

NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Scholarships Screening Test

Saturday, January 23, 2010

Time Allowed: 150 Minutes

Maximum Marks: 40

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 11 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Applied Mathematics and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.
- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of *all* the qualifying statements (amongst (a), (b), and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. **There will be no partial credit.**
- \mathbb{N} denotes the set of natural numbers, \mathbb{Z} - the integers, \mathbb{Q} - the rationals, \mathbb{R} - the reals and \mathbb{C} - the field of complex numbers. \mathbb{R}^n denotes the n -dimensional Euclidean space, which is assumed to be endowed with its 'usual' topology. The symbol \mathbb{Z}_n will denote the ring of integers modulo n . The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol I will denote the identity matrix of appropriate order. The space of continuous real valued functions on an interval $[a, b]$ is denoted by $\mathcal{C}[a, b]$ and is endowed with its usual 'sup' norm.

Section 1: Algebra

1.1 Solve the equation

$$x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$$

given that two of its roots are equal in magnitude but opposite in sign.

1.2 Let G be a group. A subgroup H of G is called *characteristic* if $\varphi(H) \subset H$ for all automorphisms φ of G . Pick out the true statement(s):

- (a) Every characteristic subgroup is normal.
- (b) Every normal subgroup is characteristic.
- (c) If N is a normal subgroup of a group G , and M is a characteristic subgroup of N , then M is a normal subgroup of G .

1.3 Let G be a group and let H and K be subgroups of G . The *commutator subgroup* (H, K) is defined as the smallest subgroup containing all elements of the form $hkh^{-1}k^{-1}$, where $h \in H$ and $k \in K$. Pick out the true statement(s):

- (a) If H and K are normal subgroups, then (H, K) is a normal subgroup.
- (b) If H and K are characteristic subgroups, then (H, K) is a characteristic subgroup.
- (c) (G, G) is normal in G and $G/(G, G)$ is abelian.

1.4 Write the following permutation as a product of disjoint cycles:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 1 & 2 \end{pmatrix}.$$

1.5 Pick out the true statement(s):

- (a) The set of all 2×2 matrices with rational entries (with the usual operations of matrix addition and matrix multiplication) is a ring which has no non-trivial ideals.
- (b) Let $R = \mathcal{C}[0, 1]$ be considered as a ring with the usual operations of pointwise addition and pointwise multiplication. Let

$$\mathcal{I} = \{f : [0, 1] \rightarrow \mathbb{R} \mid f(1/2) = 0\}.$$

Then \mathcal{I} is a maximal ideal.

- (c) Let R be a commutative ring and let \mathcal{P} be a prime ideal of R . Then R/\mathcal{P} is an integral domain.

1.6 What is the degree of the following numbers over \mathbb{Q} ?

- (a) $\sqrt{2} + \sqrt{3}$
- (b) $\sqrt{2}\sqrt{3}$

1.7 Let V be the real vector space of all polynomials of degree ≤ 3 with real coefficients. Define the linear transformation

$$T(\alpha_0 + \alpha_1x + \alpha_2x^2 + \alpha_3x^3) = \alpha_0 + \alpha_1(x + 1) + \alpha_2(x + 1)^2 + \alpha_3(x + 1)^3.$$

Write down the matrix of T with respect to the basis $\{1, x, x^2, x^3\}$ of V .

1.8 Let A be an $n \times n$ upper triangular matrix with complex entries. Pick out the true statement(s):

- (a) If $A \neq 0$, and if $a_{ii} = 0$, for all $1 \leq i \leq n$, then $A^n = 0$.
- (b) If $A \neq I$ and if $a_{ii} = 1$ for all $1 \leq i \leq n$, then A is not diagonalizable.
- (c) If $A \neq 0$, then A is invertible.

1.9 Pick out the true statement(s):

- (a) There exist $n \times n$ matrices A and B with real entries such that

$$(I - (AB - BA))^n = 0.$$

- (b) If A is a symmetric and positive definite $n \times n$ matrix, then

$$(\operatorname{tr}(A))^n \geq n^n \det(A)$$

where ‘tr’ denotes the trace and ‘det’ denotes the determinant of a matrix.

- (c) Let A be a 5×5 skew-symmetric matrix with real entries. Then A is singular.

1.10 Let A be a 5×5 matrix whose characteristic polynomial is given by

$$(\lambda - 2)^3(\lambda + 2)^2.$$

If A is diagonalizable, find α and β such that

$$A^{-1} = \alpha A + \beta I.$$

Section 2: Analysis

2.1 Let $\{a_n\}$ be a sequence of positive real numbers such that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = r < 1.$$

Can we evaluate $\lim_{n \rightarrow \infty} a_n$? If 'yes', right down that limit.

2.2 Test the following series for convergence:

(a)

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+\frac{5}{4}}}.$$

(b)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right).$$

2.3 Consider the polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

with real coefficients. Pick out the case(s) which ensure that the polynomial $p(\cdot)$ has a root in the interval $[0, 1]$.

(a) $a_0 < 0$ and $a_0 + a_1 + \cdots + a_n > 0$.

(b)

$$a_0 + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0.$$

(c)

$$\frac{a_0}{1.2} + \frac{a_1}{2.3} + \cdots + \frac{a_n}{(n+1)(n+2)} = 0.$$

2.4 Pick out the true statement(s):

(a) The function

$$f(x) = \frac{\sin(x^2)}{\sin^2 x}$$

is uniformly continuous on the interval $]0, 1[$.

(b) A continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous if it maps Cauchy sequences into Cauchy sequences.

(c) If a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous, then it maps Cauchy sequences into Cauchy sequences.

2.5 Test the following for uniform convergence:

(a) The sequence of functions

$$\left\{ \frac{x^n}{1+x^n} \right\}$$

over the interval $[0, 2]$.

(b) The series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2 + 1}$$

over \mathbb{R} .

(c) The sequence of functions

$$\{n^2 x^2 e^{-nx}\}$$

over the interval $]0, \infty[$.

2.6 Evaluate:

$$\lim_{n \rightarrow \infty} \left\{ \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \cdots \left(1 + \frac{n}{n}\right) \right\}^{\frac{1}{n}}.$$

2.7 Let $f : [-\pi, \pi] \rightarrow \mathbb{R}$ be continuous. Pick out the case(s) which imply that $f \equiv 0$.

(a)

$$\int_{-\pi}^{\pi} x^n f(x) dx = 0, \text{ for all } n \geq 0.$$

(b)

$$\int_{-\pi}^{\pi} f(x) \cos nx dx = 0, \text{ for all } n \geq 0.$$

(c)

$$\int_{-\pi}^{\pi} f(x) \sin nx dx = 0, \text{ for all } n \geq 1.$$

2.8 Evaluate:

$$\int_{\Gamma} \frac{dz}{(z^2 + 4)^2}$$

where $\Gamma = \{z \in \mathbb{C} \mid |z - i| = 2\}$, described in the anticlockwise (*i.e.* positive) direction.

2.9 Find the residue at $z = 1$ of the function:

$$f(z) = \frac{5z - 2}{z(z - 1)}.$$

2.10 Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be analytic. Which of the following conditions imply that f is a constant function?

(a) $\operatorname{Re} f(z) > 0$ for all $z \in \mathbb{C}$.

(b) $|f(z)| \in \mathbb{Z}$ for all $z \in \mathbb{C}$.

(c) $f(z) = i$ when $z = \left(1 + \frac{k}{n}\right) + i$ for every positive integer k .

Section 3: Topology

3.1 Let S^1 denote the unit circle in the plane \mathbb{R}^2 . Pick out the true statement(s):

- (a) There exists $f : S^1 \rightarrow \mathbb{R}$ which is continuous and one-one.
- (b) For every continuous function $f : S^1 \rightarrow \mathbb{R}$, there exist uncountably many pairs of distinct points x and y in S^1 such that $f(x) = f(y)$.
- (c) There exists $f : S^1 \rightarrow \mathbb{R}$ which is continuous and one-one and onto.

3.2 Which of the following metric spaces are separable?

- (a) $C[0, 1]$ with its usual 'sup-norm' topology.
- (b) The space ℓ^∞ of all bounded real sequences with the metric

$$d(x, y) = \sup_n |x_n - y_n|,$$

where $x = (x_n)$ and $y = (y_n)$.

- (c) The space ℓ^2 of all square summable real sequences with the metric

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2 \right)^{\frac{1}{2}},$$

where $x = (x_n)$ and $y = (y_n)$.

3.3 Which of the following sets are nowhere dense?

- (a) The Cantor set in $[0, 1]$.
- (b) The xy -plane in \mathbb{R}^3 .
- (c) Any countable set in \mathbb{R} .

3.4 Pick out the true statement(s).

- (a) If $f :]-1, 1[\rightarrow \mathbb{R}$ is bounded and continuous, it is uniformly continuous.
- (b) If $f : S^1 \rightarrow \mathbb{R}$ is continuous, it is uniformly continuous.
- (c) If (X, d) is a metric space and $A \subset X$, then the function $f(x) = d(x, A)$ defined by

$$d(x, A) = \inf_{y \in A} d(x, y)$$

is uniformly continuous.

3.5 Which of the following maps define a homeomorphism?

- (a) $f : \mathbb{R} \rightarrow]0, \infty[$, where $f(x) = e^x$.
- (b) $f : [0, 1] \rightarrow S^1$, where $f(t) = (\cos 2\pi t, \sin 2\pi t)$.
- (c) Any map $f : X \rightarrow Y$ which is continuous, one-one and onto, if X is compact and Y is Hausdorff.

3.6 Consider the set of all $n \times n$ matrices with real entries as the space \mathbb{R}^{n^2} . Which of the following sets are compact?

- (a) The set of all orthogonal matrices.
- (b) The set of all matrices with determinant equal to unity.
- (c) The set of all invertible matrices.

3.7 In the set of all $n \times n$ matrices with real entries, considered as the space \mathbb{R}^{n^2} , which of the following sets are connected?

- (a) The set of all orthogonal matrices.
- (b) The set of all matrices with trace equal to unity.
- (c) The set of all symmetric and positive definite matrices.

3.8 Let X be an arbitrary topological space. Pick out the true statement(s):

- (a) If X is compact, then every sequence in X has a convergent subsequence.
- (b) If every sequence in X has a convergent subsequence, then X is compact.
- (c) X is compact if, and only if, every sequence in X has a convergent subsequence.

3.9 Which of the following metric spaces are complete?

- (a) The space $\mathcal{C}^1[0, 1]$ of continuously differentiable real-valued functions on $[0, 1]$ with the metric

$$d(f, g) = \max_{t \in [0, 1]} |f(t) - g(t)|.$$

- (b) The space of all polynomials in a single variable with real coefficients, with the same metric as above.
- (c) The space $\mathcal{C}[0, 1]$ with the metric

$$d(f, g) = \int_0^1 |f(t) - g(t)| dt.$$

3.10 Classify the following alphabets into homeomorphism classes:

N, B, H, M

Section 4: Applied Mathematics

4.1 A body, falling under gravity, experiences a resisting force of air proportional to the square of the velocity of the body. Write down the differential equation governing the motion satisfied by the distance $y(t)$ travelled by the body in time t .

4.2 Reduce the following differential equation to a linear system of first order equations:

$$\frac{d^2x}{dt^2} + P(t)\frac{dx}{dt} + Q(t)x = 0.$$

4.3 The volume of the unit ball in \mathbb{R}^N is given by

$$\omega_N = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2} + 1)}$$

where $\Gamma(\cdot)$ denotes the usual gamma function. Write down the explicit value of ω_5 .

4.4 Consider the differential equation

$$(1 + x)y' = py$$

where p is a constant. Assume that the equation has a power series solution $y = \sum_{n=0}^{\infty} a_n x^n$. Write down the recurrence relation for the coefficients a_n .

4.5 In the above problem, if $y(0) = 1$, use the above series to find a closed form solution to the equation.

4.6 Classify the following partial differential operators as elliptic, parabolic or hyperbolic:

- (a) $5u_{xx} + 6u_{xy} + 2u_{yy}$.
- (b) $2u_{xx} + 6u_{xy} + 2u_{yy}$.

4.7 Let f and g be two smooth scalar valued functions. Compute

$$\operatorname{div}(\nabla f \times \nabla g).$$

4.8 Let S denote the sphere centred at the origin and of radius $a > 0$ in \mathbb{R}^3 . Write down the coordinates of the unit outward normal to S at the point $(x, y, z) \in S$.

4.9 Use Gauss' divergence theorem to evaluate

$$\int \int_S (x^4 + y^4 + z^4) dS$$

where S is the sphere mentioned in the preceding problem.

4.10 Consider the domain $[0, 1] \times [0, T]$. Let $h > 0$ and $k > 0$. Let $x_n = nh$ and $t_m = mk$ for positive integers m and n . Let $u_n^m = u(x_n, t_m)$. Write down the partial differential equation for which the following discretization defines a numerical scheme:

$$\frac{u_n^{m+1} - u_n^m}{k} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{h^2}.$$

Section 5: Miscellaneous

5.1 Let n be a fixed positive integer and let C_k denote the usual binomial coefficient ${}^n C_k$, the number of ways of choosing k objects from n objects. Evaluate:

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1}.$$

5.2 Find the number of ways $2n$ persons can be seated at 2 round tables, with n persons at each table.

5.3 Let a point (x, y) be chosen at random in the square $[0, 1] \times [0, 1]$. Find the probability that $y \geq x^2$.

5.4 Pick out the true statement(s):

- (a) If n is an odd positive integer, then 8 divides $n^2 - 1$.
- (b) If n and m are odd positive integers, then $n^2 + m^2$ is not a perfect square.
- (c) For every positive integer n ,

$$\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$$

is an integer.

5.5 Consider a circle of unit radius centered at O in the plane. Let AB be a chord which makes an angle θ with the tangent to the circle at A . Find the area of the triangle OAB .

5.6 Evaluate:

$$\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \cdots$$

5.7 Evaluate:

$$1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \cdots$$

5.8 Find the sum to n terms as well as the sum to infinity of the series:

$$\frac{1}{3} + \frac{1}{4} \cdot \frac{1}{2!} + \frac{1}{5} \cdot \frac{1}{3!} + \cdots$$

5.9 If a, b and c are all distinct real numbers, find the condition that the following determinant vanishes:

$$\begin{vmatrix} a & a^2 & 1 + a^3 \\ b & b^2 & 1 + b^3 \\ c & c^2 & 1 + c^3 \end{vmatrix}.$$

5.10 Assume that the line segment $[0, 2]$ in the x -axis of the plane acts as a mirror. A light ray from the point $(0, 1)$ gets reflected off this mirror and reaches the point $(2, 2)$. Find the point of incidence on the mirror.