NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 21, 2013

Time Allowed: 150 Minutes

Maximum Marks: 30

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 7 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.

- There are three parts to this test: Algebra, Analysis and Geometry. Each part consists of **10** questions adding up to **30** questions in all.

- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.

- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).

- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.

- **Notations**

  - $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \cdots\}$, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.
  - The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.
  - We denote by $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$), the set of all $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$).
  - The trace of a square matrix A will be denoted tr(A) and the determinant by det(A).
  - The transpose (respectively, adjoint) of a matrix A in $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will be denoted by A^T (respectively, A^*).
  - The derivative of a function f will be denoted by f'.
  - All logarithms, unless specified otherwise, are to the base e.

- **Calculators are not allowed.**

Section 1: Algebra

1.1 Which of the following statements are true?
a. Every group of order 11 is cyclic.
b. Every group of order 111 is cyclic.
c. Every group of order 1111 is cyclic.

1.2 Let S_n denote the symmetric group of order n, *i.e.* the group of all permutations of the n symbols $\{1, 2, \cdots, n\}$. Given two permutations σ and τ in S_n, we define the product $\sigma\tau$ as their composition got by applying σ first and then applying τ to the set $\{1, 2, \cdots, n\}$. Write down the following permutation in S_8 as the product of disjoint cycles:

$$(1\ 4\ 3\ 8\ 7)(5\ 4\ 8).$$

1.3 Write down all the permutations in S_4 which are conjugate to the permutation $(1\ 2)(3\ 4)$.

1.4 Let R be a ring such that $x^2 = x$ for every $x \in R$. Which of the following statements are true?
a. $x^n = x$ for every $n \in \mathbb{N}$ and every $x \in R$.
b. $x = -x$ for every $x \in R$.
c. R is a commutative ring.

1.5 For a prime number p let $\mathbb{F}_p$ denote the field consisting of $0, 1, 2, \cdots, p-1$ with addition and multiplication modulo p. Which of the following quotient rings are fields?
a.
$$\mathbb{F}_5[x]/(x^2 + x + 1)$$
b.
$$\mathbb{F}_2[x]/(x^3 + x + 1)$$
c.
$$\mathbb{F}_3[x]/(x^3 + x + 1)$$

1.6 Let V be the subspace of $\mathbb{M}_2(\mathbb{R})$ consisting of all matrices with trace zero and such that the entries of the first row add up to zero. Write down a basis for V.

1.7 Let $V \subset \mathbb{M}_n(\mathbb{R})$ be the subspace of all matrices such that the entries in every row add up to zero and the entries in every column also add up to zero. What is the dimension of V?

1.8 Let $T : \mathbb{M}_2(\mathbb{R}) \to \mathbb{M}_2(\mathbb{R})$ be the linear transformation defined by

$$T(A) = 2A + 3A^T.$$

Write down the matrix of this transformation with respect to the basis $\{E_i, 1 \le i \le 4\}$ where

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

1.9 Find the values of $\alpha \in \mathbb{R}$ such that the matrix

$$\begin{bmatrix} 3 & \alpha \\ \alpha & 5 \end{bmatrix}$$

has 2 as an eigenvalue.

1.10 Let $A = (a_{ij}) \in \mathbb{M}_3(\mathbb{R})$ be such that $a_{ij} = -a_{ji}$ for all $1 \leq i, j \leq 3$. If $3i$ is an eigenvalue of A, find its other eigenvalues.

Section 2: Analysis

2.1 Evaluate:
$$\lim_{x\to\infty} x\left(\log\left(1+\frac{x}{2}\right)-\log\left(\frac{x}{2}\right)\right).$$

2.2 Which of the following series are convergent?

a.
$$\sum_{n=1}^{\infty}(\sqrt{n^4+1}-\sqrt{n^4-1}).$$

b.
$$\sum_{n=1}^{\infty}((n^3+1)^{\frac{1}{3}}-n).$$

c.
$$\frac{1.2}{3.4.5}+\frac{2.3}{4.5.6}+\frac{3.4}{5.6.7}+\cdots$$

2.3 Find the points where the following function $f:\mathbb{R}\to\mathbb{R}$ is continuous:
$$f(x) \;=\; \begin{cases} 0 & \text{if } x \text{ is rational,} \\ \sin|x| & \text{if } x \text{ is irrational.} \end{cases}$$

2.4 Let $f:]0,\infty[\to]0,\infty[$ be differentiable at $a>0$. Evaluate:
$$\lim_{x\to a}\left(\frac{f(x)}{f(a)}\right)^{\frac{1}{\log x-\log a}}.$$

2.5 Define
$$f(x) \;=\; \begin{cases} x^2 & \text{if } x\in]0,2[\text{ and } x \text{ is rational,} \\ 2x-1 & \text{if } x\in]0,2[\text{ and } x \text{ is irrational.} \end{cases}$$

Which of the following statements are true?
a. f is not differentiable at $x=1$.
b. f is differentiable at $x=1$.
c. f is differentiable only at $x=1$.

2.6 Which of the following statements are true?
a. If
$$a_0+\frac{a_1}{2}+\frac{a_2}{3}+\cdots+\frac{a_n}{n+1}=0,$$
where $a_i\in\mathbb{R}$ for $0\le i\le n$, then the polynomial
$$a_0+a_1x+a_2x^2+\cdots+a_nx^n$$
has a root in the interval $]0,1[$.
b. If $f:[a,b]\to\mathbb{R}$ is continuous and is differentiable in $]a,b[$, where $a>0$ and if
$$\frac{f(a)}{a} = \frac{f(b)}{b},$$
then there exists $x_0\in]a,b[$ such that $x_0f'(x_0)=f(x_0)$.
c. The polynomial $x^{13}+7x^3-5$ has only one real root.

2.7 Evaluate:

$$\int_0^1 \max\{x, \frac{1}{2} - x\}\ dx.$$

2.8 In the collection of all right circular cylinders of fixed volume c, what is the relationship between the height h and the radius r of the cylinder which has the least total surface area?

2.9 Write down the Taylor series expansion about the origin valid in the interval $]-1,1[$ for the function $f(x) = \log\sqrt{1+x^2}$.

2.10 Let $a > 0$. Find the sum of the infinite series:

$$1 + \frac{(\log a)^2}{2!} + \frac{(\log a)^4}{4!} + \frac{(\log a)^6}{6!} + \cdots$$

Section 3: Geometry

3.1 Find the reflection of the point $(2, 1)$ with respect to the line $x = y$ in the xy-plane.

3.2 Find the area of the circle in the xy-plane which has its centre at the point (1,2) and which has the line $x = y$ as a tangent.

3.3 Find the incentre of the triangle in the xy-plane whose sides are given by the lines $x = 0, y = 0$ and $\frac{x}{3} + \frac{y}{4} = 1$.

3.4 Let A and B be fixed points in a plane such that the length of the line segment AB is d. Let the point P describe an ellipse by moving on the plane such that the sum of the lengths of the line segments PA and PB is a constant, ℓ. Express the length of the semi-major axis, a and the length of the semi-minor axis, b, of the ellipse in terms of d and ℓ.

3.5 Let $A = (a_{ij})$ be a non-zero 2×2 symmetric matrix with real entries. Let

$$S = \{(x, y) \in \mathbb{R}^2 \mid a_{11}x^2 + 2a_{12}xy + a_{22}y^2 = 0\}.$$

Which of the following conditions imply that S is unbounded?
a. $\det(A) > 0$.
b. $\det(A) = 0$.
c. $\det(A) < 0$.

3.6 Let $A \in \mathbb{M}_2(\mathbb{R})$ define an invertible linear transformation on $\mathbb{R}^2$. Let T be a triangle with one of its vertices at the origin and of area a. What is the area of the triangle which is the image of T under this transformation?

3.7 Find the area of the ellipse whose equation in the xy-plane is given by

$$5x^2 - 6xy + 5y^2 = 8.$$

3.8 Let a, b and c be positive real numbers. Find the equation of the sphere which passes through the origin and through the points where the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

meets the coordinate axes.

3.9 Consider the sphere $x^2 + y^2 + z^2 = r^2$. Let $(a, b, c) \neq (0, 0, 0)$ be a point in the interior of this sphere. Write down the equation of the plane whose intersection with the sphere is a circle whose centre is the point (a, b, c).

3.10 Find the area of the polygon whose vertices are represented in the complex plane by the eighth roots of unity.