquad \frac{32}{\pi} = (5\sqrt{5}-1) + \frac{47\sqrt{5}+29}{64}\left(\frac{1}{2}\right)^3\left(\frac{\sqrt{5}-1}{2}\right)^8 + \dots,$$
$$\frac{89\sqrt{5}+59}{64^2}\left(\frac{1.3}{2.4}\right)^3\left(\frac{\sqrt{5}-1}{2}\right)^{16} + \dots,$$
$$\left[q = e^{-\pi\sqrt{15}},\ 2kk' = \frac{1}{8}\left(\frac{\sqrt{5}-1}{2}\right)\right];$$



here $(5\sqrt{5}-1)$, $(47\sqrt{5}+29)$, $(89\sqrt{5}+59)$, etc. are in arithmetical progression.

14. The ordinary modular equations express the relations which hold between $k$ and $l$ when $nK'/K = L'/L$, or $q^n = Q$, where

$$q = e^{-\pi K'/K},\quad Q = e^{-\pi L'/L},$$
$$K = 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \dots .$$

There are corresponding theories in which $q$ is replaced by one or other of the functions

$$q_1 = e^{-\pi K_1'\sqrt{2}/K_1},\quad q_2 = e^{-2\pi K_2'/(K_2\sqrt{3})}\quad q_3 = e^{-2\pi K_3'/K_3},$$

where

$$K_1 = 1 + \frac{1.3}{4^2}k^2 + \frac{1.3.5.7}{4^2.8^2}k^4 + \frac{1.3.5.7.9.11}{4^2.8^2.12^2}k^6 + \dots,$$
$$K_2 = 1 + \frac{1.2}{3^2}k^2 + \frac{1.2.4.5}{3^2.6^2}k^4 + \frac{1.2.4.5.7.8}{3^2.6^2.9^2}k^6 + \dots .$$
$$K_3 = 1 + \frac{1.5}{6^2}k^2 + \frac{1.5.7.11}{6^2.12^2}k^4 + \frac{1.5.7.11.13.17}{6^2.12^2.18^2}k^6 + \dots .$$

From these theories we can deduce further series for $1/\pi$, such as

$$(31)\qquad \frac{27}{4\pi} = 2 + 17.\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot\left(\frac{2}{27}\right) + 32.\frac{1.3}{2.4}\cdot\frac{1.4}{3.6}\cdot\frac{2.5}{3.6}\cdot\left(\frac{2}{27}\right)^2 + \dots,$$

$$(32)\qquad \frac{15\sqrt{3}}{2\pi} = 4 + 37.\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot\left(\frac{4}{125}\right) + 70.\frac{1.3}{2.4}\cdot\frac{1.4}{3.6}\cdot\frac{2.5}{3.6}\cdot\left(\frac{4}{125}\right)^2 + \dots,$$

$$(33)\qquad \frac{5\sqrt{5}}{2\pi\sqrt{3}} = 1 + 12.\frac{1}{2}\cdot\frac{1}{6}\cdot\frac{5}{6}\cdot\left(\frac{4}{125}\right) + 23.\frac{1.3}{2.4}\cdot\frac{1.7}{6.12}\cdot\frac{5.11}{6.12}\cdot\left(\frac{4}{125}\right)^2 + \dots,$$

$$(34)\qquad \frac{85\sqrt{85}}{18\pi\sqrt{3}} = 8 + 141.\frac{1}{2}\cdot\frac{1}{6}\cdot\frac{5}{6}\cdot\left(\frac{4}{85}\right)^3 + 274.\frac{1.3}{2.4}\cdot\frac{1.7}{6.12}\cdot\frac{5.11}{6.12}\cdot\left(\frac{4}{85}\right)^6 + \dots,$$

(35) $$\frac{4}{\pi}=\frac{3}{2}-\frac{23}{2^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2}+\frac{43}{2^6}\cdot\frac{1.3}{2.4}\cdot\frac{1.3.5.7}{4^2.8^2}-\dots,$$

(36) $$\frac{4}{\pi\sqrt{3}}=\frac{3}{4}-\frac{31}{3.4^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2}+\frac{59}{3^2.4^5}\cdot\frac{1.3}{2.4}\cdot\frac{1.3.5.7}{4^2.8^2}-\dots,$$

(37) $$\frac{4}{\pi}=\frac{23}{18}-\frac{283}{18^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2}+\frac{543}{18^5}\cdot\frac{1.3}{2.4}\cdot\frac{1.3.5.7}{4^2.8^2}-\dots,$$

(38) $$\frac{4}{\pi\sqrt{5}}=\frac{41}{72}-\frac{685}{5.72^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2}+\frac{1329}{5^2.72^5}\cdot\frac{1.3}{2.4}\cdot\frac{1.3.5.7}{4^2.8^2}-\dots,$$

(39) $$\frac{4}{\pi}=\frac{1123}{882}-\frac{22583}{882^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2}+\frac{44043}{882^5}\cdot\frac{1.3}{2.4}\cdot\frac{1.3.5.7}{4^2.8^2}-\dots,$$

(40) $$\frac{2\sqrt{3}}{\pi}=1+\frac{9}{9}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2}+\frac{17}{9^2}\cdot\frac{1.3}{2.4}\cdot\frac{1.3.5.7}{4^2.8^2}+\dots,$$

(41) $$\frac{1}{2\pi\sqrt{2}}=\frac{1}{9}+\frac{11}{9^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2}+\frac{21}{9^5}\cdot\frac{1.3}{2.4}\cdot\frac{1.3.5.7}{4^2.8^2}+\dots,$$

(42) $$\frac{1}{3\pi\sqrt{3}}=\frac{3}{49}+\frac{43}{49^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2}+\frac{83}{49^5}\cdot\frac{1.3}{2.4}\cdot\frac{1.3.5.7}{4^2.8^2}+\dots,$$

(43) $$\frac{2}{\pi\sqrt{11}}=\frac{19}{99}+\frac{299}{99^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2}+\frac{579}{99^5}\cdot\frac{1.3}{2.4}\cdot\frac{1.3.5.7}{4^2.8^2}+\dots,$$

(44) $$\frac{1}{2\pi\sqrt{2}}=\frac{1103}{99^2}+\frac{27493}{99^6}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2}+\frac{53883}{99^{10}}\cdot\frac{1.3}{2.4}\cdot\frac{1.3.5.7}{4^2.8^2}+\dots.$$



In all these series the first factors in each term form an arithmetical progression; *e.g.*, 2, 17, 32, 47, ..., in (31), and 4, 37, 70, 103, ..., in (32). The first two series belong to the theory of $q_2$, the next two to that of $q_3$, and the rest to that of $q_1$.

The last series (44) is extremely convergent. Thus, taking only the first term, we see that

$$\frac{1103}{99^2}=.11253953678\dots,$$

$$\frac{1}{2\pi\sqrt{2}}=.11253953951\dots.$$

15. In concluding this paper I have to remark that the series

$$1-24\left(\frac{q^2}{1-q^2}+\frac{2q^4}{1-q^4}+\frac{3q^6}{1-q^6}+\dots\right),$$

which has been discussed in §§ 8–13, is very closely connected with the perimeter of an ellipse whose eccentricity is $k$. For, if $a$ and $b$ be the semi-major and the semi-minor axes, it is known that

(45) $$p=2\pi a\left\{1-\frac{1}{2^2}k^2-\frac{1^2.3}{2^2.4^2}k^4-\frac{1^2.3^2.5}{2^2.4^2.6^2}k^6-\dots\right\},$$

where $p$ is the perimeter and $k$ the eccentricity. It can easily be seen from (45) that

(46) $$p=4ak'^2\left\{K+k\frac{dK}{dk}\right\}.$$

But, taking the equation

$$q^{\frac{1}{12}}(1-q^2)(1-q^4)(1-q^6)\dots=(2kk')^{\frac{1}{6}}\sqrt{(K/\pi)},$$

and differentiating both sides logarithmically with respect to $k$, and combining the result with (46) in such a way as to eliminate $dK/dk$, we can show that

(47) $$p=\frac{4a}{3K}\left[K^2(1+k'^2)+(\tfrac{1}{2}\pi)^2\times\left\{1-24\left(\frac{q^2}{1-q^2}+\frac{2q^4}{1-q^4}+\dots\right)\right\}\right].$$

But we have shown already that the right-hand side of (47) can be expressed in terms of $K$ if $q=e^{-\pi\sqrt{n}}$, where $n$ is any rational number. It can also be shown that $K$ can be expressed in terms of $\Gamma$-functions if $q$ be of the forms $e^{-\pi n}$, $e^{-\pi n\sqrt{2}}$ and $e^{-\pi n\sqrt{3}}$, where $n$ is rational. Thus, for example, we have

$$(48)\quad \begin{cases} k=\sin\dfrac{\pi}{4}, \quad q=e^{-\pi}, \\ \qquad p=a\sqrt{\left(\dfrac{\pi}{2}\right)}\left\{\dfrac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}+\dfrac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}\right\}, \\ k=\tan\dfrac{\pi}{8}, \quad q=e^{-\pi\sqrt{2}}, \\ \qquad p=a\sqrt{\left(\dfrac{\pi}{4}\right)}\left\{\dfrac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})}+\dfrac{\Gamma(\frac{5}{8})}{\Gamma(\frac{9}{8})}\right\}, \\ k=\sin\dfrac{\pi}{12}, \quad q=e^{-\pi\sqrt{3}}, \\ \qquad p=a\sqrt{\left(\dfrac{\pi}{\sqrt{3}}\right)}\left\{\left(1+\dfrac{1}{\sqrt{3}}\right)\dfrac{\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{6})}+2\,\dfrac{\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})}\right\}, \\ \dfrac{b}{a}=\tan^2\dfrac{\pi}{8}, \quad q=e^{-2\pi}, \\ \qquad p=(a+b)\sqrt{\left(\dfrac{\pi}{2}\right)}\left\{\dfrac{1}{2}\,\dfrac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}+\dfrac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}\right\}, \end{cases}$$

and so on.

16. The following approximations for $p$ were obtained empirically :—

$$(49)\quad p=\pi\left[3(a+b)-\sqrt{\{(a+3b)(3a+b)\}}+\epsilon\right],$$

where $\epsilon$ is about $ak^{12}/1048576$ ;

$$(50)\quad p=\pi\left\{(a+b)+\frac{3(a-b)^2}{10(a+b)+\sqrt{(a^2+14ab+b^2)}}+\epsilon\right\},$$

where $\epsilon$ is about $3ak^{20}/68719476736$.