COMPLEXITY OF THE REACHABILITY PROBLEM IN SUBCLASSES OF PETRI NETS

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Certificate

Certified that the work contained in this thesis entitled “Complexity of the reachability problem in subclasses of Petri nets”, by Praveen M, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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Abstract

The reachability problem in Petri nets is known to be decidable but the algorithms known have non-primitive-recursive complexity. Lot of research has gone into identifying structural and behavioural restrictions, when imposed on Petri nets will result in efficient algorithms. We look at some of these properties, focussing on the relationship between the properties and the resulting complexity of the reachability problem in Petri nets satisfying these properties.

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Chapter 1

Introduction

Petri nets were introduced by C.A.Petri in 1962 in his PhD thesis [2]. They are one of the most popular formal models of concurrent systems. A Petri net is a mathematical model of a parallel system, in the same way that a finite automaton is a mathematical model of a sequential system. Properties such as deadlock freedom, liveness, fairness etc. can be studied formally using Petri nets.

For any mathematical model possessing high expressive power, we expect algorithms for analysis of such models to be of high complexity. Research on Petri nets has pointed out that this is indeed the case. Even though most interesting properties are decidable for arbitrary nets, the decision algorithms are extremely inefficient. Many attempts have been made to restrict Petri nets in terms of their structure and/or behaviour to get better (more efficient) algorithms. As expected, expressive power of the restricted class of nets will be lower than those of arbitrary nets.

The problem of *reachability* in Petri nets is an important one. It was observed by Hack [9] and Keller [14] that many other problems like *liveness* and *persistence* are recursively equivalent to the reachability problem. Hack has also observed in [9] that many subproblems and variations of the reachability problem like *submarking reachability problem*, *zero reachability problem* etc. are in fact recursively equivalent to it.

Mayr gave the first complete proof of decidability in 1981 in [22]. Kosaraju [15] later gave a simplified proof. The algorithm given in both [22] and [15] are quite complicated and need non primitive recursive space. Lambert [17] simplified the proof further but the algorithm is still non primitive recursive. The best known lower bound for this problem is exponential space by Lipton [21]. Many subclasses of Petri nets have been studied and primitive recursive algorithms have been obtained for the reachability problem of these subclasses. The survey by Esparza and Nielsen [7] contains a list of such results. The purpose of this thesis is to study some of these subclasses and the complexity of the reachability problem of such subclasses.

The subclasses for which complexity of the reachability problem is known can be classified as follows based on the technique used to design the algorithm for solving reachability.

- Combinatorial techniques In these subclasses, reachability algorithm is designed based on some combinatorial properties satisfied by the Petri nets in the concerned subclass. 1-safe Petri nets, whose complexity of the reachability problem was given by Cheng, Esparza and Palsberg in [4] is an example of this.

- State equation technique In these subclasses, the reachability question is studied in terms of a system of equations. For general Petri nets, existence of solution to this system of equations is a necessary condition. Properties satisfied by Petri nets in these subclasses are then used to prove that it is also sufficient and thus, the reachability problem is reduced to checking feasibility of a system of equations. Normal Petri nets, introduced by Yamasaki in [33] is an example of this type of subclass. Howell, Rosier and Yen established the complexity of the reachability problem in Normal Petri nets and the closely related Sinkless Petri nets in their paper [12].

- Linear algebraic techniques In these subclasses, results from linear algebra are used to reason about Petri nets. Live T-systems, which are studied in [6], is an example of this. Linear algebraic techniques have also been applied successfully to other problems related to Petri nets. [23] contains many such results. [19] by Lautenbach, [5] by Desel and [31] by Silva et. al. are good references for uses of linear algebraic techniques and linear programming in Petri net theory.

- Other advanced algebraic techniques In [3], Cardoza, Lipton and Meyer used results from ring of polynomials to establish the complexity of the reachability problem in a subclass of Petri nets called reversible nets.

1.1 Organization of the thesis

Rest of this thesis is organized as follows. Section 1.2 introduces the notations used throught this thesis and gives some preliminary definitions. This is followed by a result known as exchange lemma in section 1.3. In section 1.4, we mention the complexity classes that arise in connection with the reachability problem in various subclasses of Petri nets.

Chapter 2 studies various subclasses of Petri nets and their complexity of the reachability problem. Each section of this chapter is dedicated to a subclass or a family of closely related subclasses. The sections have been ordered in increasing order of the complexity of the respective reachability problem.

In Chapter 3, the concept of S-Variants is introduced. Some subclasses of Petri nets are identified based on properties pertaining to S-Variants and the complexity of the reachability problem in these subclasses is studied.

Chapter 4 concludes with a summary of the results and some potential directions for further work.

1.2 Notation and preliminaries

Let $\mathcal{Z}$ be the set of integers, $\mathcal{Z}_{0+}$ the set of natural numbers and $\mathcal{Z}_{+}$ the set of positive integers. A finite Petri net is a 4-tuple $\mathcal{N} = (P, T, Pre, Post)$ where:

- P is a set of m places,
- T is a set of n transitions,
- Pre and $Post$ are the incidence functions:

$$\begin{aligned} Pre : P \times T &\rightarrow [0..D] \text{ (representing arcs going from places to transitions)},\\ Post : P \times T &\rightarrow [0..D] \text{ (representing arcs going from transitions to places)}. \end{aligned}$$

If $t \in T$ is a transition, by ${}^\bullet t$, we denote the set of places that have arcs leading to t, i.e., ${}^\bullet t = \{p \in P \mid Pre(p,t) \neq 0\}$. ${}^\bullet t$ is called the set of *input places* of t. Similarly, $t^\bullet$ denotes the *output places* of t: $t^\bullet = \{p \in P \mid Post(p,t) \neq 0\}$. The input and output transitions of a place p (${}^\bullet p$ and $p^\bullet$ respectively) are defined similarly. For a subset $T' \subseteq T$ of transitions, ${}^\bullet T'$ ($T'^\bullet$) denotes the union of input places (output places) of all transitions in T'. The set of input and output transitions of a subset $P' \subseteq P$ of places is defined similarly.

For the purpose of complexity analysis, we assume the Petri net is presented as two matrices, one for Pre and one for $Post$. This has size $2mnd$ bits, where d is defined to be $\log D$. The $m \times n$ *incidence matrix* of the net $\mathbf{N} = [n_{ij}]$ ($1 \leq i \leq m$, $1 \leq j \leq n$) is given by $n_{ij} = Post(p_i, t_j) - Pre(p_i, t_j)$.

A function $M : P \rightarrow \mathcal{Z}_{0+}$ is called a *marking*. A *net system* $(\mathcal{N}, M_0, M_f)$ is a Petri net N with an initial marking M_0 and a final marking M_f. A transition $t \in T$ is *enabled* at marking M iff for all $p \in P : M(p) \geq Pre(p,t)$. If $t \in T$ is enabled at a marking M, then t may be *fired* yielding a new marking M' given by the equation $M'(p) = M(p) - Pre(p,t) + Post(p,t)$ for all $p \in P$ or $M' = M + \mathbf{N}\mathbf{t}$, where $\mathbf{t}$ is the characteristic vector of the transition t. $M \xrightarrow{t} M'$ denotes that M' is reached from M by firing t. For a subset $S \subseteq P$ of places, $M(S)$ denotes the sum of tokens in places of S under M: $M(S) = \sum_{p \in S} M(p)$.

A finite sequence of transitions, $\sigma = t_1 t_2 \ldots t_r$ is a *finite firing sequence* of $(\mathcal{N}, M_0, M_f)$ iff there exist markings $M_1, M_2, \ldots, M_r$ such that $\forall i$, $1 \leq i \leq r : M_{i-1} \xrightarrow{t_i} M_i$. Its *Parikh vector* $\overline{\sigma} : T \rightarrow \mathcal{Z}_{0+}$ has as the i'th component the number of occurrences in σ of transition t_i. In the above case, we have $M_r = M_0 + \mathbf{N}\overline{\sigma}$. We will also use $\overline{\sigma}(U)$ to denote the number of occurrences in σ of transitions from a subset of transitions U. For a firing sequence σ, $\mathcal{A}(\sigma)$ is called the alphabet of σ and denotes the set of transitions occurring in σ.

We say that the marking M_r is *reachable* from M_0 by firing σ: $M_0 \xrightarrow{\sigma} M_r$. The *reachability set* $R(\mathcal{N}, M_0) = \{M : P \rightarrow \mathcal{Z}_{0+} \mid \exists \text{ finite firing sequence } \sigma : M_0 \xrightarrow{\sigma} M\}$ denotes the set of all markings reachable from M_0.

Definition 1 (Reachability problem)**.** Given a net system $(\mathcal{N}, M_0, M_f)$, the reachability problem is to decide if $M_f \in R(\mathcal{N}, M_0)$. Let $m_0 = \log M_0$ and $m_f = \log M_f$. From the discussion above, it is clear that the input to the problem is of size $N = 2mnd + m_0 + m_f$ bits.

We now introduce some vector notation which is convenient for studying Petri nets. We will use **1** to denote a vector of all 1's: the dimension of the vector will be clear from the context. Similarly **0** denotes a vector of all 0's. If T is a finite set and $U \subseteq T$, then $\mathbf{e}[U]$ is the characteristic vector which has entry 1 in components corresponding to elements of U and 0 entries everywhere else. For two vectors $\mathbf{X}$ and $\mathbf{Y}$ with n components, $\mathbf{X} \geq \mathbf{Y}$ means that for $1 \leq i \leq n$, $\mathbf{X}(i) \geq \mathbf{Y}(i)$. $\mathbf{X} > \mathbf{Y}$ means that $\mathbf{X} \geq \mathbf{Y}$ and $\mathbf{X} \neq \mathbf{Y}$. $\mathbf{X} \leq \mathbf{Y}$ and $\mathbf{X} < \mathbf{Y}$ are similarly defined.

S-invariants and T-invariants of a Petri net are useful in analysis of many properties. These are fundamental in nature and used widely in analysis of Petri nets.

Definition 2 (S-invariant)**.** Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net. An integer vector (mapping) $\mathbf{K} : P \to \mathcal{Z}$ is a S-invariant iff for all $t \in T$,

$$\sum_{p \in P} \mathbf{K}(p)\,(Post(p,t) - Pre(p,t)) = 0 \ .$$

Informally, a S-invariant is a linear combination of number of tokens in each place such that its value doesn't change when transitions fire.

Definition 3 (T-invariant)**.** Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net. An integer vector (mapping) $\mathbf{J} : T \to \mathcal{Z}$ is a T-invariant iff for all $p \in P$,

$$\sum_{t \in T} \mathbf{J}(t)\,(Post(p,t) - Pre(p,t)) = 0 \ .$$

If all entries of a T-invariant $\mathbf{J}$ are non-negative (positive), then it is called a *semi-positive* (resp. *positive*) T-invariant. If σ is a finite firing sequence of $(\mathcal{N}, M_0, M_f)$ with Parikh vector $\overline{\sigma} = \mathbf{J}$, then, by definition of T-invariant, we get $M_0 \xrightarrow{\sigma} M_0$. Thus, semi-positive and positive T-invariants denote firing sequences whose net effect is zero on every place.

Sometimes, we will view Petri nets as special graphs. It will be convenient to extend of the concept of paths and circuits to Petri nets as well.

Definition 4. Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net. A nonempty sequence $x_1 x_2 \ldots x_k$ of nodes is a path of $\mathcal{N}$ if for all $1 \leq i < k$, it satisfies

$$x_i \text{ is a place } \Rightarrow Pre(x_i, x_{i+1}) \neq 0$$

and

$$x_i \text{ is a transition } \Rightarrow Post(x_{i+1}, x_i) \neq 0 \ .$$

A path leading from a node x to a node y is a circuit if no element occurs more than once and either $Post(x, y) \neq 0$ (if x is a place) or $Pre(y, x) \neq 0$ (if x is a transition). If γ is a circuit, P_γ denotes the set of places in γ.

If we ignore directions of arcs and treat places and transitions as vertices, we will get the underlying undirected graph of a Petri net. Throught this thesis, we only deal with Petri nets whose underlying undirected graph has one connected component.

1.3 Exchange Lemma

Firing sequences do not provide full information about the causal relationship between transition firings. If a marking M enables a sequence $t_1 t_2$, it is not necessarily the case that t_2 can be fired only after t_1. If ${}^\bullet t_1 \cup t_1{}^\bullet$ and ${}^\bullet t_2 \cup t_2{}^\bullet$ are disjoint, then $t_2 t_1$ is also enabled at M. The Exchange Lemma given in [6] provides a more general condition under which transitions of a firing sequence can be exchanged. We reproduce it here without proof.

Lemma 1 (Lemma 2.14 in [6])**.** *Let U and V be disjoint subsets of transitions of a Petri net satisfying ${}^\bullet U \cap V^\bullet = \emptyset$. Let σ be a firing sequence such that $\mathcal{A}(\sigma) \subseteq U \cup V$, where $\mathcal{A}(\sigma)$ denotes the alphabet of σ. If $M \xrightarrow{\sigma} M'$, then $M \xrightarrow{\sigma|_U \sigma|_V} M'$.*

1.4 Complexity classes

In this thesis, we study the complexity of the reachability problem in different sub classes of Petri nets. For each class of Petri nets, we give an upper bound and/or a lower bound for the reachability problem in that class. Upper bound will be given by giving an algorithm to solve reachability in a class of Petri nets and proving that the algorithm is in some complexity class. Lower bound will be given by proving that reachability problem for a class of Petri nets is hard for some complexity class.

Below is a list of complexity classes used in this thesis.

- **NL**: Non deterministic logarithmic space.
- **NLIN**: Non deterministic linear space.
- **P**: Deterministic polynomial time.
- **NP**: Non deterministic polynomial time.
- **PSPACE**: Deterministic polynomial space.
- **EXPTIME**: Deterministic exponential time.
- **EXPSPACE**: Deterministic exponential space.

Chapter 2

Structural and behavioural restrictions

Structural and behavioural restrictions can be imposed on Petri nets to get efficient reachability algorithms. In this chapter, we will study several such restrictions and see the complexity of the reachability problem for the resulting subclass of Petri nets.

2.1 S-systems

In this section, we study a simple class of Petri nets called *S-systems*. We will prove that reachability problem for S-systems is **NL**-hard.

S-systems are those Petri nets where every transition has exactly one input place and one output place.

Definition 5 (S-systems)**.** A Petri net $\mathcal{N} = (P, T, Pre, Post)$ is a S-net iff $|{}^\bullet t| = 1 = |t^\bullet|$ for every transition t and the range of Pre and $Post$ is $\{0, 1\}$. A net system $(\mathcal{N}, M_0, M_f)$ is a S-system iff $\mathcal{N}$ is a S-net.

If M_0 puts one token in exactly one place and zero tokens in all other places, the S-system $(\mathcal{N}, M_0, M_f)$ can be thought of as a finite state automaton over the alphabet T with set of states P. For this reason, S-systems with 1 token are also called *state machine graphs*.

The fundamental property of S-systems is that all reachable markings have the same total number of tokens as the initial marking.

Proposition 1 (Proposition 3.2 in [6])**.** *Let $\mathcal{N} = (P, T, Pre, Post)$ be a S-net and M_0 an initial marking. If M is a reachable marking, then $M_0(P) = M(P)$.*

Proof. Since $\mathcal{N}$ is a S-net, the occurrence of a transition removes a token from one place and adds a token to one place (these two places may be the same). So the total number of tokens remains unchanged. □

2.1.1 Lower bound

Now, we will prove that reachability problem for S-systems is **NL**-hard. For this, we will reduce the well known **NL**-hard problem directed s-t connectivity (see e.g., [25]) to reachability in S-systems.

Theorem 1. *The reachability problem for S-systems is* ***NL****-hard.*

Proof. We will reduce directed s-t connectivity problem to reachability in S-systems. Suppose $G = (V, E)$ is a directed graph with set of vertices V and edges E. Suppose s and u are two vertices and we are to determine if there is a path from s to u.

Define a Petri net $\mathcal{N} = (P, T, Pre, Post)$ as follows. $P = V$ and $T = E$. $Pre(p, t) = 1$ iff there is a $p' \in V$ such that $t = (p, p') \in E$, and $Pre(p, t) = 0$ otherwise. $Post(p, t) = 1$ iff there is a $p' \in V$ such that $t = (p', p) \in E$, and $Post(p, t) = 0$ otherwise. It is easy to see that $\mathcal{N}$ is a S-net. Define initial marking M_0 as $M_0(s) = 1$ and $M_0(p) = 0$ if $p \neq s$. Define final marking M_f as $M_f(u) = 1$ and $M_f(p) = 0$ if $p \neq u$. It is easy to see that u is reachable from s in G iff M_f is reachable from M_0 in $(\mathcal{N}, M_0, M_f)$. It is also easy to see that $(\mathcal{N}, M_0, M_f)$ can be computed from $G = (V, E)$ in logarithmic space. Since the constructed net system has 1 token, this hardness proof holds for state machine graphs also. □

2.1.2 Shortest sequence

Now, we will look at a result that guarantees the existence of "short" firing sequences that reach reachable markings. These results are proved in [6] and play an important role in establishing upper bound of reachability in S-systems.

Lemma 2 (Lemma 3.10 in [6])**.** *Let $\mathcal{N} = (P, T, Pre, Post)$ be a S-net, M a marking of $\mathcal{N}$, and $\mathbf{X} : T \to \mathcal{Z}_{0+}$ a vector such that $M + \mathbf{N} \cdot \mathbf{X} \geq \mathbf{0}$. If every circuit of $\mathcal{N}$ contains a transition t such that $\mathbf{X}(t) = 0$, then there is a firing sequence σ such that $M \xrightarrow{\sigma} M'$ and $\overline{\sigma} = \mathbf{X}$.*

Proof. We proceed by induction on $|\mathbf{X}|$, the sum of entries in $\mathbf{X}$.

Base. If $|\mathbf{X}| = 0$ then $\mathbf{X} = \mathbf{0}$. We can take $\sigma = \epsilon$, the empty sequence.

Step. $|\mathbf{X}| \geq 1$. Let $\langle \mathbf{X} \rangle$ denote the set of transitions t satisfying $\mathbf{X}(t) > 0$. We claim that a transition of $\langle \mathbf{X} \rangle$ is enabled at M.

Since T is finite and every circuit contains a transition that doesn't belong to $\langle \mathbf{X} \rangle$, there is a place s such that some transition $t \in s^\bullet$ belongs to $\langle \mathbf{X} \rangle$ but no transition in ${}^\bullet s$ belongs to $\langle \mathbf{X} \rangle$. Since $M + \mathbf{N} \cdot \mathbf{X} \geq \mathbf{0}$ by assumption, we have

$$\begin{array}{rcll} 0 & \leq & M(s) + \sum_{u \in {}^\bullet s} \mathbf{X}(u) - \sum_{u \in s^\bullet} \mathbf{X}(u) & \\ & = & M(s) - \sum_{u \in s^\bullet} \mathbf{X}(u) & ({}^\bullet s \cap \langle \mathbf{X} \rangle = \emptyset) \\ & \leq & M(s) - \mathbf{X}(t) & (t \in s^\bullet) \end{array}$$

Since $\mathbf{X}(t) > 0$, we have $M(s) > 0$, and therefore t is enabled at M, which proves the claim.

Let $M \xrightarrow{t} M''$. Then $M'' + \mathbf{N} \cdot (\mathbf{X} - \mathbf{e}[t]) = M + \mathbf{N} \cdot \mathbf{X} \geq 0$. We can apply the induction hypothesis to $(\mathbf{X} - \mathbf{e}[t])$, which yields a sequence $M'' \xrightarrow{\tau} M'$ satisfying $\overline{\tau} = \mathbf{X} - \mathbf{e}[t]$. Taking $\sigma = t\tau$, the result follows. □

Theorem 2 (Theorem 3.11 in [6]). *Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a S-net with n transitions. Suppose M_0 is an initial marking with $M_0(P) = b$. If M is a reachable marking then there exists a firing sequence σ of length at most $b \cdot n$ such that $M_0 \xrightarrow{\sigma} M$.*

Proof. Let $\mathbf{X} \geq \mathbf{0}$ be an integer-valued solution of the system of equations $M_0 + \mathbf{N} \cdot \mathbf{X} = M$. Such a solution exists since M is reachable from M_0. Assume further that $\mathbf{X}$ is minimal with respect to the order $\leq$.

We claim that $\langle \mathbf{X} \rangle$ doesn't contain the set of transitions of any circuit of $\mathcal{N}$. Suppose that $\mathbf{X}(t) > 0$ for all transitions t of some circuit γ. Then, from $\mathbf{X}$, we can remove one occurrence each of every transition in γ to get another vector $\mathbf{Y}$ such that $M_0 + \mathbf{N} \cdot \mathbf{Y} = M$ (this is possible since every transition has exactly one input and one output place). Then $\mathbf{Y} < \mathbf{X}$, contradicting the minimality of $\mathbf{X}$. This proves the claim.

By Lemma 2, there is a firing sequence σ such that $M_0 \xrightarrow{\sigma} M$ and $\overline{\sigma} = \mathbf{X}$. We will prove that no transition occurs more than b times in σ, which implies the result of this theorem.

Let t be an arbitrary transition of the alphabet $\mathcal{A}(\sigma)$ of σ. Define

$$\begin{aligned} U &= \{u \in \mathcal{A}(\sigma) \mid \text{there is a path } u \ldots t \text{ containing only transitions of } \mathcal{A}(\sigma)\} \\ V &= \mathcal{A}(\sigma) \setminus U \end{aligned}$$

We claim that ${}^\bullet U \cap V^\bullet = \emptyset$. Otherwise, we end up with a contradiction as follows. Suppose $p \in {}^\bullet U \cap V^\bullet$. Then, there is a transition $t_v \in V$ such that $p \in t_v{}^\bullet$. Also, there is a transition $t_u \in U$ such that $p \in {}^\bullet t_u$. This means there is a path $t_v \ldots t_u \ldots t$ containing only transitions of $\mathcal{A}(\sigma)$, contradicting the definitions of U and V. Therefore, ${}^\bullet U \cap V^\bullet = \emptyset$.

By definition, we also have that U and V are disjoint and $\mathcal{A}(\sigma) \subseteq U \cup V$. Now we can apply the Exchange Lemma, and conclude that the sequence $\sigma|_U$ is also enabled at M_0.

By definition, $t \in U$. Let s be the unique place in $t^\bullet$. Since no circuit contains only transitions of $\mathcal{A}(\sigma)$, no transition in $s^\bullet$ belongs to U. So no transition of $s^\bullet$ occurs in $\sigma|_U$. Since no place of $\mathcal{N}$ has more than b tokens in any marking reachable from M_0 (Proposition 1) and $t \in {}^\bullet s$, we can infer that t occurs at most b times in $\sigma|_U$. Therefore, t occurs at most b times in σ, which finishes the proof. □

2.1.3 Upper bound

We are now ready to give an upper bound for reachability in S-systems. Algorithm 1 is a non deterministic algorithm that decides whether a given target marking is reachable from a given initial marking in the given S-net.

Claim 1. *Suppose $(\mathcal{N}, M_0, M_f)$ is a S-system where M_f is reachable. Then at least one execution path of Algorithm 1 accepts. If M_f is not reachable, then no execution path of Algorithm 1 accepts.*

Proof. If M_f is reachable, by Theorem 2, there is a firing sequence σ of length at most $n \cdot b$ such that $M_0 \xrightarrow{\sigma} M_f$. The execution path of Algorithm 1 that guesses σ will accept.

If M_f is not reachable, then the test in line 9 of Algorithm 1 always fails in any execution path. Hence, in this case, all execution paths reject. □

Require: Net System $(\mathcal{N}, M_0, M_f)$ is a S-system.
Let $b = M_0(P)$.
Let $n =$ number of transitions in $\mathcal{N}$.
Let $i = 1$.
Let $currentMarking = M_0$.
while $i \leq n \cdot b$ **do**
  $i \leftarrow i + 1$. Non deterministically guess a transition t.
  If t is not enabled in $currentMarking$, halt and reject.
  If t is enabled in $currentMarking$, $currentMarking \leftarrow currentMarking + \mathbf{N} \cdot \mathbf{e}[t]$.
  if $currentMarking = M_f$ **then**
    Halt and accept.
  else
    Continue with the loop.
  end if
end while
We never reached M_f in the above loop. Halt and reject.
Ensure: Accept if M_f is reachable form M_0, reject otherwise.

Algorithm 1: Reachability algorithm for S-systems

Claim 2. *Algorithm 1 runs in linear space.*

Proof. Input to the algorithm requires $2mn + m_0 + m_f$ bits. Since $\log b \leq m_0$, space needed for the variable i is $\log n + m_0$ bits. Space required for the variable $currentMarking$ is m_0 bits. Space needed to guess a transition is $\log n$ bits and space needed for updating $currentMarking$ is m_0 bits. Hence, the whole algorithm runs in linear space. □

Claim 1 and Claim 2 imply that reachability problem for S-systems is in **NLIN**. For a state machine graph, there is only one token in any marking. Hence, b is Algorithm 1 is 1. Therefore, variable i needs only $\log n$ bits. Also in this case, $currentMarking$ can be represented in $\log m$ bits (just store the place in which the token is present). This will make Algorithm 1 run in log space. Along with Theorem 1, this implies that reachability problem for state machine graphs is **NL**-complete.

2.2 T-systems

Here, we study a class of Petri nets called *T-systems*. A Petri net where every place has exactly one input and one output transition is called a T-system.

Definition 6. Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net. $\mathcal{N}$ is defined to be a T-net iff for every place p, $|{}^{\bullet}p| = |p^{\bullet}| = 1$ and the range of Pre and $Post$ is $\{0, 1\}$. A net system $(\mathcal{N}, M_0, M_f)$ is a T-system iff $\mathcal{N}$ is a T-net.

We will study the complexity of reachability in a subclass of T-systems called *live T-systems*.

Definition 7. Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net. The net system $(\mathcal{N}, M_0, M_f)$ is a live system iff, for every reachable marking M and every transition t, there exists a marking M' reachable from M such that M' enables t.

If γ is a circuit and M is a marking of some Petri net $\mathcal{N}$, $M(\gamma)$ denotes the number of tokens M puts in the places of γ. The following result from [6] demonstrates a fundamental property of T-systems.

Proposition 2 (Proposition 3.14 in [6]). *Suppose γ is a circuit of a T-system $(\mathcal{N}, M_0, M_f)$. For every reachable marking M, $M(\gamma) = M_0(\gamma)$.*

Proof. Let t be a transition. If t does not belong to γ, then the firing of t doesn't change the number of tokens of any place in γ since $\mathcal{N}$ is a T-net. If t is in γ, then exactly one input place and one output place of t are in γ. So the firing of t removes one token from a place in γ and adds one token to a place in γ. In both cases, the token count of γ doesn't change. ☐

If in a net system $(\mathcal{N}, M_0, M_f)$, there is a circuit γ such that $M_0(\gamma) = 0$, then from Proposition 2, γ is never marked in any reachable marking and hence no transition in γ is ever enabled. Thus, we can conclude that in a live T-system, every circuit is initially marked.

T-invariants of T-nets have a special property. This property will also be used in the study of complexity of reachability problem in T-systems.

Proposition 3 (Proposition 3.16 in [6]). *Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a T-net. A vector $\mathbf{J} : T \to \mathcal{Z}$ is a T-invariant iff $J = (x, \ldots, x)$ for some x.*

Proof. Since $\mathcal{N}$ is a T-net, every place p has exactly one input transition t_p and one output transition t'_p. Therefore

$$\sum_{t \in T} \mathbf{J}(t)(Post(p,t) - Pre(p,t)) = \mathbf{J}(t'_p) - \mathbf{J}(t_p) \ .$$

It follows that $\mathbf{J}$ is a T-invariant iff $\mathbf{J}(t'_p) = \mathbf{J}(t_p)$ for every place p. This is the case iff $\mathbf{J}(t) = \mathbf{J}(t')$ for every two transitions t and t'; in other words, iff there exists a number x such that $\mathbf{J}(t) = x$ for every transition t of $\mathcal{N}$. ☐

Another concept used in reachability analysis is the following relationship between two markings of a Petri net.

Definition 8 (Definition 2.32 in [6]). Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net and $\mathbf{K}$ is an arbitrary rational valued vector that satisfies for all transitions $t \in T$ the following equation:

$$\sum_{p \in P} \mathbf{K}(P)(Post(p,t) - Pre(p,t)) = 0 \tag{2.1}$$

Two markings M and L of $\mathcal{N}$ are said to agree on all S-invariants if $\mathbf{K} \cdot M = \mathbf{K} \cdot L$ for every rational valued vector $\mathbf{K}$ that satisfies (2.1).

The following result from [6] characterizes markings that agree on all S-invariants.

Theorem 3 (Theorem 2.34 in [6]). *Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net. Two markings M and L of $\mathcal{N}$ agree on all S-invariants iff the equation $M + \mathbf{N} \cdot \mathbf{X} = L$ has some rational-valued solution for $\mathbf{X}$.*

Proof. ($\Rightarrow$): Since M and L agree on all S-invariants, they also agree on a basis $\{\mathbf{K}_1, \ldots, \mathbf{K}_f\}$ of the solution space of (2.1). For every vector $\mathbf{K}_j$ of this basis we have $\mathbf{K}_j \cdot (L - M) = \mathbf{0}$. A well known theorem of linear algebra states that the columns of $\mathbf{N}$ include a basis of the space of solutions of the homogeneous system

$$\mathbf{K}_j \cdot \mathbf{X} = \mathbf{0} \ \ (1 \leq j \leq f) \ .$$

Therefore, $(L - M)$ is a rational linear combination of these columns, i.e., $\mathbf{N} \cdot \mathbf{X} = (L - M)$ has some rational valued solution for $\mathbf{X}$.

($\Leftarrow$): Let $\mathbf{K}$ be any rational valued vector satisfying (2.1). Then we have $\mathbf{K} \cdot \mathbf{N} = \mathbf{0}$ and hence

$$\mathbf{K} \cdot L = \mathbf{K} \cdot M + \mathbf{K} \cdot \mathbf{N} \cdot \mathbf{X} = \mathbf{K} \cdot M \ .$$

□

2.2.1 Upper bound for live T-systems

Now, we are ready to give an upper bound on complexity of reachability problem in live T-systems. The following result from [6] gives a characterization of reachable markings, from which we can easily get a polynomial time algorithm for solving reachability.

Theorem 4 (Theorem 3.21 in [6]). *Suppose $(\mathcal{N}, M_0, M_f)$ is a live T-system. M_f is reachable iff it agrees with M_0 on all S-invariants.*

Proof. ($\Rightarrow$): Straightforward application of (2.1).

($\Leftarrow$): Theorem 3 implies that there exists a rational valued vector $\mathbf{X}$ satisfying $M_f = M_0 + \mathbf{N} \cdot \mathbf{X}$. Since $\mathbf{J} = (1, \ldots, 1)$ is a T-invariant (Proposition 3), we have $\mathbf{N} \cdot (\mathbf{X} + \lambda \mathbf{J}) = \mathbf{N} \cdot \mathbf{X}$ for every λ. Therefore, we can further assume that $\mathbf{X} \geq \mathbf{0}$.

Let T denote the set of transitions of $\mathcal{N}$. The proof is divided into two steps:

1. There exists a vector $\mathbf{Y} : T \rightarrow \mathcal{Z}_{0+}$ such that $M_f = M_0 + \mathbf{N} \cdot \mathbf{Y}$.

   Define $\mathbf{Y}$ by $\mathbf{Y}(t) = \lceil \mathbf{X}(t) \rceil$ for every transition t, where, given a rational number x, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Since $M_f = M_0 + \mathbf{N} \cdot \mathbf{X}$, we have for every place p:

   $$M_f(p) = M_0(p) + \mathbf{X}(t_1) - \mathbf{X}(t_2)$$

   where t_1 is the unique transition in ${}^\bullet p$ and t_2 is the unique transition in $p^\bullet$.

   Since both $M_f(p)$ and $M_0(p)$ are integer valued, $\mathbf{X}(t_1) - \mathbf{X}(t_2)$ is an integer. By the definition of $\mathbf{Y}$, we have $\mathbf{X}(t_1) - \mathbf{X}(t_2) = \mathbf{Y}(t_1) - \mathbf{Y}(t_2)$. So

   $$M_f(p) = M_0(p) + \mathbf{Y}(t_1) - \mathbf{Y}(t_2)$$

   and hence $M_f = M_0 + \mathbf{N} \cdot \mathbf{Y}$.

2. $M_0 \xrightarrow{*} M_f$

   By induction on $|\mathbf{Y}|$, the sum of entries of the vector $\mathbf{Y}$ defined in part 1.

   Base. $|\mathbf{Y}| = 0$. Then $\mathbf{Y} = \mathbf{0}$ and $M_f = M_0$.

   Step. $|\mathbf{Y}| \geq 1$. We first show that some transition of $support(\mathbf{Y})$ is enabled at M_0. Let P_Y be the set of places in ${}^\bullet support(\mathbf{Y})$ that are unmarked at M_0.

   If a place $p \in P_Y$ has an input transition that belongs to $support(\mathbf{Y})$, then, since p is unmarked at M_0 and $M_0 + \mathbf{N} \cdot \mathbf{Y} = M_f \geq \mathbf{0}$, some transition in ${}^\bullet p$ belongs to $support(\mathbf{Y})$ as well.

   Every circuit of $\mathcal{N}$ is marked at M_0 because the system $(\mathcal{N}, M_0, M_f)$ is live. Therefore, there exists a path of maximal length containing only places of P_Y and transitions of $support(\mathbf{Y})$. Since, as shown above, every place of P_Y has an input transition that belongs to ${}^\bullet support(\mathbf{Y})$, the path begins with a transition $t \in {}^\bullet support(\mathbf{Y})$. Moreover, no input place of t belongs to P_Y because the path has maximal length. Therefore, every input place of t is marked at M_0, and hence t is enabled.

   Let $M_0 \xrightarrow{t} M_1$. Then, $M_1 + \mathbf{N} \cdot (\mathbf{Y} - \mathbf{e}[t]) = M_f$. Since $|\mathbf{Y} - \mathbf{e}[t]| < |\mathbf{Y}|$, we can apply the induction hypothesis to M_1. So, $M_1 \xrightarrow{*} M_f$. It follows that $M_0 \xrightarrow{t} M_1 \xrightarrow{*} M_f$, which implies $M_0 \xrightarrow{*} M_f$.

□

It is now easy to get a polynomial time algorithm for solving reachability problem in live T-Systems. Suppose $(\mathcal{N}, M_0, M_f)$ is the given live T-System. From Theorem 4, we have that M_f is reachable iff M_f and M_0 agree on all S-invariants. Theorem 3 then implies that M_f is reachable iff there is a rational valued solution $\mathbf{X}$ for the system of equations $M_0 + \mathbf{N} \cdot \mathbf{X} = M_f$. This can be easily checked by Gaussian elimination method. Thus, we conclude that reachability problem for live T-Systems is in **P**.

2.3 Sinkless Petri nets

S-Systems and T-Systems we have studied till now have simple structure and are very restrictive in terms of their expressive power. Now, we will relax the requirements on structure a little and obtain a more general class of Petri nets whose reachability problem is **NP**-complete.

The most general class of Petri nets we study in this section is Sinkless Petri nets. We start with simpler classes of Petri nets called *circuit free* Petri nets and *conflict free* Petri nets and develop the idea progressively. We begin with the definitions.

Definition 9. A Petri net $\mathcal{N} = (P, T, Pre, Post)$ is defined to be circuit free iff $\mathcal{N}$ doesn't have any circuits and range of Pre and $Post$ is $\{0, 1\}$. A net system $(\mathcal{N}, M_0, M_f)$ is called circuit free iff $\mathcal{N}$ is circuit free.

Definition 10. A Petri net $\mathcal{N} = (P, T, Pre, Post)$ is defined to be conflict free iff

1. Range of Pre and $Post$ is $\{0, 1\}$.

2. For every place p for which there are at–least two transitions t_1 and t_2 with $p \in {}^\bullet t_1 \cap {}^\bullet t_2$, we also have $p \in t_1{}^\bullet \cap t_2{}^\bullet$.

Informally, in any marking of a conflict free Petri net, if transitions t_1 and t_2 are enabled, t_2 is enabled even after t_1 fires.

If γ is a circuit in a Petri net $\mathcal{N}$, let P_γ denote the set of places in γ.

Definition 11. Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net and γ is a circuit of $\mathcal{N}$. γ is defined to be a minimal circuit iff P_γ doesn't properly include the set of places of any other circuit.

Definition 12. Suppose $(\mathcal{N}, M_0, M_f)$ is a net system and γ is a circuit in $\mathcal{N}$. γ is said to have a *sink* iff there are reachable markings M,M' and a firing sequence σ such that $M(P_\gamma) \neq 0$, $M \xrightarrow{\sigma} M'$ and $M'(P_\gamma) = 0$. γ is said to be *sinkless* iff it doesn't have a sink. The net system $(\mathcal{N}, M_0, M_f)$ is sinkless iff each of its minimal circuits is sinkless and range of Pre and $Post$ is $\{0, 1\}$.

Reachability algorithm for all the classes of Petri nets studied in this section are based on the following idea. Suppose $(\mathcal{N}, M_0, M_f)$ is a net system and M_f is reachable. If σ is a firing sequence such that $M_o \xrightarrow{\sigma} M_f$, $\overline{\sigma}$ is a positive integral solution for $\mathbf{X}$ in the system of equations

$$M_0 + \mathbf{N} \cdot \mathbf{X} = M_f \tag{2.2}$$

If a structural and/or behavioural property of $\mathcal{N}$ implies that existence of positive integral solution to (2.2) is also sufficient, then reachability problem reduces to checking if there is a positive integral solution to (2.2). This is an instance of integer linear programming and hence is in **NP**.

2.3.1 Circuitless Petri nets

Circuit free Petri nets are a simple class of Petri nets where existence of positive integral solution to (2.2) is a necessary and sufficient condition for reachability. We will start with its simple proof and then move onto more complicated proofs for other classes of Petri nets.

Theorem 5 (Theorem 16 in [24])**.** *Suppose $(\mathcal{N}, M_0, M_f)$ is a circuit free net system. M_f is reachable iff there exists a nonnegative integral solution $\mathbf{X}$ satisfying* (2.2).

Proof. Only sufficiency remains to be shown. Suppose there exists such a solution $\mathbf{X}$. Let $\mathcal{N}_X$ denote the sub net of $\mathcal{N}$ consisting of transitions t such that $\mathbf{X}(t) > 0$, together with their input and output places and their connecting arcs. Let M_{0X} (M_{fX}) denote the sub vector of M_0 (M_f) for places in $\mathcal{N}_X$. Consider the net system $(\mathcal{N}_X, M_{0X}, M_{fX})$ that is circuit free. We claim that at least one transition t is enabled at M_{0X}.

Suppose for contradiction's sake that all transitions are disabled at M_{0X}. Let t be any transition and $p \in {}^\bullet t$ be an input place that doesn't have enough tokens to enable t at M_{0X}. If p has any input transition t_1, t_1 is also disabled at M_{0X}. Let $p_1 \in {}^\bullet t_1$

be an input place of t_1 that doesn't have enough tokens to enable t_1. If p_1 has any input transition t_2, t_2 is also disabled at M_{0X}. Since there are finitely many places and transitions and $\mathcal{N}$ doesn't have circuits, this backtracking ends up at a token free place p that doesn't have any input transitions but has at least one output transition t. Since p doesn't have input transitions, it will never receive any tokens. Since $\mathbf{X}(t) > 0$ and firing of t takes away a token from p, this contradicts the fact that $M_f \geq \mathbf{0}$. Thus, at least one transition t is enabled at M_{0X}.

Now, fire t. Let the resulting marking be $M' = M_0 + \mathbf{N} \cdot \mathbf{e}[t]$. If $\mathbf{X}' = \mathbf{X} - \mathbf{e}[t]$, $M_f = M' + \mathbf{N} \cdot \mathbf{X}'$, and the sub net $\mathcal{N}_{X'}$ is circuit free. Repeat the above process until $\mathbf{X}'$ becomes $\mathbf{0}$. This process will terminate in a finite number of steps since all entries of $\mathbf{X}$ are finite. This proves that M_f is reachable. □

2.3.2 Conflict free Petri nets

Now, we will prove that reachability problem for conflict free Petri nets is **NP** hard. This was first proven by Jones, Landweber and Lien in [13].

Theorem 6 (Theorem 2.6 in [13])**.** *The reachability problem for conflict free Petri nets is **NP** hard.*

Proof. We will reduce CNF-SAT problem to reachability problem for conflict free Petri nets. Let $F = C_1 \wedge C_2 \wedge \cdots \wedge C_r$ be a CNF formula where each C_i is a disjunction of (some of) the variables $x_1, \ldots, x_s$ and their negations $\overline{x_1}, \ldots, \overline{x_s}$. We construct a Petri net $\mathcal{N}_F = (P, T, Pre, Post)$ with $P = \{x_1, \ldots, x_s, C_1, \ldots, C_r\}$, i.e., one place each for each clause and one place for each variable of F. The set of transitions is $T = \{t_1, \ldots, t_s, t'_1, \ldots, t'_s, t_{C1}, \ldots, t_{Cr}\}$. Pre and $Post$ are defined as follows:

- $Pre(C_i, t_{Ci}) = 1$ for $1 \leq i \leq r$. $Pre(p, t) = 0$ for all other cases.
- $Post(x_i, t_i) = 1 = Post(x_i, t'_i)$ for $1 \leq i \leq s$. $Post(x, t) = Post(x, t') = 0$ for all other cases.
- $Post(C_i, t_j) = 1$ if variable x_j appears in clause C_i. $Post(C_i, t'_j) = 1$ if $\overline{x_j}$ appears in clause C_i. $Post(C, t) = 0$ in all other cases.

Initial marking M_0 is given by $M_0(p) = 0$ for all places p. Final marking M_f is given by $M_f(p) = 1$ for all places p.

If there is an assignment satisfying F, we can prove that M_f is reachable from M_0 as follows. If the satisfying assignment sets x_i to true, t_i can fire, putting one token in place x_i and one token each in all those places C_j where x_i appears in C_j. If x_i is set to false, t'_i can fire, putting one token in place x_i and one token each in all those places C_j where $\overline{x_i}$ appears in C_j. Thus, if a satisfying assignment exists, we can fire transitions t_i, t'_i such that each place x_i receives exactly one token and each place C_j receives at least one token. Extra tokens in C_j can then be removed by firing t_{Cj} as needed. Therefore, if F is satisfiable, there is at–least one firing sequence through which final marking M_f is reachable from M_0.

On the other hand, suppose M_f is reachable from M_0. If $M_0 \xrightarrow{\sigma} M_f$, then for each i, $1 \leq i \leq s$, exactly one of t_i, t'_i is fired in σ. If neither of them are fired, the place x_i will not have any tokens after σ fires, a contradiction. If both t_i and t'_i fire or if one/both of them fire more than once, the place x_i will have more than one token, again a contradiction (note that there are no transitions that can take away tokens from place x_i). Now, consider the assignment that assigns true to the variable x_i if transition t_i fires in σ, and assigns false to the variable x_i if transition t'_i fires in σ. This assignment satisfies all the clauses of F (otherwise, if clause C_j is not satisfied, the place C_j will not have any tokens after σ fires, a contradiction).

Therefore, F is satisfiable iff M_f is reachable from M_0 in the Petri net $\mathcal{N}_F$. It is also easy to see that $\mathcal{N}_F$, M_0 and M_f can be constructed from F in polynomial time. Since the constructed net is conflict free, it follows that reachability problem in conflict free Petri nets in **NP** hard. ☐

2.3.3 Upper bound for Sinkless Petri nets

We will now give upper bounds for reachability problem in conflict free Petri nets and sinkless Petri nets. Sinkless Petri nets were introduced by Yamasaki in [33]. Howell and Rosier [11] gave a **NP** algorithm for solving reachability in conflict free Petri nets. It was followed by Howell, Rosier and Yen's **NP** algorithm for solving reachability in normal and sinkless Petri nets [12]. Normal Petri nets are those that behave like sinkless Petri nets for all initial markings. Hence, normal Petri nets form a subclass within sinkless Petri nets. Since conflict free Petri nets are a subclass of normal Petri nets (see [18]), we will directly study the algorithm for normal and sinkless Petri nets.

In all the results that follow, *ordinary petri nets* mean those Petri nets for which range of Pre and $Post$ is $\{0, 1\}$. Following three results from [33] establish the fundamental properties based on which algorithm for reachability in sinkless Petri nets are developed.

Lemma 3 (Lemma 3.1 in [33]). *Suppose $\mathcal{N} = (P, T, Pre, Post)$ is an ordinary Petri net and M_0 is an initial marking. Let σ be a firing sequence such that $M_0 + \mathbf{N} \cdot \overline{\sigma} \geq \mathbf{0}$ and no transition occurring in σ is enabled in M_0. Then there exists a token free circuit $\gamma = p_1 t_1 \ldots p_r t_r p_1$ such that $\overline{\sigma} \geq \overline{\tau}$ where $\tau = t_1 \ldots t_r$.*

Proof. Let t be a transition occurring in σ. Since t is not enabled in M_0, there exists a place p such that $M_0(p) = 0$ and $p \in {}^\bullet t$. Since $M_0 + \mathbf{N} \cdot \overline{\sigma} \geq \mathbf{0}$ and t occurs in σ, there must be another transition $t' \in {}^\bullet p$ that also occurs in σ. Since t' is also not enabled in M_0, we will have another token free place $p' \in {}^\bullet t'$ and so on. Since there are finitely many places and transitions, this backtracking will stop when we reach back p. Consequently, we can get a token free circuit $\gamma = p_1 t_1 \ldots p_r t_r p_1$ such that $\overline{\sigma} \geq \overline{\tau}$ where $\tau = t_1 \ldots t_r$. ☐

Lemma 4 (Lemma 3.2 in [33]). *Suppose $\mathcal{N} = (P, T, Pre, Post)$ is an ordinary Petri net and M_0 is an initial marking. For every finite firing sequence σ, a rearrangement of σ is enabled at M_0 if*

1. $M_0 + \mathbf{N} \cdot \overline{\sigma} \geq \mathbf{0}$.
2. *For each firing sequence η enabled at M_0 and each circuit $\gamma = p_1 t_1 \ldots p_r t_r p_1$ with $\tau = t_1 \ldots t_r$, if $\overline{\sigma} \geq \overline{\eta\tau}$, then the token count of γ is non zero at the marking $M_0 + \mathbf{N} \cdot \overline{\eta}$.*

Proof. We proceed by induction on length of σ. If $\sigma = \epsilon$, the empty sequence, the result is clear. If $\sigma \neq \epsilon$, from Lemma 3, there exists a transition t that occurs in σ and is enabled at M_0. Let $\sigma = utu'$. Then $(M_0 + \mathbf{N} \cdot \mathbf{e}[t]) + \mathbf{N} \cdot \overline{uu'} \geq \mathbf{0}$, and $M_0 + \mathbf{N} \cdot \mathbf{e}[t]$ and uu' satisfy condition (2). By induction hypothesis, a rearrangement v of uu' is enabled at $M_0 + \mathbf{N} \cdot \mathbf{e}[t]$. Thus, tv is a rearrangement of σ and is enabled at M_0. □

Lemma 5 (Theorem 3.3 in [33])**.** *Suppose $\mathcal{N} = (P, T, Pre, Post)$ is an ordinary Petri net and M_0 is an initial marking. If there are no token free circuits in any reachable marking, then the reachability set is given by*

$$R(\mathcal{N}, M_0) = \{M \mid M = M_0 + \mathbf{N} \cdot \overline{\sigma} \textit{ for some firing sequence } \sigma\}$$

Proof. By Lemma 4, M is reachable from M_0 iff $M = M_0 + \mathbf{N} \cdot \overline{\sigma}$ for some firing sequence σ. □

Lemma 5 requires that there be no token free circuit in any reachable marking. As seen in Definition 12, sinkless net systems may have token free circuits. We are only ensured that once a minimal circuit is marked, it will never be unmarked again. Howell, Rosier and Yen [12] showed how to extend Lemma 5 to sinkless Petri nets.

Lemma 6 (Lemma 4.2 in [12])**.** *Let $\mathcal{N} = (P, T, Pre, Post)$ be a sinkless Petri net, and let $\mathcal{N}' = (P, T', Pre', Post')$ be another Petri net such that $T' \subseteq T$ and Pre' ($Post'$) is the restriction of Pre ($Post$) to $P \times T'$. Let M_0 be an initial marking of $\mathcal{N}$ and σ be a firing sequence of $\mathcal{N}$ such that $M_0 \xrightarrow{\sigma} M$ and each $t \in T'$ is enabled at some intermediate marking during firing of σ from M_0. Then the net $\mathcal{N}'$ with initial marking M doesn't have any token free circuits in any reachable marking.*

Proof. Suppose for contradiction's sake that some reachable marking M' in $R(\mathcal{N}', M)$ has a token free circuit γ'. Since M' is reachable in $\mathcal{N}'$, it must also be reachable from M in $\mathcal{N}$. Let $\sigma\sigma'$ be a firing sequence of $\mathcal{N}$ such that $M_0 \xrightarrow{\sigma} M \xrightarrow{\sigma'} M'$. Also, since γ' is a circuit in $\mathcal{N}'$, it must also be a circuit in $\mathcal{N}$. Thus, there must be a minimal circuit γ of $\mathcal{N}$ such that $P_\gamma \subseteq P_{\gamma'}$. Clearly, $M'(\gamma) = 0$. Let p be any place in γ, and let t be the transition following p in γ'. Since $t \in T'$, t must have been enabled at some marking M'' reached during the firing of σ from M_0 in $\mathcal{N}$. Thus, $M''(p) > 0$, so $M''(\gamma) > 0$. Since $M'(\gamma) = 0$ and M' is reachable from M'', γ has a sink - a contradiction. □

Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a sinkless Petri net, M_0 an initial marking and $P = \{p_1, \ldots, p_m\}$ and $T = \{t_1, \ldots, t_n\}$. Let $\tau = t_{j_1} \cdots t_{j_r}$ be any sequence of distinct transitions from T. In [12], characteristic system of inequalities for $\mathcal{N}, M_0$ and τ is defined as $S(\mathcal{N}, M_0, \tau) = S_0 \cup S_1 \cup \cdots \cup S_r$, where $S_0 = \{\mathbf{x}_0 = M_0\}$, $S_h = \{\mathbf{x}_{h-1}(i) \geq Pre(p_i, t_{j_h}), \mathbf{x}_h = \mathbf{x}_{h-1} + \mathbf{N}_h \cdot \mathbf{y}_h \mid 1 \leq i \leq m\}$, and $\mathbf{N}_h$ is $\mathbf{N}$ restricted to $\{t_{j_1}, \ldots, t_{j_h}\}$ for $1 \leq h \leq r$. The variables in S are the components of m dimensional column vectors $\mathbf{x}_0, \ldots, \mathbf{x}_r$ and the h dimensional column vectors $\mathbf{y}_h$, $1 \leq h \leq r$.

Lemma 7 (Lemma 4.3 in [12])**.** *Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a sinkless Petri net and M_o is an initial marking. Let M be any marking of $\mathcal{N}$. Then there is a firing sequence σ such that $M_0 \xrightarrow{\sigma} M$ iff there is some sequence $\tau = t_{j_1} \cdots t_{j_r}$ of distinct* *transitions in T such that $S(\mathcal{N}, M_0, \tau)$ has a non negative integer solution in which $\mathbf{x}_r = M$. Furthermore, σ and τ can be chosen such that $\sigma = \sigma_1 \dots \sigma_r$, where $M_0 = \mathbf{x}_0 \xrightarrow{\sigma_1} \mathbf{x}_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_r} \mathbf{x}_r = M$, t_{j_h} is enabled at $\mathbf{x}_{h-1}$, $\sigma_h \in \{t_{j_1}, \dots, t_{j_h}\}^*$, and $\mathbf{y}_h(h')$ gives the number of times $t_{j_{h'}}$ occurs in σ_h, for $1 \leq h' < h \leq r$.*

Proof. ($\Rightarrow$) Let $M \in R(\mathcal{N}, M_0)$. Then for some firing sequence σ, $M_0 \xrightarrow{\sigma} M$. Let $\sigma = \sigma_1 \dots \sigma_r$ such that for $1 \leq h \leq r$, σ_h begins with some transition t_{j_h} that doesn't occur in $\sigma_1, \dots, \sigma_{h-1}$ and σ_h contains only transitions from the set $\{t_{j_1}, \dots, t_{j_h}\}$. Let $\tau = t_{j_1} \cdots t_{j_r}$. By letting $\mathbf{y}_h(h')$ be the number of times $t_{j_{h'}}$ occurs in σ_h, $1 \leq h' \leq h \leq r$, it is easily seen that $S(\mathcal{N}, M_0, \tau)$ has a non negative integer solution for which $M_0 = \mathbf{x}_0 \xrightarrow{\sigma_1} \mathbf{x}_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_r} \mathbf{x}_r = M$.

($\Leftarrow$) We will show by induction on r that for any sequence $\tau_r = t_{j_1} \cdots t_{j_r}$ of r distinct transitions from T and any marking M, if $S(\mathcal{N}, M_0, \tau_r)$ has a non negative integer solution for which $x_r = M$, then $M_0 = \mathbf{x}_0 \xrightarrow{\sigma_1} \mathbf{x}_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_r} \mathbf{x}_r = M$ such that t_{j_h} is enabled at x_{h-1}, $\sigma_h \in \{t_{j_1}, \dots, t_{j_h}\}^*$, and $t_{j_{h'}}$ occurs $\mathbf{y}_h(h')$ times in σ_h for $1 \leq h' \leq h \leq r$. The lemma will then follow.

Base. Let $r = 0$. Then $S = (\mathcal{N}, M_0, \tau_0) = \{\mathbf{x}_0 = M_0\}$. Clearly, the only solution to S is M_0, and $M_0 \xrightarrow{\epsilon} M_0$.

Hypothesis. Let r be some positive integer, and assume that for any sequence of distinct transitions $\tau = t_{j_1} \cdots t_{j_{r-1}}$ from T and any marking M, if $S(\mathcal{N}, M_0, \tau_{r-1})$ has a non negative integer solution in which $\mathbf{x}_{r-1} = M$, then $M_0 = \mathbf{x}_0 \xrightarrow{\sigma_1} \mathbf{x}_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_{r-1}} \mathbf{x}_{r-1} = M$ such that t_{j_h} is enabled at x_{h-1}, $\sigma_h \in \{t_{j_1}, \dots, t_{j_h}\}^*$, and $t_{j_{h'}}$ occurs $\mathbf{y}_h(h')$ times in σ_h for $1 \leq h' \leq h \leq r-1$.

Step. Let $\tau_r = t_{j_1} \cdots t_{j_r}$ be a sequence of distinct transitions in T, M be any marking of $\mathcal{N}$, $\tau_{r-1} = t_{j_1} \dots t_{j_{r-1}}$, and $T' = \{t_{j_1}, \dots, t_{j_r}\}$. Then $S(\mathcal{N}, M_0, \tau_r) = S(\mathcal{N}, M_0, \tau_{r-1}) \cup \{\mathbf{x}_{r-1}(i) \geq \mathit{Pre}(p_i, t_{j_r}), \mathbf{x}_r = \mathbf{x}_{r-1} + \mathbf{N}_r \cdot \mathbf{y}_r \mid 1 \leq i \leq m\}$. Suppose $S(\mathcal{N}, M_0, \tau_r)$ has a non negative integer solution in which $x_r = M$. Since any solution of $S(\mathcal{N}, M_0, \tau_r)$ is clearly a solution to $S(\mathcal{N}, M_0, \tau_{r-1})$, from the induction hypothesis, $M_0 = \mathbf{x}_0 \xrightarrow{\sigma_1} \mathbf{x}_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_{r-1}} \mathbf{x}_{r-1}$ such that t_{j_h} is enabled at x_{h-1}, $\sigma_h \in \{t_{j_1}, \dots, t_{j_h}\}^*$, and $t_{j_{h'}}$ occurs $\mathbf{y}_h(h')$ times in σ_h for $1 \leq h' \leq h \leq r-1$. Let $\sigma' = \sigma_1 \dots \sigma_{r-1}$. Since $\mathbf{x}_{r-1}(i) \geq \mathit{Pre}(p_i, t_{j_r})$ for $1 \leq i \leq m$, t_{j_r} is enabled at $\mathbf{x}_{r-1}$. Thus, each $t \in T'$ is enabled at some intermediate marking during firing of σ' from M_0. So from Lemma 6, $(P, T', \mathit{Pre}', \mathit{Post}')$ has no token free circuit in any marking reachable from $\mathbf{x}_r$, where Pre' (Post') is Pre (Post) restricted to $P \times T'$. Since $M = \mathbf{x}_r = \mathbf{x}_{r-1} + \mathbf{N}_r \cdot \mathbf{y}_r$ for some non negative integer vector $\mathbf{y}_r$, from Lemma 4, $\mathbf{x}_{r-1} \xrightarrow{\sigma_r} \mathbf{x}_r = M$ from some $\sigma_r \in T'^*$ containing $\mathbf{y}_r(h')$ occurrences of $t_{j_{h'}}$ for $1 \leq h' \leq r$. Thus, $M_0 = \mathbf{x}_0 \xrightarrow{\sigma_1} \mathbf{x}_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_r} \mathbf{x}_r = M$ such that t_{j_h} is enabled at x_{h-1}, $\sigma_h \in \{t_{j_1}, \dots, t_{j_h}\}^*$, and $t_{j_{h'}}$ occurs $\mathbf{y}_h(h')$ times in σ_h for $1 \leq h' \leq h \leq r$. □

Now, we are ready to prove that reachability problem for sinkless Petri nets is in **NP**.

Theorem 7 (Theorem 4.2 in [12]). *The reachability problem for sinkless (normal) Petri nets is **NP** complete.*

Proof. From Theorem 6, the reachability problem for conflict free Petri nets is **NP** hard. Since any conflict free Petri net is normal [18], it follows that the reachability problem for normal Petri nets is **NP** hard. Since any normal Petri net is sinkless, we need only show that reachability problem for sinkless Petri nets is in **NP**. We use the following non deterministic algorithm for deciding reachability in any given sinkless net system $(\mathcal{N}, M_0, M_f)$. First, guess a sequence τ of r distinct transitions from the set of transitions T. Then construct $S(\mathcal{N}, M_0, \tau)$ in polynomial time. Next, construct $S = S(\mathcal{N}, M_0, \tau) \cup \{\mathbf{x}_r = M_f\}$. Since integer linear programming is in **NP**, we can guess a solution to S and verify it in polynomial time. Clearly S has a non negative integer solution iff $S(\mathcal{N}, M_0, \tau)$ has a non negative integer solution in which $\mathbf{x}_r = M_f$. From Lemma 7, there is a τ such that $S(\mathcal{N}, M_0, \tau)$ has a non negative integer solution in which $\mathbf{x}_r = M_f$ iff $M_f \in R(\mathcal{N}, M_0)$. Therefore, the reachability problem for sinkless (normal) Petri nets in **NP** complete. □

Most of the results in this chapter are for ordinary Petri nets. As observed by Yamasaki in [33], Lemma 3, Lemma 4, Lemma 5 can be made to work with little modifications for Petri nets where range of Pre is $\{0, 1\}$ but range of $Post$ is $\mathcal{Z}_{0+}$. Consequently, Lemma 6, Lemma 7 and Theorem 7 also go through for such Petri nets.

2.4 Free choice Petri nets

In this chapter, we will look at free choice Petri nets. In [8], Esparza proved that reachability in live and bounded free choice Petri nets is **NP**-complete.

Definition 13. A Petri net $\mathcal{N} = (P, T, Pre, Post)$ is a *free choice* Petri net if $Pre(p, t) \neq 0$ implies $Pre(p', t') \neq 0$ for any $p' \in {}^\bullet t$ and $t' \in p^\bullet$. A net system $(\mathcal{N}, M_i, M_f)$ if a free choice net system iff $\mathcal{N}$ is a free choice Petri net. If, in addition, every reachable marking of $(\mathcal{N}, M_0, M_f)$ enables at least one transition, then it is called a live free choice system. If, in addition, there exists a positive integer b such that every reachable marking puts at most b tokens in each place, then a live system $(\mathcal{N}, M_0, M_f)$ is called a live and bounded free choice system. It is sometimes referred to as live and b-bounded free choice system.

Free choice Petri nets as defined here are sometimes referred to as extended free choice nets in the literature. In a free choice net, if transitions t_1 and t_2 have a common input place and t_1 is enabled at some marking, then t_2 is enabled at that marking too.

The **NP** hardness lower bound for reachability in live and bounded free choice Petri nets is divided into two parts. In the first part, the CNF-satisfiability problem is reduced to what is called the constrained reachability problem of live and bounded free choice Petri nets. In the second part, constrained reachability problem is reduced to the reachability problem in live and bounded free choice Petri nets.

Definition 14 ([8]). The *constrained reachability* problem is defined as follows.

Given: A live and bounded free choice Petri net system $(\mathcal{N}, M_0, M_f)$, two subsets $T_{=1}$ and $T_{\geq 1}$ of transitions of $\mathcal{N}$.

To decide: Is there a firing sequence σ such that $M_0 \xrightarrow{\sigma} M_f$ and σ contains each transition of $T_{=1}$ exactly once and each transition of $T_{\geq 1}$ at least once?

2.4.1 Lower bound for constrained reachability problem

Given a CNF formula ϕ, a literal is a boolean variable or it's negation. A clause is a disjunction of literals and ϕ is the conjunction of clauses. A CNF formula is identified with the set of clauses that appear in it. A clause is identified with the set of literals that appear in it.

Given a CNF formula $\phi = \{C_1, C_2, \ldots, C_c\}$, over variables $x_1 \ldots, x_v$, we assume w.l.o.g. that no clause contains both a variable and it's negation, and that for every $1 \leq i \leq v$, there is a clause that contains either x_i or $\overline{x_i}$. In [8], Esparza constructed a system $(\mathcal{N}, M_0, M_f)$, two subsets $T_{=1}$ and $T_{\geq 1}$ of transitions. $\mathcal{N}$ is constructed in several steps. To begin with, we have the empty net. At each step, new places, transitions or even new subnets are added to $\mathcal{N}$.

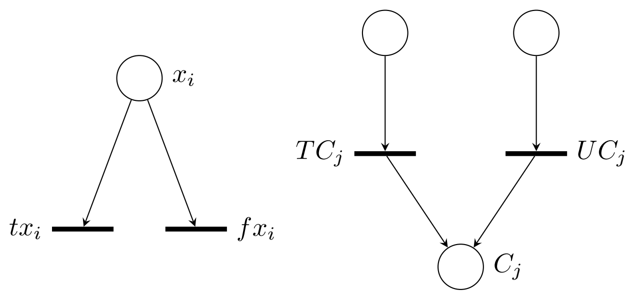

Figure 2.1: The net Nx_i (left) and NC_j (right)

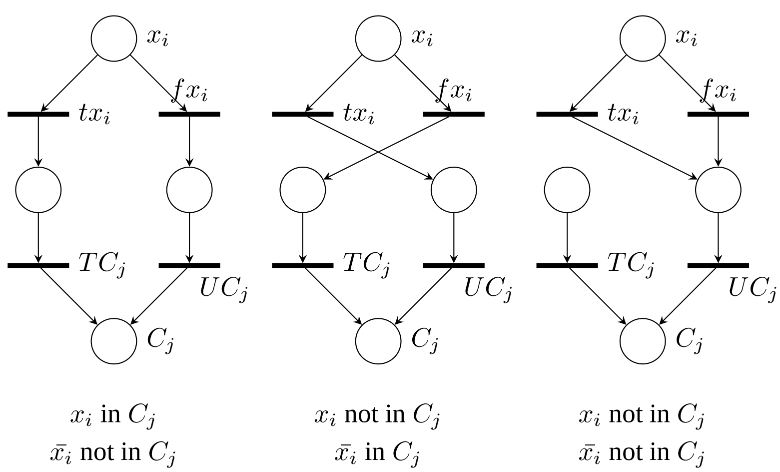

Figure 2.2: Connection from Nx_i to NC_j

- For every variable x_i, add to $\mathcal{N}$ the net Nx_i shown in Fig. 2.1.

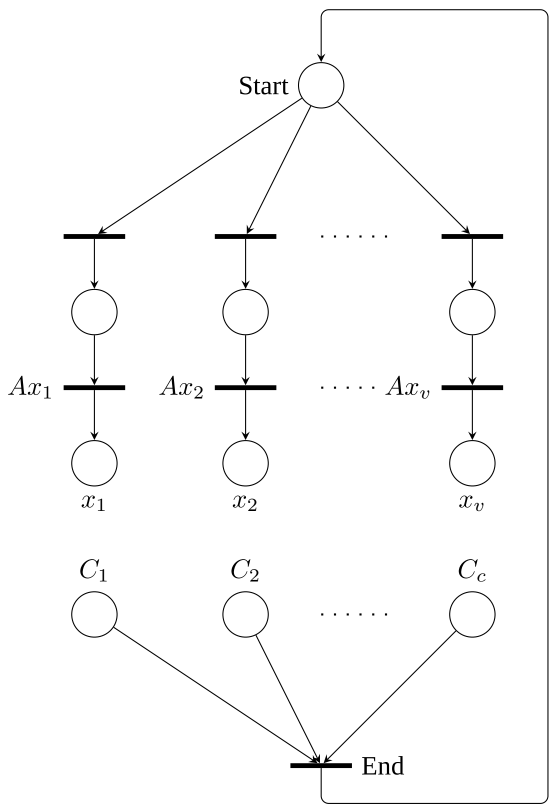

Figure 2.3: Connection from NC_1,...,NC_m to Nx_1, ...,Nx_n

- For every clause C_j, add to $\mathcal{N}$ the net NC_j as shown in Fig. 2.1.
- For each variable x_i and every clause C_j, connect the net Nx_i to the net NC_j as shown in Fig. 2.2, according to three possible cases: (1) x_i appears in C_j but $\overline{x_i}$ does not; (2) $\overline{x_i}$ appears in C_j but x_i does not; (3) Neither x_i not $\overline{x_i}$ appears in C_j.
- Connect the places $C_1, \dots, C_c$ to the places $x_1, \dots, x_v$ as shown in Fig. 2.3.

This concludes the construction of $\mathcal{N}$. M_0 and M_f are both equal to the marking that puts one token in the place *Start* and no tokens anywhere else. Finally, $T_{\geq 1} = \{TC_1, \dots, TC_c\}$ and $T_{=1} = \{Ax_1, \dots, Ax_v\}$.

It is a routine exercise to prove that $(\mathcal{N}, M_0, M_f)$ constructed above is a live and 1-bounded free choice Petri net. The intuition behind this construction is briefly explained below. Let σ be a firing sequence of $(\mathcal{N}, M_0, M_f)$ in which transitions of $T_{=1}$ occur exactly once and transitions of $T_{\geq 1}$ occur at least once.

- The occurrence of the transition Ax_i denotes that x_i is going to be assigned a truth value.
- The nets Nx_i are used to determine the assignment of the variables. Since the transitions of the set $T_{=1}$ occur exactly once in σ, for every $1 \leq i \leq v$ either tx_i or fx_i occurs in σ, but not both. In this way, σ determines a unique truth assignment A_σ defined by: $A_\sigma(x_i) = \mathrm{true}$ if tx_i occurs in σ, and $A_\sigma(x_i) = \mathrm{false}$ if fx_i occurs in σ.
- After assigning a value to a variable, σ updates the truth values of the clauses. The connections between each pair of nets Nx_i and NC_j are chosen with the following intended meaning: the occurrence of TC_j in σ sets C_j to true, while occurrence of UC_j leaves it unchanged. Therefore, C_j is true under A_σ iff the transition TC_j occurs at least once in σ.

The following lemma follows from the above construction.

Lemma 8 ([8]). *Constrained reachability problem in live and bounded free choice nets is **NP**-hard.*

Proof. Given a CNF formula ϕ, construct the corresponding system $(\mathcal{N}, M_0, M_f)$ as explained above. It is easy to see that this construction can be done in polynomial time.

Suppose there is a firing sequence $M_0 \xrightarrow{\sigma} M_f$ satisfying the conditions of the problem. Since every transition of $T_{\geq 1}$ occurs at least once in σ, the truth assignment A_σ makes all clauses true, which implies that ϕ is satisfiable.

Conversely, let ϕ be a satisfiable formula. We take an assignment that makes ϕ true and use it to construct a firing sequence σ such that $M_0 \xrightarrow{\sigma} M_f$ and σ satisfies the conditions of the problem. The sequence σ is the concatenation of sequences $\sigma_1, \sigma_2, \dots, \sigma_v$. Each σ_i starts with firing of one of the output transitions of the place Start, followed by the corresponding Ax_i transition and the transition tx_i or fx_i, according to the assignment, and ends with the transition End. Due to the way nets Nx_i and NC_j are connected, σ contains every transition of $T_{\geq 1}$ at least once. □

2.4.2 Constrained reachability to reachability

We now see how to reduce constrained reachability to reachability problem in live and bounded free choice Petri nets. Given a live and 1-bounded free choice system $(\mathcal{N}, M_0, M_f)$, two subsets of transitions $T_{=1}$ and $T_{\geq 1}$, Esparza [8] constructed a system $(\mathcal{N}', M_0', M_f')$ such that M_f is reachable from M_0 in $\mathcal{N}$ under the constraints given by $T_{=1}$ and $T_{\geq 1}$ iff M_f' is reachable from M_0' in $\mathcal{N}'$.

In order to define $(\mathcal{N}', M_0', M_f')$, some building blocks and a composition operation is defined. The blocks are shown in Fig. 2.4 and Fig. 2.5. The following two lemmata

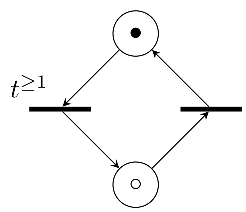

Figure 2.4: The system $(\mathcal{N}^{\geq 1}, M_0^{\geq 1}, M^{\geq 1})$

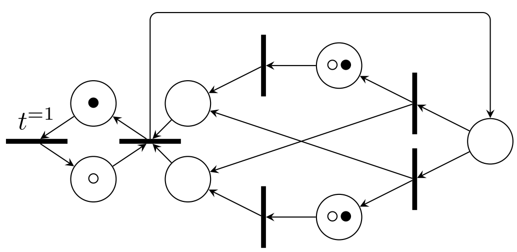

Figure 2.5: The system $(\mathcal{N}^{=1}, M_0^{=1}, M^{=1})$

can be easily proved by inspecting all enabled firing sequences.

Lemma 9 ([8]). *Let $(\mathcal{N}^{\geq 1}, M_0^{\geq 1}, M^{\geq 1})$ and $t^{\geq 1}$ be as shown in Fig. 2.4. $(\mathcal{N}^{\geq 1}, M_0^{\geq 1}, M^{\geq 1})$ is a live and 1-bounded free choice system and satisfies the following property: there exists a firing sequence σ with $M_0^{\geq 1} \xrightarrow{\sigma} M^{\geq 1}$ containing r times the transition $t^{\geq 1}$ iff $r \geq 1$.*

Lemma 10 ([8]). *Let $(\mathcal{N}^{=1}, M_0^{=1}, M^{=1})$ and $t^{=1}$ be as shown in Fig. 2.5. $(\mathcal{N}^{=1}, M_0^{=1}, M^{=1})$ is a live and 1-bounded free choice system and satisfies the following property: there exists a firing sequence σ with $M_0^{=1} \xrightarrow{\sigma} M^{=1}$ containing r times the transition $t^{=1}$ iff $r = 1$.*

The composition operation is defined on nets in the following way. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two disjoint nets, and let t_1 and t_2 be transitions of $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively. The *merge* of t_1 and t_2 is the operation consisting of the following three parts.

- Put $\mathcal{N}_1$ and $\mathcal{N}_2$ side by side.
- Remove t_1 and t_2 together with their incident arcs.
- Add a new transition t; let the preset (postset) of t be the union of presets (postsets) of t_1 and t_2.

Let $\mathcal{N}$ be the net obtained after the above operation. The set of places of $\mathcal{N}$ is the disjoint union of places of $\mathcal{N}_1$ and $\mathcal{N}_2$. Therefore, a marking of $\mathcal{N}$ is characterized by its' projection onto these two sets of places. We denote by (M_1, M_2) that marking that projects onto markings M_1 of $\mathcal{N}_1$ and M_2 of $\mathcal{N}_2$. The composition operation is extended to systems as follows: the system obtained after the merge of transitions t_1 and t_2 of the systems $(\mathcal{N}_1, M_{01}, M_{f1})$ and $(\mathcal{N}_2, M_{02}, M_{f2})$ is the system $(\mathcal{N}, (M_{01}, M_{02}), (M_{f1}, M_{f2}))$, where $\mathcal{N}$ is the net defined above. We are now ready to construct $(\mathcal{N}', M_0', M_f')$. Take $(\mathcal{N}, M_0, M_f)$ and merge iteratively each transition of $T_{\geq 1}$ with $t^{\geq 1}$ of a fresh copy of $(\mathcal{N}^{\geq 1}, M_0^{\geq 1}, M^{\geq 1})$. Then, merge iteratively each transition of $T_{=1}$ with the transition $t^{=1}$ of a fresh copy of $(\mathcal{N}^{=1}, M_0^{=1}, M^{=1})$. Using Lemma 9 and Lemma 10, the following lemma can be proven.

Lemma 11 ([8]). *Let $(\mathcal{N}, M_0, M_f)$, $T_{\geq 1}$ and $T_{=1}$ be an instance of constrained reachability problem, and let $\mathcal{N}', M_0', M_f'$ be as described above. $(\mathcal{N}', M_0', M_f')$ is a live and 1-bounded free choice system, and M_f can be reached from M_0 satisfying the constraints corresponding to $T_{\geq 1}$ and $T_{=1}$ iff M_f' can be reached from M_0'.*

For a rigorous proof the above lemma, reader is referred to the original paper [8].

The reductions from CNF satisfiability to constrained reachability and from constrained reachability to reachability in live and 1-bounded free choice Petri nets proves the **NP**-hardness of the problem.

2.4.3 Upper bound

Esparza [8] also proved that reachability problem in live and bounded free choice Petri nets is in **NP**. The main result used for this is the following theorem due to Lee, Kodama and Kumagai [20].

Theorem 8. *Let $(\mathcal{N}, M_0, M_f)$ be a live and bounded free choice system. M_f is reachable from M_0 iff the following conditions hold:*

1. *The equation $M_f = M_0 + \mathbf{NX}$ has an integer solution, and*
2. *Every trap of the subnet of $\mathcal{N}$ generated by the transitions of the support of $\mathbf{X}$ is marked at M_0.*

Now, the following non-deterministic polynomial time algorithm can be used for the reachability problem (from [8]).

1. Guess a subset T' of transitions of $\mathcal{N}$.
2. Check that every trap of the subnet of $\mathcal{N}$ generated by T' is marked at M_0 (a polynomial algorithm for this problem can be found in [6]).

3. Guess a solution of $M_f = M_0 + \mathbf{N}\mathbf{X}$ where $\mathbf{X}$ has support T' (it is well known that a solution exists iff a solution of polynomial size exists, see for instance [10]).

2.5 1-Safe Petri nets

In this section, we will study the class of 1-Safe Petri nets. In a 1-Safe net system, every reachable marking has at most 1 token in every place.

Definition 15. A net system $(\mathcal{N}, M_0, M_f)$ is defined to be 1-Safe iff every reachable marking has at most 1 token in every place.

The reachability problem for 1-safe net systems was proved to be **PSPACE** complete by Cheng, Esparza and Palsberg in [4].

2.5.1 Lower bound

Theorem 9 (Theorem 6 in [4])**.** *The reachability problem for 1-Safe Petri nets is **PSPACE** hard.*

Proof. The proof is by reducing the problem of Quantified Boolean Formula to reachability of 1-Safe Petri nets in polynomial time. It is well known that Quantified Boolean Formula is **PSPACE** complete.

The problem is, given a well formed quantified Boolean formula

$$F = (Q_1 x_1)(Q_2 x_2)\cdots(Q_r x_r)E$$

where E is a Boolean expression involving the variables $x_1, x_2, \ldots, x_r$ and each Q_i is either $\exists$ or $\forall$, we have to decide if F is true.

If we are given a quantified Boolean formula F, then we construct a 1-Safe net system $(\mathcal{N}, M_0, M_f)$ such that M_f is reachable iff F is true.

Before constructing the net and the marking, we write F, in polynomial time, into an equivalent closed formula G generated by the grammar:

$$P ::= x \mid \neg P \mid P \wedge P \mid \exists x.P$$

and such that all bound variables in G are distinct.

The construction of the net for G is illustrated in Fig. 2.6, Fig. 2.7 and Fig. 2.8. The construction is compositional. The only complication is the interpretation of variables.

The net G contains the places:

$$\{P\_in, P\_T, P\_F \mid \text{P is an occurrence of a subformula of } G\} \cup$$
$$\{x\_is\_T, x\_is\_F \mid x \text{ is bound in } G\}$$

Transitions are defined in Table 2.1. $x\_in + x\_is\_T \rightarrow x\_T + x\_is\_T$ denotes a transition with input places x_in, x_is_T and output places x_T, x_is_T, with all arc weights equal to 1. For readability, when in the following we name places and

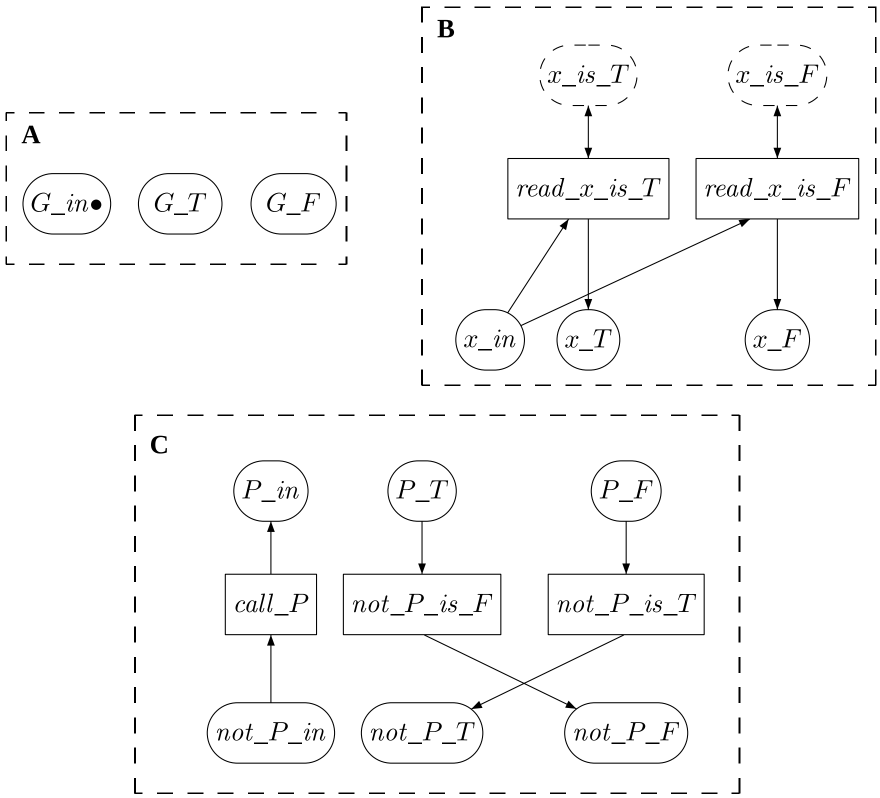

Figure 2.6: Reduction from QBF

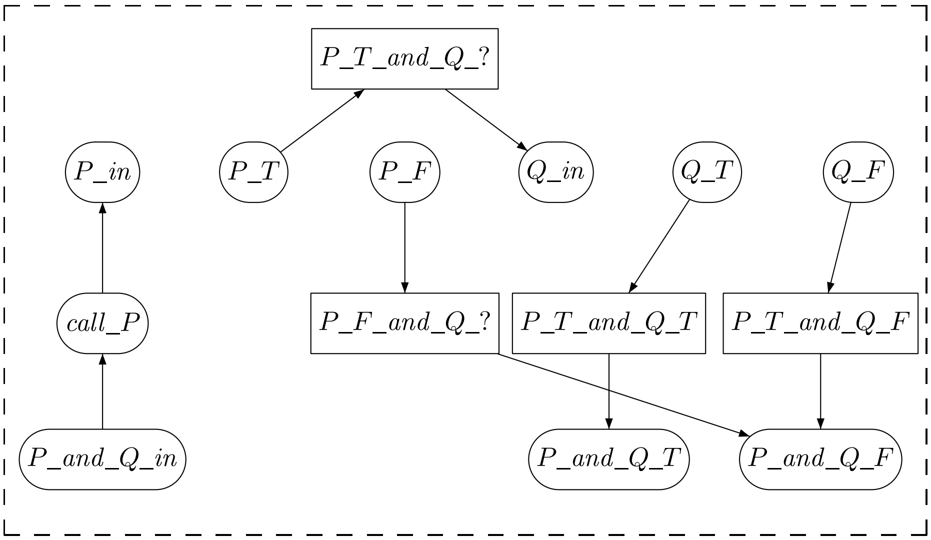

Figure 2.7: Reduction from $P \wedge Q$

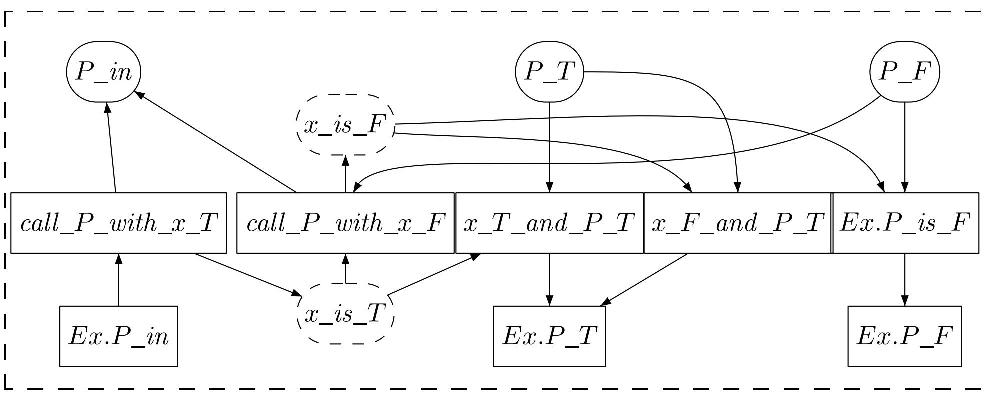

Figure 2.8: Reduction from $\exists x.P$

Occurence	Transitions		
x	$read\_x\_is\_T$	:	$x\_in + x\_is\_T \to x\_T + x\_is\_T$
	$read\_x\_is\_F$	:	$x\_in + x\_is\_F \to x\_F + x\_is\_F$
$\neg P$	$call\_P$	:	$not\_P\_in \to P\_in$
	$not\_P\_is\_F$	:	$P\_T \to not\_P\_F$
	$not\_P\_is\_T$	:	$P\_F \to not\_P\_T$
$P \wedge Q$	$call\_P$	:	$P\_and\_Q\_in \to P\_in$
	$P\_T\_and\_Q\_?$	:	$P\_T \to Q\_in$
	$P\_F\_and\_Q\_?$	:	$P\_F \to P\_and\_Q\_F$
	$P\_T\_and\_Q\_T$	:	$Q\_T \to P\_and\_Q\_T$
	$P\_T\_and\_Q\_F$	:	$Q\_F \to P\_and\_Q\_F$
$\exists x.P$	$call\_P\_with\_x\_T$	:	$Ex.P\_in \to P\_in + x\_is\_T$
	$call\_P\_with\_X\_F$	:	$X\_is\_T + P\_F \to x\_is\_F + P\_in$
	$x\_T\_and\_P\_T$	:	$x\_is\_T + P\_T \to Ex.P\_T$
	$x\_F\_and\_P\_T$	:	$x\_is\_F + P\_T \to Ex.P\_T$
	$Ex.P\_is\_F$	:	$x\_is\_F + P\_F \to Ex.P\_F$

Table 2.1: Definition of transitions

transitions, we write not_P for $\neg P$, P_and_Q for $P \wedge Q$ and $Ex.P$ for $\exists x.P$.

The initial marking is one token in G_in and zero tokens in all other places.

Intuitively, when P_in receives a token, the checking of truth of P begins. When either P_T ("true") or P_F (" false") becomes marked, this checking is completed. Let us consider in turn the construction for each of the productions in the above grammar.

First, consider variable x, see Fig. 2.6, box **B**. The places x_is_T ("x is true") and x_is_F ("x is false") are not part of the net for x but are included to indicate that they will be added when treating quantification that binds x. Note that all occurrences of the same variable x share these two places. The two transitions implement the reading of the current value of x.

Second, consider a negation $\neg P$, see Fig. 2.6, box **C**. The transition $call\_P$ transfers the "control" to the subnet for P. The two other transitions implement the negation.

Third, consider a conjunction $P \wedge Q$, see Fig. 2.7. The transition $call\_P$ transfers the "control" to the subnet for P. The four other transitions implement the conjunction.

Fourth, consider an existential quantification $\exists x.P$, see Fig. 2.8. The places x_is_T ("x is true") and x_is_F ("x is false") are the ones we mentioned above. The transition $call\_P\_with\_x\_T$ assigns true to x and transfers the "control" to the subnet for P. In case P was not true, the transition $call\_P\_with\_x\_F$ assigns false to x and again transfers "control" to the subnet of P.

The final marking M_f puts one token in G_T and zero tokens in all other places. It is now easy to see that the final marking is reachable in the net for G iff G is true. □

2.5.2 Upper bound

Now, we will give a **PSPACE** algorithm for solving reachability problem in 1-Safe Petri nets. This algorithm is also given in [4]. Along with the hardness for **PSPACE**, this will imply that reachability problem for 1-Safe net systems is **PSPACE** complete.

Given a net system $(\mathcal{N}, M_0, M_f)$ that is 1-Safe with m places and n transitions, Algorithm 2 solves reachability.

Require: Net System $(\mathcal{N}, M_0, M_f)$ has m places and is 1-Safe.
1: Let $m =$ number of places in $\mathcal{N}$.
2: Let $n =$ number of transitions in $\mathcal{N}$.
3: Let $i = 1$.
4: Let $currentMarking = M_0$.
5: **while** $i \leq 2^m$ **do**
6:     $i \leftarrow i + 1$. Non deterministically guess a transition t.
7:     If t is not enabled in $currentMarking$, halt and reject.
8:     If t is enabled in $currentMarking$, $currentMarking \leftarrow currentMarking + \mathbf{N} \cdot \mathbf{e}[t]$.
9:     **if** $currentMarking = M_f$ **then**
10:         Halt and accept.
11:     **else**
12:         Continue with the loop.
13:     **end if**
14: **end while**
15: We never reached M_f in the above loop. Halt and reject.
Ensure: Accept if M_f is reachable form M_0, reject otherwise.

Algorithm 2: Reachability algorithm for 1-Safe net systems

Claim 3. *Suppose $(\mathcal{N}, M_0, M_f)$ is a 1-Safe net system where M_f is reachable and $\mathcal{N}$ has m places. Then at least one execution path of Algorithm 2 accepts. If M_f is not reachable, then no execution path of Algorithm 2 accepts.*

Proof. Since there are m places in $\mathcal{N}$ and it is 1-Safe, there are at most 2^m possible distinct markings of $\mathcal{N}$. If M_f is reachable, there is a firing sequence σ such that $M_0 \xrightarrow{\sigma} M_f$. If length of σ is more than 2^m, at least one marking repeats at two positions in the sequence. We can now remove the transitions between these two positions and get a shorter firing sequence that is enabled at M_0 and reaches M_f. We can repeat this process till we get a firing sequence of length at most 2^m. The execution path of Algorithm 2 that guesses this short path will accept.

If M_f is not reachable, then the test in line 9 of Algorithm 2 always fails in any execution path. Hence, in this case, all execution paths reject. □

Claim 4. *Algorithm 2 runs in polynomial space.*

Proof. Input to the algorithm requires $2mn + m_0 + m_f$ bits. Since maximum value stored in i is 2^m, space needed for the variable i is m bits. Space required for the variable $currentMarking$ is m bits. Space needed to guess a transition is $\log n$ bits and space needed for updating $currentMarking$ is m bits. Hence, the whole algorithm runs in polynomial space. □

Claim 3 and Claim 4 imply that reachability problem for 1-Safe nets is in **PSPACE**.

Chapter 3

S-Variants

In this chapter, we will introduce the concept of S-variants and study how S-variants can help in designing algorithms for certain classes of Petri nets. We will characterize the nets that have S-variants and see how far this technique can be pushed with Petri nets that do not have S-variants.

All proofs of general Petri net reachability use a vector $\mathbf{v} \geq \mathbf{1}$ as one of the sufficient conditions for reachability (for example in the conditions for Kosaraju's sufficiency theorem, cf. [27]). An S-variant generalizes this kind of vector with weights.

Definition 16 (S-variants)**.** Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net. An integer vector (mapping) $\mathbf{V} : P \to \mathcal{Z}$ is an S-variant iff for all $t \in T$,

$$\sum_{p \in P} \mathbf{V}(p)\,(Post(p,t) - Pre(p,t)) \geq 1.$$

Informally, an S-variant is a linear combination of number of tokens in each place of the net such that the value of this combination strictly increases with the firing of any transition. If $\mathbf{V}$ is an S-variant and M is a marking, then we define $\mathbf{V}(M) = \sum_{p \in P} \mathbf{V}(p)M(p)$.

3.1 Petri nets with S-variants

In this section, we examine those Petri nets for which S-variants exist. We first give a characterization of such nets in terms of the T-invariant-less nets studied by Kostin [16]. We then give a polynomial space algorithm for solving the reachability problem in such net systems.

3.1.1 Characterizing nets with S-variants

If $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net and $\mathbf{N}$ is its incidence matrix, an S-variant $\mathbf{V}$ is an integral solution to the system of inequalities

$$\mathbf{N}^T \mathbf{v} \geq \mathbf{1}$$

Also, a T-invariant $\mathbf{J}$ is an integral solution to the system of equations

$$\mathbf{Nj} = 0$$

The following theorem characterizes nets that have S-variants by relating it to the non-existence of semi-positive T-invariants.

Theorem 10. *A Petri net has an S-variant iff it does not have any non-trivial semi-positive T-invariant.*

Proof. Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net and $\mathbf{N}$ is its incidence matrix. $\mathcal{N}$ has an S-variant iff the system of inequalities $\mathbf{N}^T\mathbf{v} \geq \mathbf{1}$ has an integral solution. $\mathbf{N}^T\mathbf{v} \geq \mathbf{1}$ has an integral solution iff it has a rational solution (a rational solution can be converted to an integral one by multiplying the solution vector by the least common multiple of the denominators). $\mathbf{N}^T\mathbf{v} \geq \mathbf{1}$ has a rational solution iff $-\mathbf{N}^T\mathbf{v} \leq -\mathbf{1}$ has a rational solution.

Now, we can apply a variant of the Farkas lemma (corollary 7.1e of Schrijver [29]). $-\mathbf{N}^T\mathbf{v} \leq -\mathbf{1}$ has a rational solution iff for each rational row vector $\mathbf{y} \geq \mathbf{0}$ with $\mathbf{y} \cdot -\mathbf{N}^T = \mathbf{0}$, $\mathbf{y} \cdot -\mathbf{1} \geq \mathbf{0}$. But for any non-zero rational row vector $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \cdot -\mathbf{1} < \mathbf{0}$. Hence, $-\mathbf{N}^T\mathbf{v} \leq -\mathbf{1}$ has a rational solution iff for each non-zero rational row vector $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \cdot -\mathbf{N}^T \neq \mathbf{0}$.

Since a rational solution to $\mathbf{y} \cdot -\mathbf{N}^T = \mathbf{0}$ can be converted to an integral solution by multiplying the solution vector by the LCM of the denominators, we can conclude that $-\mathbf{N}^T\mathbf{v} \leq -\mathbf{1}$ has a rational solution iff for each non-zero integral row vector $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \cdot -\mathbf{N}^T \neq \mathbf{0}$. Equivalently, we can state that $\mathcal{N}$ has S-variants iff for each non-zero integral vector $\mathbf{j} \geq 0$, $-\mathbf{Nj} \neq \mathbf{0}$.

By the fact that T-invariants are exactly the solutions of the equation system $\mathbf{Nj} = \mathbf{0}$, we conclude that $\mathcal{N}$ has S-variants iff it does not have any non-trivial semi-positive T-invariant. □

Checking whether a Petri net has S-variant is equivalent to checking if $\mathbf{N}^T\mathbf{v} \geq \mathbf{1}$ has an integral solution. Since any rational solution to $\mathbf{N}^T\mathbf{v} \geq \mathbf{1}$ can be converted to an integral solution, $\mathbf{N}^T\mathbf{v} \geq \mathbf{1}$ has an integral solution iff it has a rational solution. The later can be checked in polynomial time using linear programming. Therefore, checking whether a given Petri net has S-variant can be done in polynomial time.

The example net in Fig. 3.1 is a T-invariant-less net and hence, it has S-variants. E.g., $\mathbf{V} = [3, 2, 0, 1, 2]^T$ is an S-variant for this net.

3.1.2 Reachability algorithm

Kostin gave an algorithm to decide reachability of T-invariant-less nets [16]. However there is no complexity analysis in this paper. The main idea behind our reachability algorithm for nets with S-variants is somewhat different and we are able to analyze its complexity. Given an initial and target marking, an S-variant can be used to bound the length of any firing sequence that starts at the initial marking and ends at the given target marking. The following proposition makes this formal.

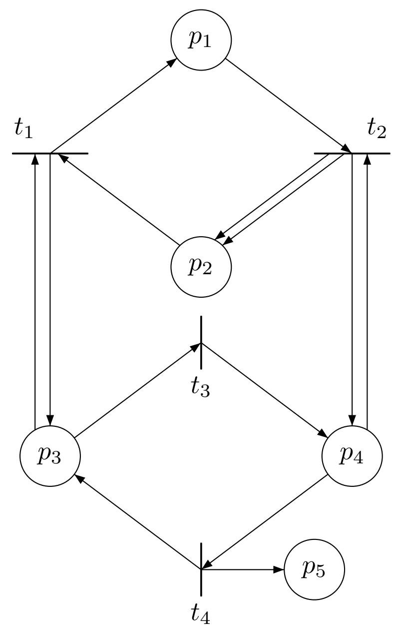

Figure 3.1: Hopcroft and Pansiot's example net

Proposition 4. *Suppose $(\mathcal{N}, M_0, M_f)$ is a net system. If σ is a finite firing sequence of $(\mathcal{N}, M_0, M_f)$ such that $M_0 \xrightarrow{\sigma} M$ and if $\mathbf{V}$ is an S-variant of $\mathcal{N}$, then the length of σ is at most $\mathbf{V}(M) - \mathbf{V}(M_0)$.*

Proof. Suppose $\sigma = t_1 t_2 \dots t_r$ and $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \dots \xrightarrow{t_r} M_r = M$. By definition of S-variant, for $1 \leq i \leq r$, $\mathbf{V}(M_i) \geq \mathbf{V}(M_{i-1}) + 1$. Substituting for all i's starting from r, we get $\mathbf{V}(M_r) \geq \mathbf{V}(M_0) + r$. Hence, $r \leq \mathbf{V}(M) - \mathbf{V}(M_0)$. □

The complexity analysis is based on the following result.

Proposition 5 (Borosh and Treybig [1], Theorem 5)**.** *Let $\mathbf{A}$ be a $n \times r$ integer matrix and $\mathbf{B}$ be an $n \times 1$ integer matrix. Suppose $\mathbf{x}$ denotes a $r \times 1$ vector of variables and the system of equations $\mathbf{A}x = \mathbf{B}$ has a non-trivial positive integral solution. Also suppose that R is the maximum of absolute values of all minors of the augmented matrix $[A \mid B]$. If $\mathbf{A}$ is a full row rank matrix and $n \leq r$, then the system of equations $\mathbf{A}x = \mathbf{B}$ has a positive integral solution where each entry of the solution is at most $Rn + nrR^2$.*

To give an upper bound for our algorithm, we need the following lemma that gives bounds for the S-variant itself. A similar result was used by Rackoff [26] to give exponential space upper bound for the boundedness problem of Petri nets.

Lemma 12. *Suppose $\mathcal{N}$ is a Petri net having an S-variant with m places and n transitions, and its incidence matrix $\mathbf{N}$ has entries in $[0..D]$. Then $\mathcal{N}$ has an S-variant $\mathbf{V}$ such that the absolute value of each entry of $\mathbf{V}$ is $O\left(mn^2(n!)^2D^{2n}\right)$.*

Proof. Since $\mathcal{N}$ has S-variants, an integral solution exists for the system $\mathbf{N}^T\mathbf{v} \geq \mathbf{1}$. If $\mathbf{I}$ is the identity matrix, then an integral solution exists for $\mathbf{N}^T\mathbf{v} \geq \mathbf{1}$ iff a positive integral solution exists for the system of equations $\left[\mathbf{I} \mid \mathbf{N}^T \mid -\mathbf{N}^T\right] \mathbf{v}' = -\mathbf{1}$.

Now we can use Proposition 5 on the system of equations $\left[\mathbf{I} \mid \mathbf{N}^T \mid -\mathbf{N}^T\right] \mathbf{v}' = -\mathbf{1}$. $\left[\mathbf{I} \mid \mathbf{N}^T \mid -\mathbf{N}^T\right]$ has full row rank due to the presence of $\mathbf{I}$. The number of columns of this coefficient matrix is $r = n + 2m \geq n$. The absolute value of minors of the augmented matrix is upper bounded by $R \leq n!D^n$. Hence, Proposition 5 is applicable and $Rn + nrR^2 \leq nn!D^n + n(n+2m)n!^2D^{2n}$. Thus, we have a positive integral solution with each entry being $O\left(mn^2(n!)^2D^{2n}\right)$.

It can be easily seen that a positive integral solution to $\left[\mathbf{I} \mid \mathbf{N}^T \mid -\mathbf{N}^T\right] \mathbf{v}' = -\mathbf{1}$ can be converted to an integral solution for $\mathbf{N}^T\mathbf{v} \geq \mathbf{1}$ without affecting the bounds. So we conclude that if $\mathcal{N}$ has an S-variant, it has one with the absolute value of each entry being $O\left(mn^2(n!)^2D^{2n}\right)$. □

Now, Proposition 4 and Lemma 12 can be combined to give an exponential upper bound on the length of firing sequences.

Proposition 6. *Suppose $(\mathcal{N}, M_0, M_f)$ is a net system with S-variants. Suppose n is the number of transitions and m the number of places in $\mathcal{N}$ and $[0..D]$ is the range of entries in $\mathbf{N}$. If σ is a finite firing sequence of $\mathcal{N}$ such that $M_0 \xrightarrow{\sigma} M_f$, the length of σ is at most $O(m^2n^2(n!)^2D^{2n}max(M_0, M_f))$,*

Proof. Since $\mathcal{N}$ has S-variants, Lemma 12 shows that there is an S-variant $\mathbf{V}$ such that the absolute value of each entry of $\mathbf{V}$ is $O(mn^2(n!)^2D^{2n})$. By Proposition 4, we have

$$\begin{aligned}
\text{length of } \sigma &\leq \mathbf{V}(M_f) - \mathbf{V}(M_0) \\
&= \sum_{p \in P} \mathbf{V}(p)\left(M_f(p) - M_0(P)\right) \\
&\leq \sum_{p \in P} O(mn^2(n!)^2D^{2n})max(M_0, M_f) \\
&\leq mO(mn^2(n!)^2D^{2n})max(M_0, M_f) \\
\text{length of } \sigma &\leq O(m^2n^2(n!)^2D^{2n}max(M_0, M_f))\ .
\end{aligned}$$

□

By Proposition 6, it is easy to see that the following non-deterministic algorithm is correct.

1. Given a net system $(\mathcal{N}, M_0, M_f)$ with S-variants, initialize variable i to 0 and also store the initial marking M_0 in a variable $currentMarking$.
2. while $i \leq$ bound given by Proposition 6, repeat the following steps.
    (a) Non-deterministically guess some transition $t \in T$.

(b) If t is not enabled in the marking stored in $currentMarking$, reject and halt.

(c) If t is enabled, update $currentMarking$ to the marking resulting from firing t from the marking previously stored in $currentMarking$.

(d) if the marking now stored in $currentMarking$ is equal to the target marking, accept and halt, else increment i and continue with the loop.

3. If the target marking is never reached in the above loop, reject.

Now we analyze the space complexity of the above algorithm. The algorithm needs space to store the variables i and $currentMarking$. The maximum value in i will be $O(m^2n^2(n!)^2D^{2n}max(M_0, M_f))$. Hence, the space required for it is $O(\log m + n\log n + 2nd + m_0 + m_f)$ bits. Since D is the maximum number of tokens that can be added to a place by one transition and the maximum number of transitions we consider is $O(m^2n^2(n!)^2D^{2n}max(M_0, M_f))$, the maximum value that will be stored for each place in the variable $currentMarking$ is bounded by $O(m^2n^2(n!)^2D^{2n+1}max(M_0, M_f))$. Hence, the space required for the variable $currentMarking$ is bounded by $O(m\log m + mn\log n + mnd + m(m_0 + m_f))$ bits. It is easy to see that space needed for guessing transitions and calculating resulting markings is dominated by the space required for the variables i and $currentMarking$. Thus, the whole algorithm runs in space $O(m\log m + mn\log n + mnd + m(m_0 + m_f))$ bits.

The input to the algorithm is $mnd + m_0 + m_f$ bits. So the space needed by the algorithm is bounded by a polynomial in the size of the input. By the well known theorem of Savitch [28], we have a deterministic polynomial space algorithm that solves the reachability problem in T-invariant-less Petri nets.

3.2 Lower bound

Now, we will prove that reachability for nets with S-variants is **PSPACE**-hard. In section 2.5, we saw that reachability problem for 1-Safe nets is **PSPACE**-complete. To prove that it is **PSPACE** hard, we saw that there is a polynomial time reduction from Quantified Boolean Formulas to reachability in 1-Safe nets. Here, we will prove that the resulting 1-Safe net also has an S-variant, which proves that reachability for nets with S-variants is **PSPACE**-hard.

Note that nets in Fig. 3.2 and Fig. 3.3 are exactly the same as the nets in section 2.5. We have presented them here in a slightly different format to make it easier to see that these nets have S-variants. Since we are using exactly the same nets as given in section 2.5, we will not repeat proof of correctness of the reduction here, but just prove that resulting nets have S-variants.

In what follows, it will be convenient to think of S-variant as a linear combination of number of tokens in each place of the net. If $\mathcal{N}$ is a net with 3 places p_1, p_2 and p_3, then the expression $2p_1 + 3p_2 - 4p_3$ denotes the S-variant $[2, 3, -4]^T$. An expression like $2p_1 + 3p_2 - 4p_3$ is an S-variant iff value of this expression strictly increases whenever any transition in the net fires.

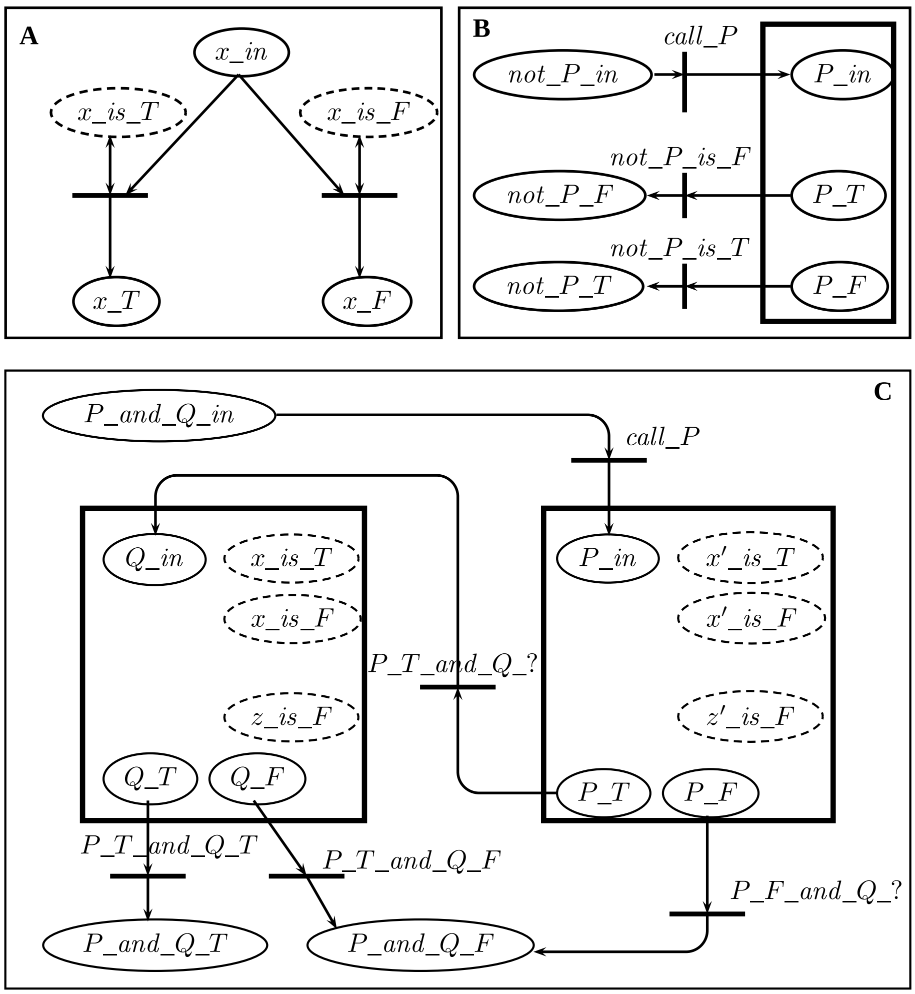

Figure 3.2: Reduction from $P \wedge Q$

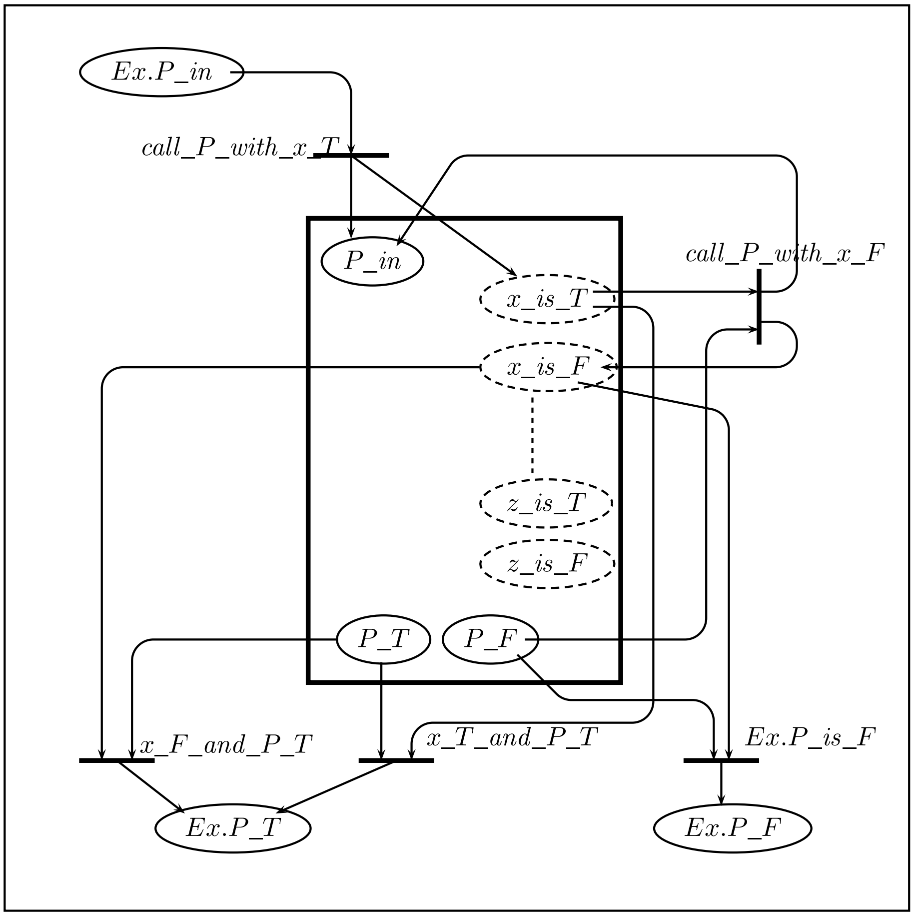

Figure 3.3: S-Variant for $\exists x.P$

Theorem 11. *Suppose G is a Quantified Boolean Formula and $\mathcal{N}_G$ is the corresponding net as given in section 2.5. Then, for each $i \geq 1$, $\mathcal{N}_G$ has a S-variant $\mathbf{V}(i)$ satisfying the following properties:*

1. *$\mathbf{V}(i) + k \cdot G\_in$ is a S-variant of G for $0 \leq k \leq i-1$.*
2. *The co-efficient of G_in in $\mathbf{V}(i)$ is 0.*
3. *There exists a finite number $j_G(i)$ such that removing a token from G_T or G_F decreases $\mathbf{V}(i) + k \cdot G\_in$ by at most $j_G(i)$ for $0 \leq k \leq i-1$.*
4. *The co-efficient of X_is_T and x_is_F in $\mathbf{V}(i)$ is 0, for any unbounded variable x in G.*

Proof. By induction on structure of G.

Base. $G = x$ for some variable x. See box **A** in Fig. 3.2. It is easy to see that $\mathbf{V}(i) = i \cdot x\_T + i \cdot x\_F$ is a S-variant that satisfies all the required properties with $j_x(i) = i$.

Step. $G = \neg P$. See box **B** in Fig. 3.2. By induction hypothesis, we have a S-variant $\mathbf{V}_P(i)$ for the net corresponding to P that satisfies all the stated properties. The S-variant for the net corresponding to $\neg P$ is given by

$$\mathbf{V}(i) = \mathbf{V}_P(i+1) + i \cdot P\_in + (j_P(i+1)+1) \cdot not\_P\_T + (j_P(i+1)+1) \cdot not\_P\_F \ .$$

Since $\mathbf{V}_P(i+1) + i \cdot P\_in$ is a S-variant for $\mathcal{N}_P$ (the net corresponding to P) and transitions inside $\mathcal{N}_P$ do not affect the places not_P_T and not_P_F, $\mathbf{V}(i)$ strictly increases whenever any transition inside $\mathcal{N}_P$ fires. When the transition $call\_P$ fires, $\mathbf{V}(i)$ increases due to the presence of $i \cdot P\_in$. When the transition $not\_P\_is\_F$ (resp. $not\_P\_is\_T$) fires, $\mathbf{V}_i$ increases due to the presence of $\mathbf{V}_P(i+1) + (j_P(i+1)+1) \cdot not\_P\_F$ (resp. $\mathbf{V}_P(i+1) + (j_P(i+1)+1) \cdot not\_P\_T$). It is easy to see that $\mathbf{V}(i) + k \cdot not\_P\_in$ is a S-variant of $\mathcal{N}_{\neg P}$ for $0 \leq k \leq i-1$. In this case, $j_G(i) = j_P(i+1)+1$.

$G = P \wedge Q$. See box **C** in Fig. 3.2. The S-variant for the net corresponding to $P \wedge Q$ is given by

$$\begin{aligned}\mathbf{V}(i) \quad = \quad & \mathbf{V}_P(i+1) + i \cdot P\_in + \mathbf{V}_Q(j_P(i+1)+2) + (j_P(i+1)+1) \cdot Q\_in \\ & + (j_P(i+1) + j_Q(j_P(i+1)+2)+1) \cdot (P\_and\_Q\_T + P\_and\_Q\_F) \ .\end{aligned}$$

Since $\mathbf{V}_P(i+1) + i \cdot P\_in$ is a S-variant for $\mathcal{N}_P$ and transitions inside $\mathcal{N}_P$ do not affect places in $\mathcal{N}_Q$ or $P\_and\_Q\_in$, $P\_and\_Q\_T$ or $P\_and\_Q\_F$, $\mathbf{V}(i)$ increases whenever any transition in $\mathcal{N}_P$ fires. Since $\mathbf{V}_Q(j_P(i+1)+2) + (j_P(i+1)+1) \cdot Q\_in$ is a S-variant for $\mathcal{N}_Q$ and transitions inside $\mathcal{N}_Q$ do not affect places in $\mathcal{N}_P$ or $P\_and\_Q\_in$, $P\_and\_Q\_T$ or $P\_and\_Q\_F$, $\mathbf{V}(i)$ increases whenever any transition in $\mathcal{N}_Q$ fires. When $call\_P$ fires, $\mathbf{V}(i)$ increases due to the presence of $i \cdot P\_in$. When transitions $P\_F\_and\_Q\_?$ or $P\_T\_and\_Q\_F$ fires, $\mathbf{V}(i)$ increases due to the presence of $(j_P(i+1) + j_Q(j_P(i+1)+2)+1) \cdot P\_and\_Q\_F$. When transition $P\_T\_and\_Q\_?$ fires, $\mathbf{V}(i)$ increases due to the presence of $(j_P(i+1)+1) \cdot Q\_in$. When transition $P\_T\_and\_Q\_T$ fires, $\mathbf{V}(i)$ increases due to the presence of $(j_P(i+1) + j_Q(j_P(i+$

$1)+2)+1) \cdot P\_and\_Q\_T$. It is easy to see that $\mathbf{V}(i)+k \cdot P\_and\_Q\_in$ is a S-variant of $\mathcal{N}_{P \wedge Q}$ for $0 \leq k \leq i-1$. In this case, $j_G(i) = j_P(i+1) + j_Q(j_P(i+1)+2)+1$.

$G = \exists x.P$. See Fig. 3.3. The S-variant for the net corresponding to $\exists x.P$ is given by

$$\begin{aligned}\mathbf{V}(i) &= \mathbf{V}_P(i+1) + i \cdot P\_in + (2+2j_P(i+1)) \cdot Ex.P\_T \\ &\quad +(1+j_P(i+1)) \cdot x\_is\_F + (2+2j_P(i+1)) \cdot Ex.P\_F \ .\end{aligned}$$

Since $\mathbf{V}_P(i+1) + i \cdot P\_in$ is a S-variant for $\mathcal{N}_P$ and co-efficient of x_is_F is 0 in $\mathbf{V}_P(i+1)$ (since x is not bound inside P), $\mathbf{V}(i)$ increases when any transition inside $\mathcal{N}_P$ fires (note that transitions inside $\mathcal{N}_P$ don't change token count of x_is_F, $Ex.P\_T$ and $Ex.P\_F$). When transition $call\_P\_with\_x\_T$ fires, $\mathbf{V}(i)$ increases due to the presence of $i \cdot P\_in$. When transition $x\_T\_and\_P\_T$ fires, $\mathbf{V}(i)$ increases due to the presence of $(2+2j_P(i+1)) \cdot Ex.P\_T$. When transition $call\_P\_with\_x\_F$ fires, $\mathbf{V}(i)$ increases due to the presence of $(1+j_P(i+1)) \cdot x\_is\_F$. When transition $x\_F\_and\_P\_T$ fires, $\mathbf{V}(i)$ increases due to the presence of $(2+2j_P(i+1)) \cdot Ex.P\_T$. When transition $Ex.P\_is\_F$ fires, $\mathbf{V}(i)$ increases due to the presence of $(2+2j_P(i+1)) \cdot Ex.P\_F$. It is easy to see that $\mathbf{V}(i)+k \cdot Ex.P\_in$ is a S-variant of $\mathcal{N}_{Ex.P}$ for $0 \leq k \leq i-1$. In this case, $j_G(i) = 2+2j_P(i+1)$.

□

3.3 Petri nets with partial S-variants

In this section, we examine the applicability of S-variants technique for nets that do not have S-variants. First, we identify transitions with some properties, which will motivate the extension of S-variants. Then, we identify some properties satisfied by a more general class of nets and use those properties to implement a slightly modified version of the S-variant algorithm for reachability. We then identify some bottlenecks that prevent this technique from being applied to general Petri nets.

The net in Fig. 3.4 is used as a motivating example for understanding these properties. This is a modified version of the net in Fig. 3.1 (the place p_5 has been removed from that net). The firing sequence $t_3 t_4$ represents a semi-positive T-invariant and thus it does not have S-variants.

3.3.1 Transitions that make progress

The principal idea behind the S-variant technique is to use the changes caused in the marking by a transition to get a bound on the number of firings of that transition. If a transition is part of some semi-positive T-invariant, then the action of that transition can be cancelled by firing other transitions that are part of the T-invariant. The S-variant technique will not work for such transitions. The rest of this section formalizes this notion.

Definition 17. Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net and $t \in T$ is a transition. t is said to be progressive if it is not part of any semi-positive T-invariant.

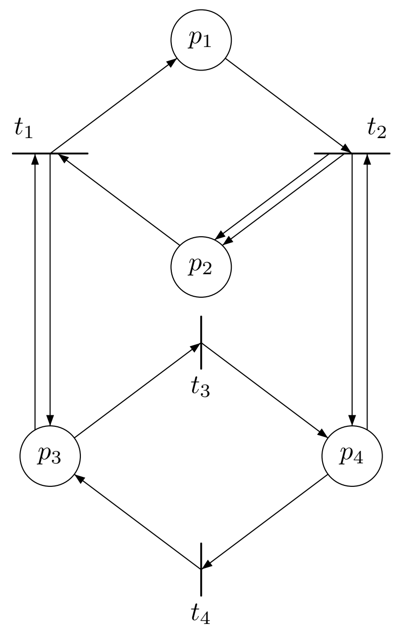

Figure 3.4: Net system with progressive transitions, $M_0 = (0, 1, 1, 0)$

In Fig. 3.4, $U = \{t_1, t_2\}$ is the set of progressive transitions.

Definition 18. Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net and $\emptyset \neq U \subseteq T$ is a nonempty subset of progressive transitions. An integer vector (mapping) $\mathbf{V} : P \to \mathcal{Z}$ is a partial S-variant iff it satisfies the following properties:

1. For all $t \in U$, $\sum_{p \in P} \mathbf{V}(p)(Post(p,t) - Pre(p,t)) \geq 1$.
2. For all other $t \in T \setminus U$, $\sum_{p \in P} \mathbf{V}(p)(Post(p,t) - Pre(p,t)) \geq 0$.

As before, if M is a marking, $\mathbf{V}(M)$ denotes $\sum_{p \in P} \mathbf{V}(p)M(p)$. If $t \in U$ and $M \xrightarrow{t} M'$, then $\mathbf{V}(M') \geq \mathbf{V}(M)+1$. If $t \in T \setminus U$ and $M \xrightarrow{t} M'$, then $\mathbf{V}(M') \geq \mathbf{V}(M)$.

For the net in Fig. 3.4, a partial S-variant is $\mathbf{V} = [3, 2, 0, 0]^T$, for the set of progressive transition $U = \{t_1, t_2\}$. If we think of p_i as denoting the number of tokens in the place p_i at any time, then the expression $3p_1 + 2p_2$ denotes the partial S-variant. The value of this expression increases by 1 if either t_1 or t_2 fires. It remains unchanged if t_3 or t_4 fires.

Again, it will be useful to think of a partial S-variant as an integral solution to a system of inequalities. Suppose for a Petri net $\mathcal{N} = (P, T, Pre, Post)$, $\mathbf{N}$ is the incidence matrix. If $\mathbf{N}^T$ is represented as

$$\mathbf{N}^T = \left[\begin{array}{c} \mathbf{N}_1 \\ \hline \mathbf{N}_2 \end{array} \right]$$

where $\mathbf{N}_1$ represents progressive transitions and $\mathbf{N}_2$ represents other transitions, then a partial S-variant is an integral solution to the system of inequalities

$$\left[\frac{\mathbf{N}_1}{\mathbf{N}_2}\right] \mathbf{v} \geq \left[\frac{\mathbf{1}}{\mathbf{0}}\right] \quad . \tag{3.1}$$

The following theorem characterizes nets having partial S-variants by relating them to positive T-invariants.

Theorem 12. *A Petri net has a partial S-variant iff it does not have any positive T-invariant.*

Proof. Suppose $\mathcal{N}$ is a Petri net with incidence matrix $\mathbf{N}$. If $\mathcal{N}$ has a partial S-variant, it has at least one progressive transition. Since progressive transitions are not part of any semi-positive T-invariant, $\mathcal{N}$ cannot have positive T-invariants.

On the other hand, if $\mathcal{N}$ does not have positive T-invariants, there must be a non-empty subset U of transitions that are not part of any semi-positive T-invariant. Let us represent $\mathbf{N}$ as $\left[\mathbf{N}_1^T \mid \mathbf{N}_2^T\right]$, where $\mathbf{N}_1^T$ represents transitions in U and $\mathbf{N}_2^T$ represents the other transitions. Absence of transitions from U in every semi-positive T-invariant means that the following system of equations does not have integral solutions.

$$\begin{aligned} \left[\mathbf{N}_1^T \mid \mathbf{N}_2^T\right] \mathbf{j} &= \mathbf{0} \\ \mathbf{j} &\geq \mathbf{0} \\ \mathbf{j}^T \left[\frac{\mathbf{1}}{\mathbf{0}}\right] &> \mathbf{0} \ . \end{aligned} \tag{3.2}$$

We can again apply a variant of the Farkas lemma and convert rational solutions to integral solutions to conclude that absence of integral solutions to equation (3.2) means presence of integral solution to equation (3.1). This means that a partial S-variant exists for $\mathcal{N}$. □

3.3.2 Modified algorithm for nets with specific properties

As we have seen before, nets that do not have S-variants will have semi-positive T-invariants. Since presence of semi-positive T-invariants indicate presence of firing sequences that can repeat arbitrarily many times without changing the marking reached, it is not possible to get an upper bound on the length of firing sequences. On the other hand, since we are only interested in knowing whether a given target marking can be reached, we could restrict ourselves to just those firing sequences where such repetitions do not occur. If a Petri net has the property that any firing sequence can be replaced by another one where such "useless" repetitions are removed, we can extend the S-variant technique to that net. In the rest of this section, we will formalize this notion.

We now define the properties characterizing the class of nets to which we aim to extend the S-variant technique.

Definition 19. Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net with n transitions. The net system $(\mathcal{N}, M_0, M_f)$ is called S-varying if it satisfies the following properties:

1. The set U of progressive transitions of $\mathcal{N}$ is not empty.
2. The non progressive transitions $V = T \setminus U$ are bounded by progressive ones, that is, there is a function f such that for every firing sequence σ with $M_0 \xrightarrow{\sigma} M$, there is another firing sequence τ with $M_0 \xrightarrow{\tau} M$ and $\overline{\tau}(V) \leq f(\overline{\tau}(U), M_0, D, n, m)$, where n is the number of transitions, m is the number of places and D is the maximum entry in the incidence matrix $\mathbf{N}$ of $\mathcal{N}$.

The first property above states that there are some progressive transitions so that we can track them in terms of upper bounding the number of their occurrences in firing sequences. The second property states that there may be transitions that are part of T-invariants, but the number of their occurrences will be bounded by the progressive transitions in those firing sequences where the "useless" repetitions are removed. In our example net in Fig. 3.4, $V = \{t_3, t_4\}$. Using different functions in this definition leads to different complexities for the reachability algorithm of the corresponding class of nets.

In Fig. 3.4, the firing sequence $t_3 t_4$ represents a semi-positive T-invariant and it can repeat arbitrarily many times without affecting the marking reached. However, any such firing sequence σ can be replaced by another one τ where between any two firings of t_3, there is at least one occurrence of t_2. Thus, $\tau(t_3) \leq 2\overline{\tau}(t_2) + 1$. In addition, once t_3 fires, t_4 can fire at most once before t_3 fires again. Hence, $\overline{\tau}(t_4) \leq \overline{\tau}(t_3) + 1$. Thus, the net in Fig. 3.4 is S-varying with $\overline{\tau}(V) \leq 2\overline{\tau}(U) + 1$.

We will use these properties satisfied by S-varying net systems to bound the length of firing sequences that need to be considered for reachability.

Lemma 13. *Suppose $(\mathcal{N}, M_0, M_f)$ is an S-varying net system with n transitions, m places and incidence matrix entries in the range $[0..D]$. There exists a partial S-variant $\mathbf{V}$ of $\mathcal{N}$ such that the absolute value of each entry of $\mathbf{V}$ is $O(mn^2(n!)^2 D^{2n})$.*

Proof. Since $(\mathcal{N}, M_0, M_f)$ is an S-varying net system, $\mathcal{N}$ has at least one progressive transition. Since this transition is not part of any semi-positive T-invariant, $\mathcal{N}$ does not have any positive T-invariant. Therefore Theorem 12 implies that a partial S-variant exists. If U is the set of progressive transitions of $\mathcal{N}$, then equation (3.1) has an integral solution, where $\mathbf{N}_1$ represents progressive transitions and $\mathbf{N}_2$ represents the other transitions. Equation (3.1) has integral solution iff the following system of equations has a positive integral solution.

$$\left[\mathbf{I} \mid \frac{\mathbf{N}_1}{\mathbf{N}_2} \mid \frac{-\mathbf{N}_1}{-\mathbf{N}_2}\right] \mathbf{v}' = \left[\frac{-\mathbf{1}}{\mathbf{0}}\right] \tag{3.3}$$

The coefficient matrix in equation (3.3) has full row rank due to the presence of I. The number of rows is n and the number of columns $r = n + 2m \geq n$. The absolute value of minors of the augmented matrix is upper bounded by $R \leq n!D^n$. Hence Proposition 5 is applicable and $Rn + nrR^2 \leq nn!D^n + n(n+2m)n!^2 D^{2n}$. Thus, we have a positive integral solution with each entry being $O\left(mn^2(n!)^2 D^{2n}\right)$.

It is easy to see that a positive integral solution to equation (3.3) can be converted into an integral solution of equation (3.1) without affecting the bounds. Hence, there exists an S-variant $\mathbf{V}$ with absolute value of each entry bounded by $O\left(mn^2(n!)^2 D^{2n}\right)$. ☐

Lemma 14. *Suppose $(\mathcal{N}, M_0, M_f)$ is an S-varying net system with m places and set of n transitions T and the entries of its incidence matrix in the range $[0..D]$. Suppose the set V of non-progressive transitions are bounded by the function $f(\overline{\tau}(U), M_0, D, m, n)$. If M_f is reachable from M_0, then there exists a firing sequence σ of length at most $O(m^2n^4(n!)^2D^{2n}max(M_0, M_f)) + f\left(O(m^2n^4(n!)^2D^{2n}max(M_0, M_f)), D, n, m\right)$ and $M_0 \xrightarrow{\sigma} M_f$.*

Proof. Since M_f is reachable, there exists a firing sequence σ that satisfies the properties of τ mentioned in property 2 of Definition 19. We partition the set of transitions T into two sets U and V. We then give upper bounds for $\overline{\sigma}(U)$ and $\overline{\sigma}(V)$, leading to the conclusion of the lemma.

- (Bound for $\overline{\sigma}(U)$) Suppose $\sigma = t_1t_2 \cdots t_r$ and $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \cdots \xrightarrow{t_r} M_r = M_f$. By Lemma 13, there exists a partial S-variant $\mathbf{V}$ with absolute value of each entry bounded by $O\left(mn^2(n!)^2D^{2n}\right)$. By definition of partial S-variant, we have for all $1 \leq i \leq r$, if $t_i \in U$, then $\mathbf{V}(M_i) \geq \mathbf{V}(M_{i-1}) + 1$ and if $t_i \in T \setminus U$, then $\mathbf{V}(M_i) \geq \mathbf{V}(M_{i-1})$. Starting at $i = r$ and substituting for i all the way up to $i = 1$, we get $\mathbf{V}(M_r) \geq \mathbf{V}(M_0) + \overline{\sigma}(U)$. Thus, $\overline{\sigma}(U) \leq \mathbf{V}(M) - \mathbf{V}(M_0) \leq O\left(m^2n^2(n!)^2D^{2n}max(M_0, M_f)\right)$.

- (Bound for $\overline{\sigma}(V)$) By property 2 of Definition 19, $\overline{\sigma}(V) \leq f(\overline{\sigma}(U), M_0, D, n, m)$.

By combining the three bounds, we can see that the length of σ is at most $O(m^2n^4(n!)^2 D^{2n}max(M_0, M_f)) + f\left(O(m^2n^4(n!)^2D^{2n}max(M_0, M_f)), D, n, m\right)$. ☐

By Lemma 14, we can see that the algorithm given in section 3.1.2 works in this case also with the same kind of complexity analysis, but with different values. In particular, if f is at most polynomial in $\overline{\sigma}(U)$, M_0, D and exponential in n, m, this will lead to a polynomial space algorithm.

3.3.3 An example of an S-varying net system

In section 2.3.3, we saw an NP algorithm for reachability in sinkless Petri nets. Although that section considers only those Petri nets where weights on edges are 0 or 1, many of the results, including the reachability algorithm, can be extended to Petri nets where nonnegative weights are allowed in the $Post$ function.

The Petri net in Fig. 3.4 provides an example of a S-varying net system but it is *sinkless* in this extended sense. In this section, we twist our example a little bit to the one in Fig. 3.5 to give an S-varying Petri net which is not sinkless.

The progressive transitions of this net are $U = \{t_1, t_2, t_5\}$ and the non progressive transitions are $V = \{t_3, t_4\}$. A partial S-variant for this Petri net is $\mathbf{V} = [3, 2, 0, 0, 4]^T$. We also have $\overline{\sigma}(V) \leq 2\overline{\sigma}(U) + 1$.

3.3.4 Bottlenecks for general Petri nets

Here, we identify some reasons why it is difficult to extend the S-variant technique to other Petri nets in general. The principal idea behind this technique is to use the

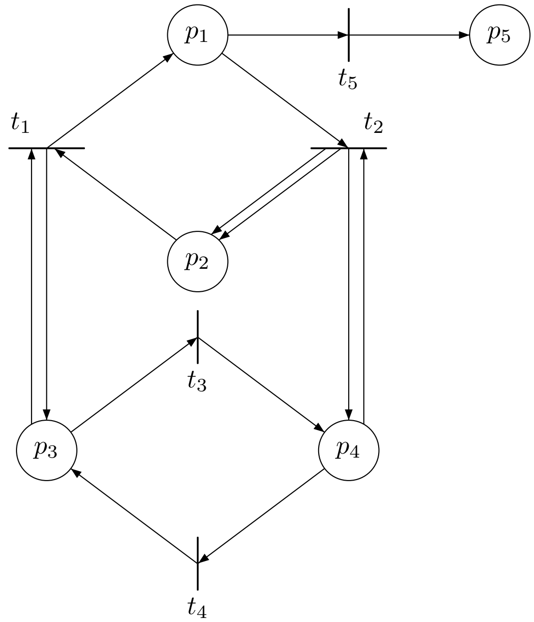

Figure 3.5: S-varying net system that is not sinkless

changes caused in the marking by a transition to get a bound on the number of firings of that transition. If there are no progressive transitions, every transition is part of a semi-positive T-invariant. Thus, the action of every transition can be "cancelled" by other transitions. So, even though some transitions fire, their net effect on the marking is zero and this makes it impossible to apply the S-variant technique. The presence of progressive transitions is important for this technique to work.

To see the difficulties in extending this technique, we need the notion of *initial synchronic lead* introduced by Silva and Colom [30].

Definition 20 (Silva and Colom)**.** Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net with n transitions, and T_i and T_j are non-empty subsets of T. Let $\mathbf{W_i}, \mathbf{W_j}$ be n-vectors of natural numbers such that $support(\mathbf{W_i}) \subseteq T_i$ and $support(\mathbf{W_j}) \subseteq T_j$. Let $\mathbf{W_{ij}} = \mathbf{W_i} - \mathbf{W_j}$. The initial synchronic lead $ISL(\mathbf{W_{ij}})$ is defined as

$$ISL(W_{ij}) = sup\left\{\mathbf{W_{ij}}^T\overline{\sigma} \mid \sigma \in L_{M_0}(\mathcal{N})\right\}$$

where $L_{M_0}(\mathcal{N})$ is the set of firing sequences σ that satisfy the equation $M_0 + \mathbf{N}\overline{\sigma} \geq \mathbf{0}$.

$ISL(\mathbf{W_{ij}})$ gives an upper bound on the weighted count of transition firings from T_i that can be fired in terms of the weighted count of transition firings from T_j in any firing sequence of $(\mathcal{N}, M_0)$.

Now, suppose the subset of progressive transitions U is not empty. $T \setminus U$ is the set of "cancellable" transitions. The main idea of the extended algorithm in section 3.3.2

is to first bound the occurrences from U and use this to bound the occurrences from $T \setminus U$. To do this, we need a relation between the number of occurrences from $T \setminus U$ and from U in any firing sequence. Formally speaking, we are looking for a finite value of $ISL(a\mathbf{e}[T \setminus U] - b\mathbf{e}[U])$ for some natural numbers a and b greater than 0. Silva and Colom [30] give a necessary and sufficient condition for $ISL(\mathbf{W}_{ij})$ to be finite.

Theorem 13 (Silva and Colom [30], Theorem 5.2)**.** *Let T_i and T_j be non-empty subsets of T of a Petri net $\mathcal{N} = (P, T, Pre, Post)$ with n transitions and incidence matrix $\mathbf{N}$. Let $\mathbf{W}_i, \mathbf{W}_j$ be n-vectors of natural numbers such that $support(\mathbf{W}_i) \subseteq T_i$ and $support(\mathbf{W}_j) \subseteq T_j$. $ISL(\mathbf{W}_i - \mathbf{W}_j)$ is finite iff for all $\mathbf{X}$ such that $\mathbf{NX} \geq \mathbf{0}$, $(\mathbf{W}_i - \mathbf{W}_j)^T\mathbf{X} \leq \mathbf{0}$ is satisfied.*

In particular, the condition in Theorem 13 requires that if a firing sequence σ denotes a T-invariant, the number of firings of T_i with weight $\mathbf{W}_i$ should not be greater than the number of firings of T_j with weight $\mathbf{W}_j$ in σ. For our purpose, $T_i = T \setminus U$ and $T_j = U$. With this definition of T_i and T_j, we can see that $ISL(a\mathbf{e}[T \setminus U] - b\mathbf{e}[U])$ is not finite for non-zero a.

It is to be noted that $ISL(\mathbf{W}_i - \mathbf{W}_j)$ takes into account every firing sequence σ where $M_0 + \mathbf{N}\overline{\sigma} \geq \mathbf{0}$, M_0 being the initial marking and $\mathbf{N}$ being the incidence matrix. In particular, linear algebraic techniques cannot make a distinction between firing sequences of the kind σ and τ given in property 2 of Definition 19. So we have to look for structural and/or behavioural properties to apply this technique for solving the reachability problem.

3.4 Partially bounded S-varying nets

In this section, we will look at S-varying nets with a special structural property that will guarantee the existence of **PSPACE** algorithm for reachability in such nets. We first need the notion of structural boundedness.

Definition 21. A net system $(\mathcal{N}, M_0, M_f)$ is said to be bounded if there exists a positive integer b such that all markings reachable from M_0 puts at most b tokens in each place. A Petri net $\mathcal{N} = (P, T, Pre, Post)$ is structurally bounded if $\mathcal{N}$ is bounded for any initial marking.

Definition 22. A Petri net $\mathcal{N} = (P, T, Pre, Post)$ is said to be partially bounded S-varying if it satisfies the following properties.

1. $\mathcal{N}$ has a non empty subset $U \subseteq T$ of progressive transitions.
2. $\mathcal{N}$, when restricted to $T \setminus U$ is structurally bounded.

It is known that [23] a Petri net $\mathcal{N}$ is structurally bounded iff the system of inequalities $\mathbf{N}^T\mathbf{y} \leq \mathbf{0}$ has a positive integral solution (all components of the solution must be greater than 0). Since a positive rational solution to $\mathbf{N}^T\mathbf{y} \leq \mathbf{0}$ can be converted to a positive integral solution, recognizing if a Petri net is structurally bounded can be done in polynomial time.

Suppose a net system $(\mathcal{N}, M_0, M_f)$ has n transitions and m places. Suppose $\mathcal{N}$ is partially bounded S-varying with set of progressive transitions U. Suppose a firing sequence σ is enabled such that $M_0 \xrightarrow{\sigma} M_f$. To prove that the class of partially bounded S-varying nets have **PSPACE** reachability algorithm, it is enough to prove that in $(\mathcal{N}, M_0, M_f)$, there is a firing sequence τ enabled at M_0 such that $M_0 \xrightarrow{\tau} M_f$ and there is a function f such that $\overline{\tau}(T \setminus U) \leq f(\overline{\tau}(U), M_0, D, n, m)$ and f is at most polynomial in $\overline{\tau}(U)$, M_0 and D and at most exponential in n and m.

Lemma 15. *Suppose a Petri net $\mathcal{N} = (P, T, Pre, Post)$ is structurally bounded. Then there exists a positive integral solution to the system of inequalities $\mathbf{N}^T y \leq \mathbf{0}$ where each component of the solution is $O\left(mn^2(n!)^2 D^{2n}\right)$.*

Proof. Since $\mathcal{N}$ is structurally bounded, there is a positive integral solution to $\mathbf{N}^T y \leq \mathbf{0}$. Therefore, there is a positive integral solution to

$$\left[\frac{\mathbf{N}^T}{-\mathbf{I}}\right] \mathbf{v} \leq \left[\frac{\mathbf{0}}{-\mathbf{1}}\right]$$

Let us denote the above system of inequalities by $\mathbf{A}\mathbf{v} \leq \mathbf{b}$. This means there is a semi positive integral solution to the system of equations

$$[\mathbf{I} \mid \mathbf{A}] \left[\frac{\mathbf{v_1}}{\mathbf{v}}\right] = \mathbf{b}$$

We can apply Proposition 5 to the above system of equations to conclude that there is a solution with each component bounded by $O\left(mn^2(n!)^2 D^{2n}\right)$. ☐

Lemma 16. *Suppose $(\mathcal{N}, M_0, M_f)$ is partially bounded S-varying with progressive transitions U. Let $V = T \setminus U$ and $\mathcal{N}_V$ be the Petri net $\mathcal{N}$ restricted to V. Suppose $\mathbf{y}$ is a positive integral vector such that $\mathbf{N_V}^T \mathbf{y} \leq \mathbf{0}$. Suppose a firing sequence σ fires at M_0 such that $\overline{\sigma}(U) = k$. Then, for any intermediate marking M reached during firing of σ, $M^T \mathbf{y} \leq M_0^T \mathbf{y} + (\mathbf{kD})^T \mathbf{y}$, where D is the maximum entry in the incidence matrix $\mathbf{N}$ of $\mathcal{N}$.*

Proof. Decompose the firing sequence σ as $\sigma = \sigma_0 u_1 \sigma_1 \cdots u_k \sigma_k$, where $u_1, \ldots, u_k$ are the progressive transitions and $\sigma_0, \ldots, \sigma_k$ are made up of the non-progressive transitions. Suppose the marking M is reached just after firing u_j or during firing of σ_j. We will prove by induction on j that $M^T \mathbf{y} \leq M_0^T \mathbf{y} + (\mathbf{jD})^T \mathbf{y}$. The result then follows since $j \leq k$.

For the base case $j = 0$, M is reached by firing transitions in $\mathcal{N}_V$ only. We have $M = M_0 + \mathbf{N_V} \overline{\sigma_0}$. Therefore, $M^T = M_0^T + \overline{\sigma_0}^T \mathbf{N_V}^T$ and hence $M^T \mathbf{y} = M_0^T \mathbf{y} + \overline{\sigma_0}^T \mathbf{N_V}^T \mathbf{y}$. Since $\mathbf{N_V}^T \mathbf{y} \leq \mathbf{0}$ and $\overline{\sigma_0} \geq \mathbf{0}$, we have $M^T \mathbf{y} \leq M_0^T \mathbf{y}$.

For the induction step, suppose $M_0 \xrightarrow{\sigma_0} \cdots \xrightarrow{\sigma_j} M_j \xrightarrow{u_{j+1}} M_{j+1} \xrightarrow{\sigma'_{j+1}} M$. By induction hypothesis, $M_j^T \mathbf{y} \leq M_0^T \mathbf{y} + (\mathbf{jD})^T \mathbf{y}$. Since D is the maximum entry in the incidence matrix $\mathbf{N}$ of $\mathcal{N}$, firing of u_{j+1} can add at most D tokens to any place. Hence, $M_{j+1} \leq M_j + \mathbf{D}$. Therefore, we have $M_{j+1}^T \mathbf{y} \leq M_0^T \mathbf{y} + ((\mathbf{j+1})\mathbf{D})^T \mathbf{y}$. Marking M was reached from M_{j+1} by firing only transitions in $\mathcal{N}_V$. Therefore, we have $M = M_{j+1} + \mathbf{N_V} \overline{\sigma_{j+1}}'$. Therefore, $M^T \mathbf{y} = M_{j+1}^T \mathbf{y} + \overline{\sigma_{j+1}}'^T \mathbf{N_V}^T \mathbf{y}$. Since $\mathbf{N_V}^T \mathbf{y} \leq \mathbf{0}$ and $\overline{\sigma_{j+1}}' \geq 0$, we get $M^T \mathbf{y} \leq M_{j+1}^T \mathbf{y} \leq M_0^T \mathbf{y} + ((\mathbf{j+1})\mathbf{D})^T \mathbf{y}$. This completes the induction and hence the proof. ☐

Theorem 14. *Suppose $(\mathcal{N}, M_0, M_f)$ is a partially bounded S-varying net. Suppose U is the set of progressive transitions of $\mathcal{N}$. If M_f is reachable from M_0, then there exists a firing sequence τ with $M_0 \xrightarrow{\tau} M_f$ such that $\overline{\tau}(T \setminus U) \leq f(\overline{\tau}(U), D, M_0, n, m)$, where f is a function that is polynomial in $\overline{\tau}(U)$, D and M_0 and exponential in m and n.*

Proof. Since M_f is reachable, there is a firing sequence σ such that $M_0 \xrightarrow{\sigma} M_f$. Let σ consist of the components $\sigma_0 u_1 \sigma_1 u_2 \sigma_2 \cdots u_k \sigma_k$ for some k, where $u_1, u_2, \ldots, u_k$ are progressive transitions and $\sigma_0, \sigma_1, \ldots, \sigma_k$ consist of non-progressive transitions. Now, define firing sequence τ as $\tau = \tau_0 u_1 \tau_1 u_2 \tau_2 \cdots u_k \tau_k$, where τ_i is same as σ_i but with subsequences that start and end with the same marking removed. Thus, when τ_i is fired, all the intermediate markings that are reached are distinct from each other. It is also easy to verify that $M_0 \xrightarrow{\tau} M_f$.

Suppose $V = T \setminus U$ is the set of non-progressive transitions and $\mathcal{N}_V$ is the Petri net $\mathcal{N}$ restricted to V. From Lemma 15, there exists a positive integral vector β where each component is greater than 0 and bounded by $O\left(mn^2(n!)^2 D^{2n}\right)$, such that $\mathbf{N_V}^T \beta \leq \mathbf{0}$. From Lemma 16, for any intermediate marking M reached during the firing of τ, $M^T\beta \leq M_0^T\beta + \mathbf{kD}^T\beta$. Hence, for any place p, $M(p) \leq (M_0^T\beta + \mathbf{kD}^T\beta)/\beta(p)$. Therefore, during the firing of τ, each place will accumulate at most $M_0^T\beta + \mathbf{kD}^T\beta$ tokens. Since there are m places, total number of distinct markings possible is $(M_0^T\beta + \mathbf{kD}^T\beta)^m$. Since there are $k+1$ sequences τ_i that make up all firings of non-progressive transitions in V, we get $\overline{\tau}(V) \leq (k+1)(M_0^T\beta + \mathbf{kD}^T\beta)^m$. This is the function f required and it is easy to verify that this is polynomial in $\overline{\tau}(U) = k$, M_0 and D and exponential in m and n. □

3.5 More examples of S-varying nets

Valk and Vidal-Naquet [32] have given a family of Petri nets to demonstrate that the bound of a bounded Petri net can be non-primitive recursive. We will now show that all nets in this family are S-varying, though not partially bounded S-varying. In this family, there is one net $\mathcal{N}_i$ for each non-negative integer i. They are defined inductively as seen in Fig. 3.6.

The initial marking of $\mathcal{N}_i$ for any $i \geq 0$ is one token in b_i, n tokens in c_i and no tokens anywhere else. The working of this family of nets can be understood as follows. We start with $\mathcal{N}_0$ first. With one token in b_0 and n tokens in c_0, transition v_0 fires once to get a token in q_0. Transition t_0 can now fire n times to put $2n$ tokens in p_0. Transition w_0 can now fire once to put a token in s_0. Transition u_0 can now fire $2n$ times to put $2n$ tokens in c_0. Transition x_0 can now fire once to put a token in e_0. Thus, $\mathcal{N}_0$ began with one token in b_0 and ended with one token in e_0, and in the process, number of tokens in c_0 was doubled.

Working of $\mathcal{N}_i$ can now be understood in terms of $\mathcal{N}_{i-1}$ as follows. Suppose $f_{i-1}(n)$ is an upper bound on the number of tokens c_{i-1} can accumulate when $\mathcal{N}_{i-1}$ has finished with its' token in e_{i-1}, when it had n tokens in c_{i-1} at the beginning. With one token in b_i and n tokens in c_i, transition t_i can fire n times to put n tokens each in c_{i-1} and d_i. Firing x_i will now "initiate" $\mathcal{N}_{i-1}$ and it will "finish" with at most $f_{i-1}(n)$ tokens in c_{i-1}. $\mathcal{N}_{i-1}$ can be initiated again by firing r_i. This time, $\mathcal{N}_{i-1}$ finishes with

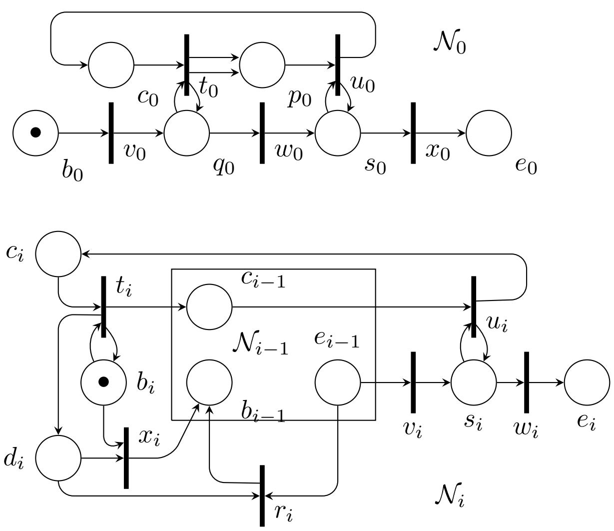

Figure 3.6: Valk and Vidal-Naquets' family of nets

at most $f_{i-1}^2(n)$ tokens in c_{i-1}. $\mathcal{N}_{i-1}$ can be initiated a maximum of n times like this to accumulate a maximum of $f_{i-1}^n(n)$ tokens in c_{i-1}. Now, transition v_i can be fired once to get a token into s_i. Transition u_i can be fired as many times as required to shift tokens from c_{i-1} to c_i. Finally, transition w_i can be fired once to get a token in e_i. Thus, $\mathcal{N}_i$ began with one token in b_i and n tokens in c_i and finished with at most $f_i(n) = f_{i-1}^n(n)$ tokens in c_i.

The family $f_i(n)$ of upper bounds on number of tokens in c_i can be defined as follows.

$$\begin{aligned} f_0(n) &= 2n \\ f_{i+1}(n) &= f_i^n(n) \end{aligned}$$

It is well known that this family of functions dominate any primitive recursive function. We will now show that any net in this family is a S-varying net and hence there is a **PSPACE** algorithm for solving reachability in any of these nets.

Proposition 7. *Suppose $\mathcal{N}_i$ is a net from the above family, $i \geq 1$. Suppose σ_i is any firing sequence satisfying the following properties:*

1. $\overline{\sigma_i}(t_i) = \overline{\sigma_i}(r_i) = \overline{\sigma_i}(u_i) = 1$ *and* $\overline{\sigma_i}(x_i) = \overline{\sigma_i}(v_i) = \overline{\sigma_i}(w_i) = 0$.
2. $\overline{\sigma_i}(r_k) = 1$ *for* $1 \leq k \leq i-1$.
3. $\overline{\sigma_i}(t_k) = \overline{\sigma_i}(u_k) = i - k + 1$ *for* $1 \leq k \leq i-1$.
4. $\overline{\sigma_i}(x_k) = \overline{\sigma_i}(v_k) = \overline{\sigma_i}(w_k) = i - k$ *for* $1 \leq k \leq i-1$.
5. $\overline{\sigma_i}(t) = 0$ *for all other transitions* t.

Then the net effect of firing σ_i is to add i places to b_0 and remove i tokens from e_0.

Proof. We will prove by induction on j that firing transitions of σ_i in nets at level $i-j$ to i will result in $j+1$ tokens being added to b_{i-j-1} and $j+1$ tokens being removed from e_{i-j-1}. The result then follows by taking $j = i-1$.

In what follows, we will represent the effect of firing a transition by expressions. $jw_i : -js_i + je_i$ means that firing transition w_i j times results in removal of j tokens from place s_i and addition of j tokens to place e_i.

When $i = 1$, t_1, r_1 and u_1 are the only transitions in $\overline{\sigma_i}$ and is not covered by the induction. In this case, effect of firing σ_i is given by the following expressions:

$$\begin{aligned} t_1 &: -c_1 + c_0 + d_1 \\ r_1 &: -d_1 - e_0 + b_0 \\ u_1 &: -c_0 + c_1 \\ total &: b_0 - e_0 \end{aligned}$$

We will now begin with base case of the induction, $j = 1$. We want to prove that effect of firing transitions of σ_i that are at level $i-1$ and i result in $2b_{i-2} - 2e_{i-2}$. This can be readily seen by observing that following expressions give the effect of each transition.

$$\begin{aligned} t_i &: -c_i + c_{i-1} + d_i \\ r_i &: -d_i - e_{i-1} + b_{i-1} \\ u_i &: -c_{i-1} + c_i \\ r_{i-1} &: -d_{i-1} - e_{i-2} + b_{i-2} \\ 2t_{i-1} &: -2c_{i-1} + 2c_{i-2} + 2d_{i-1} \\ 2u_{i-1} &: -2c_{i-2} + 2c_{i-1} \\ x_{i-1} &: -b_{i-1} - d_{i-1} + b_{i-2} \\ v_{i-1} &: -e_{i-2} + s_{i-1} \\ w_{i-1} &: -s_{i-1} + e_{i-1} \\ total &: 2b_{i-2} - 2e_{i-2} \end{aligned}$$

For the induction step, assume that firing transitions in σ_i that are at levels $i-j$ through i results in $(j+1)b_{i-j-1} - (j+1)e_{i-j-1}$. We want to prove that firing transitions in σ_i that are at levels $i-j-1$ through i results in $(j+2)b_{i-j-2} - (j+$

$2)e_{i-j-2}$. This can again be verified by the following expressions.

$$\begin{array}{rcl}
\text{level } i-j \text{ and higher} & : & (j+1)b_{i-j-1} - (j+1)e_{i-j-1} \\
r_{i-j-1} & : & -d_{i-j-1} - e_{i-j-2} + b_{i-j-2} \\
(j+2)t_{i-j-1} & : & -(j+2)c_{i-j-1} + (j+2)c_{i-j-2} + (j+2)d_{i-j-1} \\
(j+2)u_{i-j-1} & : & -(j+2)c_{i-j-2} + (j+2)c_{i-j-1} \\
(j+1)x_{i-j-1} & : & -(j+1)b_{i-j-1} - (j+1)d_{i-j-1} + (j+1)b_{i-j-2} \\
(j+1)v_{i-j-1} & : & -(j+1)e_{i-j-2} + (j+1)s_{i-j-1} \\
(j+1)w_{i-j-1} & : & -(j+1)s_{i-j-1} + (j+1)e_{i-j-1} \\
\textit{total} & : & (j+2)b_{i-j-2} - (j+2)e_{i-j-2}
\end{array}$$

□

Proposition 8. *In the net $\mathcal{N}_i$, except x_i, v_i, w_i, t_0 and u_0, all other transitions non-progressive transitions.*

Proof. It is sufficient to prove that all other transitions are part of some semi-positive T-invariant. For this, it is sufficient to take σ_i defined in proposition 7 and add iv_0, iw_0 and ix_0 transitions to it. The resulting firing sequence τ_i when fired, doesn't change the number of tokens in any place. Hence, $\overline{\tau_i}$ is a semi-positive T-invariant whose support contains all transitions except x_i, v_i, w_i, t_0 and u_0. □

Proposition 9. *For $\mathcal{N}_i$, there exists a partial S-variant whose support consists of all the progressive transitions.*

Proof. The only progressive transitions are t_0, u_0, x_i, v_i and w_i. The partial S-variant $\mathbf{V}$ is defined as follows.

1. $\mathbf{V}(c_0) = 3$.
2. $\mathbf{V}(p_0) = 2$.
3. $\mathbf{V}(c_j) = 3$, $1 \leq j \leq i$.
4. $\mathbf{V}(b_i) = -1$.
5. $\mathbf{V}(s_i) = 1$.
6. $\mathbf{V}(e_i) = 2$.
7. $\mathbf{V}(p) = 0$ for all other places p.

Table 3.1 considers each transition and verifies that it satisfies the requirement for $\mathbf{V}$ being a partial S-variant according to definition. □

To establish that $\mathcal{N}_i$ is S-varying, we need to prove that number of firings of non-progressive transitions is bounded by a function of number of firings of progressive transitions and input size. In what follows, the set of places $\{p_0, c_0, \ldots, c_i\}$ plays an important role. For convenience of notation, we introduce the following definition.

Transition	Places affected	$\sum_{p \in P} \mathbf{V}(\mathbf{p})(Post(p,t) - Pre(p,t))$
t_0	c_0, p_0	1
u_0	c_0, p_0	1
$t_j, 1 \leq j \leq i-1$	c_j, c_{j-1}, d_j	0
$x_j, 1 \leq j \leq i-1$	b_j, d_j, b_{j-1}	0
$r_j, 1 \leq j \leq i-1$	e_{j-1}, d_j, b_{j-1}	0
$v_j, 1 \leq j \leq i-1$	e_{j-1}, s_j	0
$u_j, 1 \leq j \leq i-1$	c_{j-1}, c_j	0
$w_j, 1 \leq j \leq i-1$	s_j, e_j	0
t_i	c_i, c_{i-1}, d_i	0
r_i	e_{i-1}, d_i, b_{i-1}	0
u_i	c_{i-1}, c_i	0
x_i	b_i, d_i, b_{i-1}	1
v_i	e_{i-1}, s_i	1
w_i	s_i, e_i	1

Table 3.1: Effect of transitions on $\mathbf{V}$

Definition 23. For the net $\mathcal{N}_i$, C_i is the set of places $C_i = \{p_0, c_0, \ldots, c_i\}$. $t(C_i)$ is the total number of tokens in all the places in C_i. For a marking M, $M(C_i) = \sum_{p \in C_i} M(p)$.

Proposition 10. *Suppose $\mathcal{N}_i$ has some initial marking M_i satisfying the following properties:*

1. $M_i(b_i) = 1$.
2. $M_i(b_j) = M_i(s_j) = M_i(e_j) = 0$, $0 \leq j \leq i-1$.
3. $M_i(q_0) = 0$.
4. $M_i(s_i) = M_i(e_i) = 0$.

Suppose a firing sequence σ fires and $\overline{\sigma}(t_0) = k$ such that $M_i \xrightarrow{\sigma} M$. Then we have $M(C_i) \leq M_i(C_i) + k$.

Proof. Among all transitions of $\mathcal{N}_i$, t_0 is the only one that can increase $t(C_i)$. We will now prove the result by induction on k.

For the base case $k = 0$, the result is a direct conclusion of the above observation. For the induction step, suppose $\overline{\sigma}(t_0) = k+1$. Let us split σ as follows: $M_i \xrightarrow{\sigma_0} M_1 \xrightarrow{t_0} M_2 \xrightarrow{\sigma_1} M$ where $\overline{\sigma_1}(t_0) = 0$. By induction hypothesis, we get $M_1(C_i) \leq M_i(C_i) + k$. By inspecting the action of firing t_0, we can conclude that $M_2(C_i) \leq M_1(C_i) + 1$. Again by the observation made at the beginning of this proof, we can conclude that $M(C_i) \leq M_2(C_i)$. Therefore, we have $M(C_i) \leq M_i(C_i) + k + 1$. This completes the induction step and hence the proof. ∎

Proposition 11. *Suppose the net $\mathcal{N}_i$ (for some $i \geq 1$) has the initial marking with n_l tokens in c_i, 1 token in b_i and 0 tokens in all other places. In this initial marking,* *suppose a firing sequence σ is fired. If $\overline{\sigma}(t_0) = k$, then, for every $0 \le j \le i-1$, $\overline{\sigma}(t_{i-j}) \le (n_l + k)^{j+1}$.*

Proof. By induction on j. For the base case $j = 0$, we need to show that $\overline{\sigma}(t_i) \le n_l + k$. This is true since in the given initial marking, t_i can fire at most n_l times.

By induction hypothesis, assume that $\overline{\sigma}(t_{i-j}) \le (n_l + k)^{j+1}$. For the induction step, we need to show that $\overline{\sigma}(t_{i-j-1}) \le (n_l + k)^{j+2}$. Note that for t_{i-j-1} to fire, there must be a token in b_{i-j-1}. Once there is a token in b_{i-j-1}, t_{i-j-1} can fire as many times as there are tokens in c_{i-j-1}. Once all tokens in c_{i-j-1} are exhausted, t_{i-j-1} can fire again only when more tokens are added to c_{i-j-1}. For adding more tokens to c_{i-j-1}, the token in b_{i-j-1} has to be removed. Therefore, for t_{i-j-1} to fire once more after exhausting all tokens in c_{i-j-1}, a token needs to be added to b_{i-j-1}. Let us call the period between adding a token to b_{i-j-1} and adding one token to b_{i-j-1} next time as one round. In one round, t_{i-j-1} can fire at most as many times as there are tokens in c_{i-j-1} at the beginning of the round (to add more tokens to c_{i-j-1}, the token in b_{i-j-1} has to be removed and this takes us to the next round). By proposition 10, c_{i-j-1} has at most $n_l + k$ tokens at any time. Thus, $\overline{\sigma}(t_{i-j-1})$ is bounded by $n_l + k$ times number of times a token can be added to b_{i-j-1}.

Now, the only transitions that can tokens to b_{i-j-1} are x_{i-j} and r_{i-j}. For every firing of x_{i-j} or r_{i-j}, a token is removed from d_{i-j}. Therefore, total number of times a token can be added to b_{i-j-1} is upper bounded by total number of tokens that can be added to d_{i-j}. The only transition that can add tokens to d_{i-j} is t_{i-j}. By induction hypothesis, $\overline{\sigma}(t_{i-j}) \le (n_l + k)^{j+1}$. Therefore, $\overline{\sigma}(t_{i-j-1}) \le (n_l + k)(n_l + k)^{j+1} = (n_l + k)^{j+2}$. This completes the induction and hence the proof. ☐

Proposition 12. *Suppose the net $\mathcal{N}_i$, $i \ge 1$ has the initial marking with n_l tokens in c_i, 1 token in b_i and 0 tokens in all other places. In this initial marking, suppose a firing sequence σ is fired such that $\overline{\sigma}(t_0) = k$. Then the following are true.*

1. $\overline{\sigma}(x_{i-j}) \le (n_l + k)^j$, $0 \le j \le i-1$.
2. $\overline{\sigma}(r_{i-j}) \le (n_l + k)^{j+1}$, $0 \le j \le i-1$.
3. $\overline{\sigma}(v_{i-j}) \le (n_l + k)^j$, $0 \le j \le i-1$.
4. $\overline{\sigma}(w_{i-j}) \le (n_l + k)^j$, $0 \le j \le i-1$.
5. $\overline{\sigma}(u_{i-j}) \le (n_l + k)^{j+1}$, $0 \le j \le i-1$.
6. $\overline{\sigma}(v_0), \overline{\sigma}(w_0), \overline{\sigma}(x_0) \le (n_l + k)^i$.

Proof.

1. Every firing of x_{i-j} needs one token to be added to b_{i-j}. If $j = 0$, then x_{i-j} can fire only once. Otherwise, only transitions that can add tokens to b_{i-j} is x_{i-j+1} and r_{i-j+1}. Every firing of x_{i-j+1} or r_{i-j+1} needs one token to be added to d_{i-j+1}. The only transition that can add tokens to d_{i-j+1} is t_{i-j+1}. By proposition 11, t_{i-j+1} can fire at most $(n_l + k)^j$ times. Therefore, x_{i-j} can fire at most $(n_l + k)^j$ times.

2. Every firing of r_{i-j} needs one token to be added to d_{i-j}. The only transition that can add tokens to d_{i-j} is t_{i-j}. Since by proposition 11, t_{i-j} can fire at most $(n_l + k)^{j+1}$ times, r_{i-j} can fire at most $(n_l + k)^{j+1}$ times.

3. Between any two firings of x_{i-j}, v_{i-j} can fire at most once. Since x_{i-j} can fire at most $(n_l + k)^j$ times, v_{i-j} can also fire at most $(n_l + k)^j$ times.

4. Between any two firings of v_{i-j}, w_{i-j} can fire at most once. Since v_{i-j} can fire at most $(n_l + k)^j$ times, w_{i-j} can also fire at most $(n_l + k)^j$ times.

5. Firing u_{i-j} needs a token to be present in s_{i-j}. Once a token is added to s_{i-j}, u_{i-j} can fire as many times as there are tokens in c_{i-j-1}. By proposition 10, c_{i-j-1} will have at most $(n_l + k)$ tokens. Therefore, number of times u_{i-j} can be fired is bounded by $(n_l + k)$ times the number of times a token can be added to s_{i-j}. v_{i-j} is the only transition that can add tokens to s_{i-j} and it can fire at most $(n_l + k)^j$ times. Therefore, u_{i-j} can fire at most $(n_l + k)^{j+1}$ times.

6. Every firing of v_0, w_0 or x_0 needs one token to be added to b_0. Only transitions that can add tokens to b_0 are x_1 and r_1. Every firing of x_1 or r_1 needs one token to be added to d_1. t_1 is the only transition that can add tokens to d_1 and t_1 can fire at most $(n_l + k)^i$ times by proposition 11. Hence, v_0, w_0 and x_0 can fire at most $(n_l + k)^i$ times.

□

In terms of Definition 19, $U = \{t_0, u_0, x_i, v_i, w_i\}$ and $V = T \setminus U$. $\overline{\tau}(V)$ is bounded by a function that is polynomial in $\overline{\tau}(U)$ and exponential in n, the number of transitions. As noted in the discussion after Lemma 14, this leads to a polynomial space algorithm for reachability problem in any net $\mathcal{N}_i$.

Chapter 4

Conclusion

We surveyed results that have been obtained for the complexity of the reachability problem in various subclasses of Petri nets. Table 4.1 summarizes the results seen in the previous two chapters.

Subclass	**Technique used**	**Complexity of the reachability problem**
S-Systems	Combinatorial techniques	**NL** - hard, with an algorithm in **NLIN**
Live T-Systems	Linear algebraic techniques	**P**
Normal and Sinkless Petri nets	State equation technique	**NP** - complete
Live and bounded Free choice Petri nets	State equation technique	**NP** - complete
1-Safe Petri nets	Combinatorial techniques	**PSPACE** - complete
T-invariant-less and partially bounded S-varying nets *	Linear algebraic techniques	**PSPACE** - complete

Table 4.1: Summary of results

The subclasses marked with * in Table 4.1 have been newly introduced here.

The aim is to find subclasses of Petri nets where the reachability problem is at least exponential space hard. So far, the only subclass known meeting this criteria is reversible nets, studied in [3]. We tried to identify such a subclass using some specific Petri nets as motivating examples. First is the Petri net with non-semilinear reachability set given by Hopcroft and Pansiot (Fig. 3.1) and the second one is the Petri net with non-primitive-recursive bound (Fig. 3.6). However, it turns out that both these nets have reachability algorithm running in **PSPACE**, when final marking is taken to be part of the input. In the second case, final marking itself can be non-primitive-recursive w.r.t. initial marking and size of the net. So, in order to find subclasses where reachability is at least exponential space hard, we need to find nets where the initial and final markings are small in size but a very long firing sequence with large intermediate markings is needed to reach the final marking.

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