Entanglement entropy in all dimensions

S. Shankaranarayanan

IISER-Trivandrum

THIRUVANANTHAPURAM

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With: S. Braunstein, S. Das — Work in Progress

S. Das, S. Sur — 0806.0402

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Outline

- Black-hole thermodynamics and puzzles
- Entanglement entropy
  - What is entanglement? How to quantify entanglement?
  - Example – 2 coupled HOs
  - Why is entanglement entropy relevant for black-hole entropy?
- Entanglement entropy of black-hole space-times
  - Setup
  - Assumptions
- Subleading corrections in 4-dimensions
- Entanglement in higher dimensions
  - Asymptotic limit and non-convergence of von Neumann entropy
  - Rényi entropy
  - Results in higher dimensions
- Discussions

Black-hole thermodynamics

- Classically, entropy of black-hole is infinite; temperature is zero.
- Semi-classical limit [classical gravity; quantum matter fields]

$$S_{\mathrm{BH}} = \left(\frac{k_B}{4}\right)\left(\frac{\mathcal{A}_{\mathrm{H}}}{\ell_P^2}\right) \qquad \mathcal{A}_{\mathrm{H}} \text{ horizon area} \qquad \ell_P^2 \equiv \frac{\hbar G}{c^3}$$

$$T_{\mathrm{H}} = \left(\frac{\hbar c}{k_B}\right)\left(\frac{\kappa}{2\pi}\right) \qquad k \text{ surface gravity}$$

Hawking radiation

Black-hole thermodynamics

- Semi-classical limit [classical gravity; quantum matter fields]

$$S_{\rm BH} = \left(\frac{k_B}{4}\right)\left(\frac{\mathcal{A}_{\rm H}}{\ell_P^2}\right) \qquad \mathcal{A}_{\rm H} \;\; \text{horizon area} \qquad \ell_P^2 \equiv \frac{\hbar G}{c^3}$$

$$T_{\rm H} = \left(\frac{\hbar c}{k_B}\right)\left(\frac{\kappa}{2\pi}\right) \qquad k \;\; \text{surface gravity} \qquad \text{Hawking radiation}$$

- Properties
  - Unlike ideal gas [$S \propto V$], $S_{\rm BH}$ is not extensive
  - $S_{\rm BH}$ is large $S_{\rm BH} = 1.05 \times 10^{77} \left(\frac{M}{M_\odot}\right)^2$ $\quad S_{\rm Univ} \sim 2.35 \times 10^{88}$

    Couple of hundred thousand solar mass black holes can contain as much entropy as is free in the entire universe.

Dimensional analysis – Planck units

Gravity action

Semi-classical entropy

Dimensional analysis – Planck units

Gravity action

- 4-D Gravity action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \equiv \frac{[L]^2}{\ell_{\rm Pl}^2}$$

Semi-classical entropy

- Bekenstein-Hawking entropy

$$S_{\rm BH} = \frac{A}{4\ell_{\rm Pl}^2} \equiv \frac{[L]^2}{\ell_{\rm Pl}^2}$$

Gravity action

- 4-D Gravity action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \equiv \frac{[L]^2}{\ell^2_{\mathrm{Pl}}}$$

- General gravity action

$$\begin{aligned} S &= \frac{1}{16\pi G_D}\int d^Dx\sqrt{-g}[R+\alpha R^2] \\ &\equiv \frac{[L]^{D-2}}{\ell^{D-2}_{\mathrm{Pl}}}\left[1+\frac{\alpha}{[L]^2}\right] \end{aligned}$$

Semi-classical entropy

- Bekenstein-Hawking entropy

$$S_{\mathrm{BH}} = \frac{A}{4\ell^2_{\mathrm{Pl}}} \equiv \frac{[L]^2}{\ell^2_{\mathrm{Pl}}}$$

- Noether charge entropy

$$\begin{aligned} S_{Noether} &= \frac{A_D}{4\ell^2_{\mathrm{Pl}}}\left[1+\alpha\, A_D^{2/(D-2)}\right] \\ &\equiv \frac{[L]^{D-2}}{\ell^{D-2}_{\mathrm{Pl}}}\left[1+\frac{\alpha}{[L]^2}\right] \end{aligned}$$

Black-hole entropy — puzzles

$$S = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g}[R + \alpha R^2] \equiv \frac{[L]^{D-2}}{\ell_{\rm Pl}^{D-2}} \left[1 + \frac{\alpha}{[L]^2}\right]$$

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Puzzles

- Gravity action and semiclassical BH entropy have same dimensional form

  How does the quantum corrections change these relations?

  Example: Ideal gas

$$\frac{S_{\rm ideal}}{k_B\, N} = \ln\left(V\, T^{3/2}\right) \qquad \text{Classical expression}$$

$$\frac{S_{\rm ST}}{k_B\, N} = \ln\left(V\, T^{3/2}\right) + \frac{1}{2}\ln\frac{M^3}{N^5} + \frac{3}{2}\ln\left(\frac{4\pi k_B}{3\hbar^2}\right) \qquad \text{Sackur-Tedrode equation}$$

Black-hole entropy — puzzles

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Puzzles

- Gravity action and semiclassical BH entropy have same dimensional form

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- How does $S_{\rm BH}$ fit in with the standard view of the statistical origin?

$$S \stackrel{?}{=} k_B \ln(\#\text{ of microstates})$$

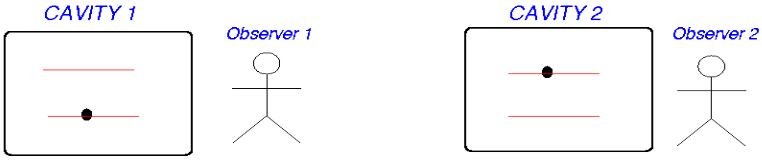
- Are quantum DOF that lead to $S_{\rm BH}$ and corrections, identical or different?

Black-hole entropy — approaches

- Bottom-up approach: Counting BH states assuming funda. structure
  - D-Branes: weak coupling string states [Strominger & Vafa '96]
  - Spin-networks: Chern-Simons theory on the boundary of horizon [Ashtekar et al '98]
  - Conformal symmetry: CFT close to the horizon [Carlip '99]
- Top-down approach: Semi-classical
  - Noether charge: bifurcate Killing horizon [Wald '93]
  - Entanglement entropy ['t hooft '85, Bombelli et al '86]

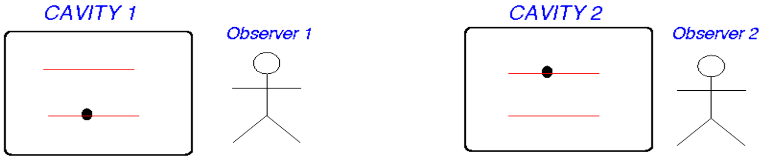
Entanglement entropy

Entanglement



Cavities: Assume one mode field with single photon excitation

Entanglement



Cavities: Assume one mode field with single photon excitation

- Statistical correlations: separable states

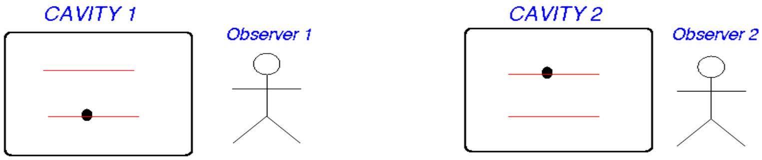
$$|\Phi\rangle_{12} = |\phi\rangle_1 \bigotimes |\psi\rangle_2$$

Properties:

1) States can be prepared by local operation

2) Cavities are uncorrelated $\Longrightarrow$ measurement of 1 do not provide information about 2

Entanglement



Cavities: Assume one mode field with single photon excitation

- Quantum correlations: entangled states

$$|\Phi\rangle_{12} = \frac{1}{\sqrt{2}}\left[|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2\right]$$

Properties:

1) States can not be prepared by local operation

2) Cavities can simultaneously exist in more than one state

3) Correlated cavities $\Longrightarrow$ state of 1 determines state of 2

Entanglement entropy

- Pure states:

  $|\Phi\rangle_{12} = \sum_{ij} \lambda_{ij} |i\rangle_1 |j\rangle_2 \qquad |\Phi\rangle_{12} = \sum_{\alpha} \omega_\alpha |\alpha\rangle_1 |\alpha\rangle_2 \Longrightarrow$ Schmidt decomposition

  Any two-body system can always be Schmidt decomposed.

  Coefficient ω_α: gives the degree of entanglement with $\sum_\alpha |\omega_\alpha|^2 = 1$

  Example: $\omega_\alpha = \frac{1}{\sqrt{\alpha}}$ is maximally entangled state

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- Measure of entanglement: Von-Neumann entropy

other measures

$$S = -Tr(\rho \ln[\rho]) = -\sum_n p_n \ln(p_n) \qquad \int_{-\infty}^{\infty} dx' \rho(x, x') f_n(x') = p_n f_n(x)$$

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$$\rho_{12} = |\Phi\rangle_{12} \langle\Phi|_{12} \implies S_{12} = 0 \qquad \rho_1 = Tr_2[\rho_{12}] \implies S_1 = S_2$$

Example – coupled harmonic oscillators

- Hamiltonian $$H = \frac{1}{2}\left[p_1^2 + p_2^2 + k_0\left(x_1^2 + x_2^2\right) + \underset{\substack{\uparrow \\ \text{interaction term}(>0)}}{k_1}\left(x_1 - x_2\right)^2\right]$$

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- Assume ground state

$$\begin{aligned}\psi_0\left(x_1, x_2\right) &= \frac{\left(\omega_+\omega_-\right)^{1/4}}{\pi^{1/2}}\exp\left[-\frac{\omega_+ x_+^2 + \omega_- x_-^2}{2}\right] \\ &= \frac{\left(\omega_+\omega_-\right)^{1/4}}{\pi^{1/2}}\exp\left[-\frac{1}{4}\left(\omega_+(x_1+x_2)^2 + \omega_-(x_1-x_2)^2\right)\right]\end{aligned}$$

$\uparrow$

entangled state

$$x_\pm = \frac{x_1 \pm x_2}{\sqrt{2}}; \quad \omega_+ = \sqrt{k_0}; \quad \omega_- = \sqrt{k_0 + 2k_1}$$

entropy of the total system is zero

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  interaction term(> 0)

- Trace over oscillator 1

$$\rho_2\left(x_2, x_2'\right) = \int_{-\infty}^{\infty} dx_1 \psi_0\left(x_1, x_2\right)\psi_0^\star\left(x_1, x_2'\right)$$

$$= \sqrt{\frac{\gamma - \beta}{\pi}} \exp\left[-\frac{\gamma}{2}\left(x_2^2 + x_2'^2\right) + \beta x_2 x_2'\right]$$

$$\beta = \frac{\omega_-(1 - R^2)^2}{4(1 + R^2)}; \qquad \gamma = \omega_- \frac{1 + 6R^2 + R^4}{4(1 + R^2)}$$

$$\xi = \left(\frac{1 - R}{1 + R}\right)^2 < 1; \qquad R^2 = \frac{\omega_+}{\omega_-} < 1$$

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  interaction term(> 0)

- eigen functions and values

$$\int_{-\infty}^{\infty} dx' \rho(x, x') f_n(x') = p_n f_n(x)$$

$$f_n(x) = H_n\left(\sqrt{\alpha}x\right)\exp\left(-\frac{\alpha x^2}{2}\right) \qquad p_n = (1-\xi)\,\xi^n$$

Example – coupled harmonic oscillators

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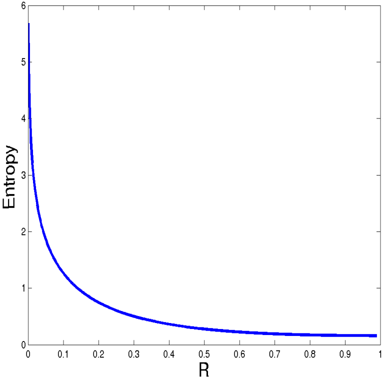
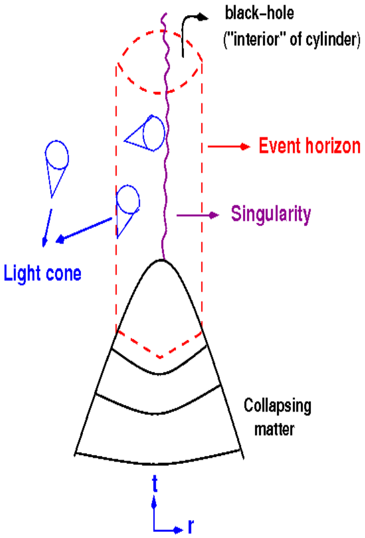


- Entanglement entropy (S_{ent})

$$S_{\text{ent}}(\xi) = -\sum_{n=0}^{\infty} p_n \ln(p_n) = -\ln[1-\xi] - \frac{\xi \ln \xi}{1-\xi}$$

- strongly coupled ($R = 0$) $S_{\text{ent}} \to \infty$
- uncoupled ($R = 1$) $S_{\text{ent}} \to 0$

Entanglement and black-hole entropy — connection



- before collapse, entanglement entropy is zero.

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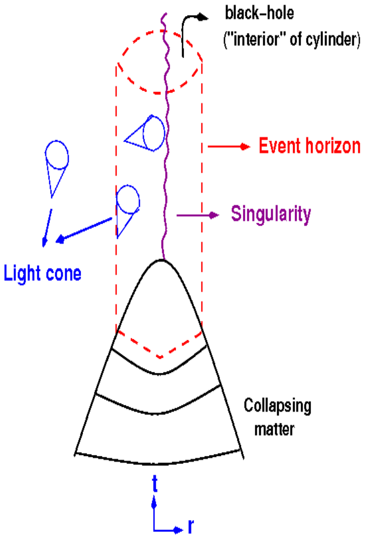

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- after the collapse, $S_{\mathrm{BH}} \neq 0$

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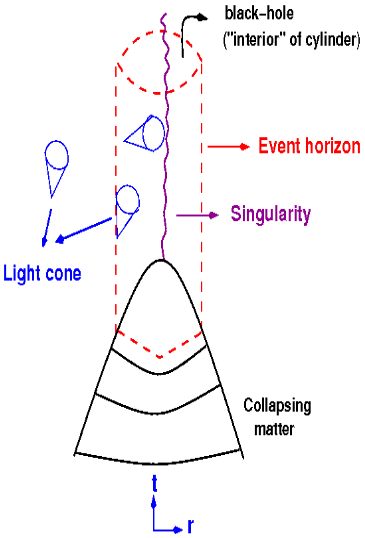

- before collapse, entanglement entropy is zero.
- after the collapse, $S_{\rm BH} \neq 0$
- observers at infinity have no information about the quantum DOF inside horizon.

$$\Longrightarrow \qquad S_{\rm ent} \neq 0$$

Both entropies are associated with the existence of horizons

- Both are pure quantum effects, with no classical analogue.

The setup and assumptions

Entanglement entropy — set up

Das, SS & Sur '08

- Metric perturbations of black-hole space-time correspond to test scalar field in these space-time. [Chandrasekhar, Mathematical Theory of BHs]

▸ Technical details

Entanglement entropy — set up

Das, SS & Sur '08

- Consider scalar fields in the following $(D+2)-$d line-element:

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr}{f(r)} + r^2 d\Omega_D^2 && \text{Schwarzschild coordinate} \\ &= -d\tau^2 + [1-f(r)]d\xi^2 + r^2 d\Omega_D^2 && \text{Lemâitre coordinate} \end{aligned}$$

where

$$\xi - \tau = \int \frac{dr}{\sqrt{1-f(r)}}$$

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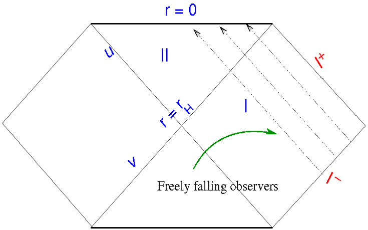

- non-singular at the horizon $(r_{\rm H})$;
  covers regions I,II in conformal diagram
- ξ is space-like everywhere;
  r is space-like only for $r > r_{\rm H}$.
- explicitly time dependent;
  correspond to freely falling observer

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where

$$\xi - \tau = \int \frac{dr}{\sqrt{1 - f(r)}}$$

- Hamiltonian of scalar field (φ) in the above background

$$H(\tau) = \frac{1}{2}\int_\tau^\infty \frac{d\xi}{\sqrt{1-f(r)}} \left[\frac{\Pi_{lm}^2}{r^D} + r^D \left(\partial_\xi \varphi_{lm}\right)^2 + l(l+D-1)[1-f(r)]\varphi_{lm}^2\right]$$

$$\varphi_{lm}(r) = r\int d\Omega\, Y_{lm}(\theta,\phi)\varphi(\vec{r}); \quad \Pi_{lm}(r) = r\int d\Omega\; Y_{lm}(\theta,\phi)\Pi(\vec{r})$$

Entanglement entropy — procedure

- Quantization: Schrödinger representation provides simple description of states for time-dependent systems

  basis vectors satisfy $\hat{\varphi}(\tau,\xi)|\,\varphi(\xi),\tau\,\rangle = \varphi(\xi)|\,\varphi(\xi),\tau\,\rangle$

  wave functional satisfy functional Schrödinger equation

$$i\frac{\partial\Psi}{\partial\tau} = \int_{\tau}^{\infty} d\xi\, H(\tau)\,\Psi[\varphi(\xi),\tau]$$

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$$i\frac{\partial\Psi}{\partial\tau} = \int_{\tau}^{\infty} d\xi\, H(\tau)\,\Psi[\varphi(\xi),\tau]$$

- As in any field theory, $S_{\rm ent}$ has ultra-violet divergences. UV divergence can be absorbed in the Newton's constant.

Technical details

Entanglement entropy — procedure

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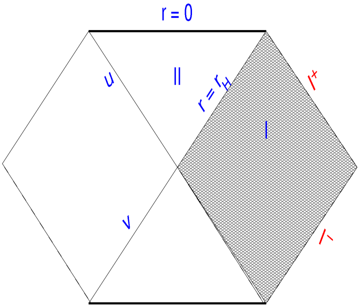
- Integrate over the region $[r_{\mathrm{H}},\infty)$ to obtain entanglement entropy



Assumption

- Hamiltonian evolves adiabatically i. e. for the Hamiltonian

$$
\begin{aligned}
H(\tau) &= \frac{1}{2}\int_{\tau}^{\infty}\frac{d\xi}{\sqrt{1-f(r)}}\left[\frac{\Pi_{_{lm}}^2}{r^2}+r^2\left(\partial_\xi\varphi_{_{lm}}\right)^2+l(l+1)[1-f(r)]\varphi_{_{lm}}^2\right] \\
&= \Pi^2+\omega^2(\tau)\varphi^2
\end{aligned}
$$

$$
\left|\frac{1}{\omega^2}\frac{d\omega}{dt}\right| \ll 1
$$

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- Technically: Evolution of the late-time modes leading to Hawking particles are negligible

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- $\Psi[\varphi(\xi),\tau]$ has a weak time-dependence

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- $\Psi[\varphi(\xi),\tau]$ has a weak time-dependence

  The Hamiltonian at a fixed Lemaître time $\tau=\tau_0\equiv 0$ reduces to

$$
H_{_F}=\frac{1}{2}\int_0^{\infty}dr\left\{\pi^2(r)+r^2\left[\frac{\partial}{\partial r}\left(\frac{\varphi(r)}{r}\right)\right]^2+\frac{l(l+1)}{r^2}\,\varphi^2(r)\right\}
$$

$\hat{\varphi}$ are time-independent
$\Psi[\varphi]$ satisfies time-independent functional Schrödinger equation.

Results for 4-dimensions

Entanglement entropy in 4-d

- can not be computed analytically $\Longrightarrow$ need to discretize Hamiltonian

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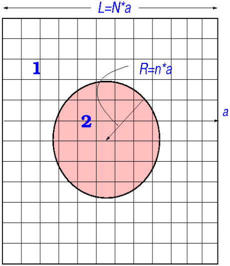


discretize in a spherical lattice $[L = N \times a]$

$$H_{lm} = \frac{1}{2a}\sum_{j=1}^{N}\left[\pi_{lm,j}^2 + \left[j+\frac{1}{2}\right]^2\left[\frac{\varphi_{lm,j}}{j} - \frac{\varphi_{lm,j+1}}{j+1}\right]^2 + \frac{l(l+1)}{j^2}\,\varphi_{lm,j}^2\right]$$

$$= \quad \frac{1}{2a}\sum_{j=1}^{N}\pi_j^2 + \frac{1}{2a}\sum_{i,j=1}^{N}\varphi_i K_{ij}\varphi_j$$

N-coupled harmonic oscillators

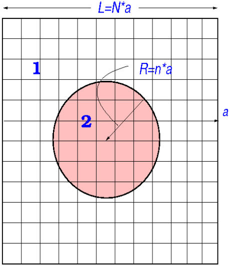
Entanglement entropy in 4-d

- can not be computed analytically $\Longrightarrow$ need to discretize Hamiltonian

$$
\begin{aligned}
K_{ij} \quad &= \frac{1}{i^2}\left[l(l+1) + \frac{9}{4}\,\delta_{i1}\delta_{j1} + \left[N-\frac{1}{2}\right]^2 \delta_{iN}\delta_{jN}\right] \\
&\frac{1}{i^2}\left[\left(i+\frac{1}{2}\right)^2 + \left(i-\frac{1}{2}\right)^2\right]\delta_{i,j(i\neq 1,N)} \\
&\underbrace{- \left[\frac{(j+\frac{1}{2})^2}{j(j+1)}\right]\delta_{i,j+1} - \left[\frac{(i+\frac{1}{2})^2}{i(i+1)}\right]\delta_{i,j-1}}_{\text{nearest neighbour interaction}}
\end{aligned}
$$

$$
K = \begin{pmatrix}
\times & \times & & & \\
\times & \times & \times & & \\
 & \times & \times & \times & \\
 & & \times & \times & \times \\
 & & & \times & \times
\end{pmatrix}
$$

Entanglement entropy in 4-d



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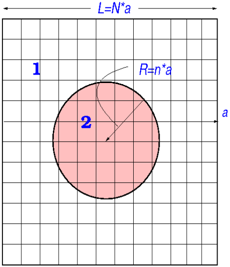
N-coupled harmonic oscillators

- The most general quantum state for N-coupled HOs

$$\Psi(x_1 \ldots x_N) = \left[\frac{|\Omega|}{\pi^N} \right]^{1/4} \exp\left[-\frac{x^T \cdot \Omega \cdot x}{2} \right] \times \prod_{i=1}^{N} \frac{1}{\sqrt{2^{\nu_i} \nu_i}} H_{\nu_i} \left(K_{D\,i}^{\frac{1}{4}} \, \bar{x}_i \right)$$

This will not provide the physics behind the result!

Entanglement entropy in 4-d



discretize in a spherical lattice $[L = N \times a]$

$$H_{lm} = \frac{1}{2a}\sum_{j=1}^{N}\left[\pi_{lm,j}^2 + \left[j+\frac{1}{2}\right]^2\left[\frac{\varphi_{lm,j}}{j} - \frac{\varphi_{lm,j+1}}{j+1}\right]^2 + \frac{l(l+1)}{j^2}\,\varphi_{lm,j}^2\right]$$

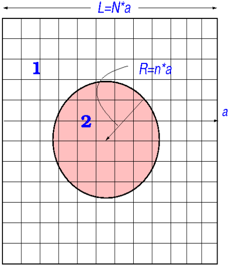
$$= \quad \frac{1}{2a}\sum_{j=1}^{N}\pi_j^2 + \frac{1}{2a}\sum_{i,j=1}^{N}\varphi_i K_{ij}\varphi_j$$

N-coupled harmonic oscillators

- Make simple choices of the state. Our choice:

$$\Psi \quad = \quad \underset{\substack{\uparrow \\ \text{vacuum state}}}{c_0\Psi_0} + \underset{\substack{\uparrow \\ \text{1-particle state}}}{c_1\Psi_1} \qquad\qquad |c_0|^2 + |c_1|^2 = 1$$

Entanglement entropy in 4-d



discretize in a spherical lattice $[L = N \times a]$

$$H_{lm} = \frac{1}{2a} \sum_{j=1}^{N} \left[\pi^2_{lm,j} + \left[j + \frac{1}{2} \right]^2 \left[\frac{\varphi_{lm,j}}{j} - \frac{\varphi_{lm,j+1}}{j+1} \right]^2 + \frac{l(l+1)}{j^2}\, \varphi^2_{lm,j} \right]$$

$$= \quad \frac{1}{2a} \sum_{j=1}^{N} \pi_j^2 + \frac{1}{2a} \sum_{i,j=1}^{N} \varphi_i K_{ij} \varphi_j$$

N-coupled harmonic oscillators

- The reduced density matrix is

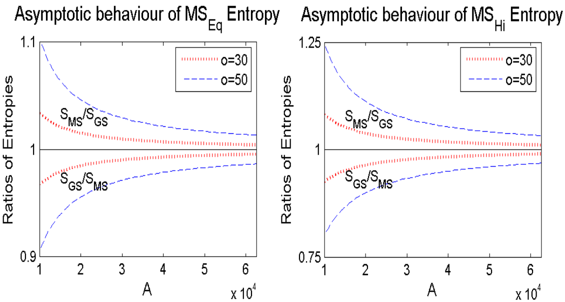
$$\rho[\varphi_2(t); \varphi_2(t')] = \int \mathcal{D}\varphi_1\, \Psi[\varphi_1(x,t); \varphi_2(x,t)] \Psi^\star[\varphi_1(x,t'); \varphi_2(x,t')]$$

- Entanglement entropy is $S = \sum_{\ell=0}^{\infty} (2\ell+1)\, S_\ell(n, N)$

▸ Higher dimensions

Power-law corrections

Das, SS & Sur '07

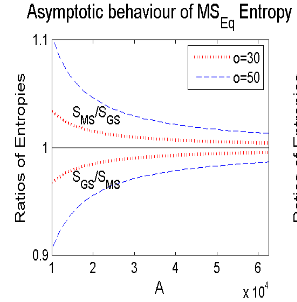


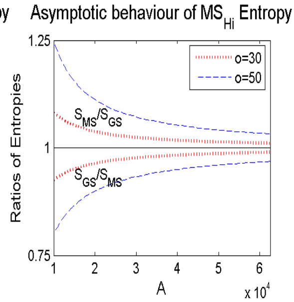
$N = 300, n = 100 - 200$

$MS_{EQ} \rightarrow |c_0| = |c_1| = 0.7$

$MS_{Hi} \rightarrow |c_0| = 0.5,$
$|c_1| = 0.87$

Power-law corrections





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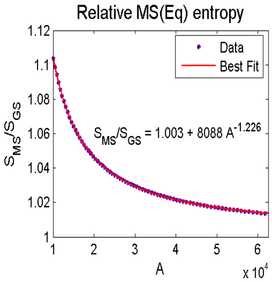
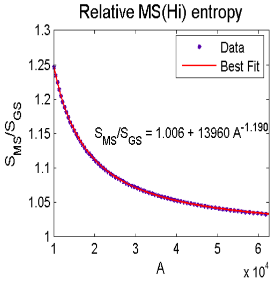




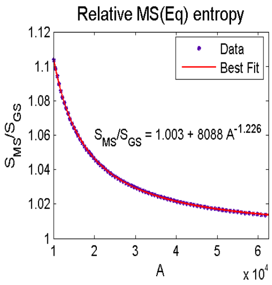
$$S_{\rm MC} = S_{\rm BH}\left(1 + \frac{a_1}{A^\beta}\right)$$

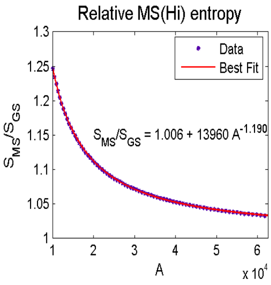
$$\beta > 0, \quad a_1 \propto |c_1|$$

Technical details

Power-law corrections

Das, SS & Sur '07





$$S_{\rm MC} = S_{\rm BH}\left(1 + \frac{a_1}{A^\beta}\right)$$

$$\beta > 0, \quad a_1 \propto |c_1|$$

Technical details

Salient features

- For large horizon area ($A \gg \ell_{\rm Pl}^2$), area law is recovered.
- For small horizon area, large deviation from the area law
- For $c_1 \to 0$, $S_{\rm MC} \to S_{\rm BH}$. $S_{\rm MC}$ of vacuum state leads to the area law and 1-particle state contribute to the power-law corrections.
- Interpretation: $A \gg \ell_{\rm Pl}^2$ difficult to excite modes, vacuum contribution High-energy limit, field modes can be excited, deviations from $S_{\rm BH}$

Higher dimensions

Asymptotic limit and non-convergence Braunstein, Das & SS '11

- 4-d space-times

4-dimensions

  - S_ℓ is independent of N (for $N > 60$).
  - For large values of ℓ: $$S_\ell \sim \frac{n(n+1)(2n+1)^2}{64\ell^2(\ell+1)^2}$$

  Large ℓ do not contribute to entropy; $S_{\rm ent}$ converge

- higher dimensional space-times

$$S(n,N,q) = \sum_\ell (2\ell + D - 1)\,\mathcal{W}(\ell)\,S_l(n,N,q) \qquad \mathcal{W}(\ell) = \left(\frac{(\ell+D-2)!}{(D-1)!\;\ell!}\right)$$

  - In large ℓ limit, we have $$S_\ell \sim \frac{n^{D/2}(n+1)^{D/2}(2n+1)^D}{2^{D+4}\ell^2(\ell+1)^2}$$
  - However, $S(n,N)$ does not converge for large ℓ.

Look for a different measure for entanglement

New measure – Rényi entropy

- Rényi entropy is defined as

◂ vonNeumann

$$S_R(q) = \frac{1}{1-q} \log \left(\sum_{i=1}^{n} p_i^q \right) \qquad q \text{ is parameter}$$

- Properties
  - In $q \to 1$ limit, $$\lim_{q\to 1} \frac{1}{1-q} \ln \sum_{i=1}^{N} p_i^q = -\sum_{i=1}^{N} p_i \ln p_i$$
  - It is nonnegative
  - It vanishes when the uncertainty is the smallest (only one $p_i \neq 0$)
  - It takes on the maximal value $\ln n$ when all n probabilities are equal
  - It is extensive; satisfies the following condition. If p_{ij} is a product of two probability distributions $p_{ij} = p_i^{(a)} p_j^{(b)}$

$$S_R(q) = \frac{1}{1-q} \ln \left(\sum_i p_i^{(a)} \sum_j p_j^{(b)} \right) = S_R^{(a)}(q) + S_R^{(b)}(q)$$

S_R in higher dimensions

Braunstein, Das & SS '11

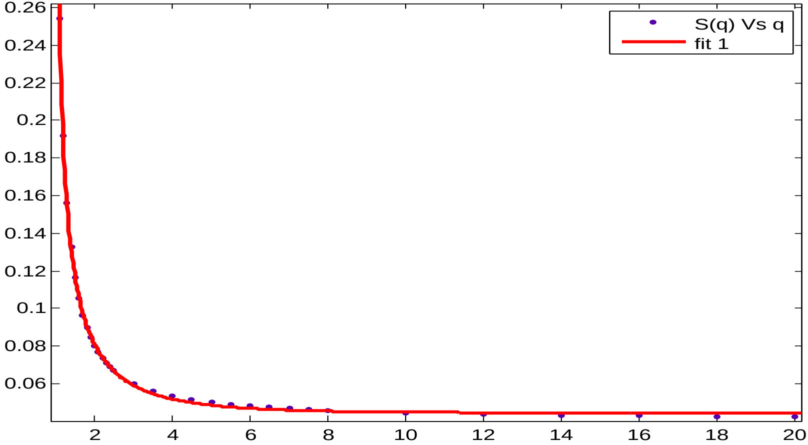
- In 4-d, for large ℓ, Renyi entropy gives:

$$S_R \sim \sum_{\ell} \frac{1}{\ell^{4q-1}} + \text{convergent part}$$

converges for $q > 1/2$; diverges for $q \leq 1/2$

S_R in higher dimensions

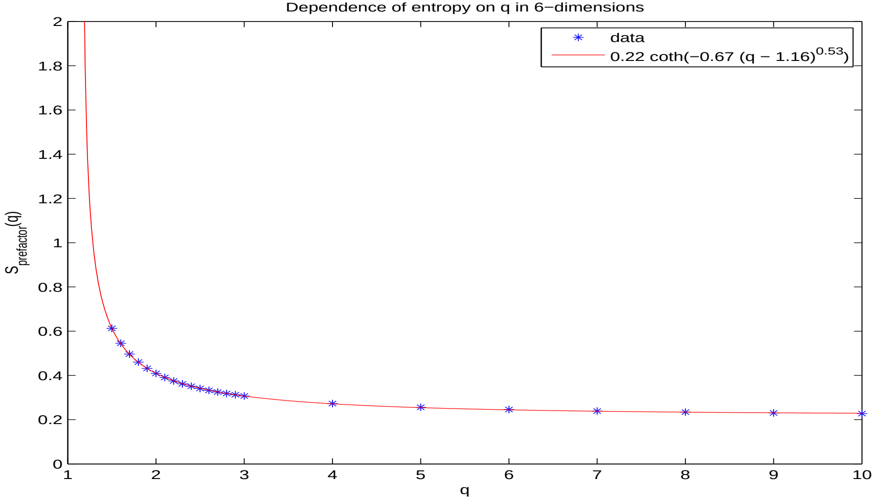
Braunstein, Das & SS '11



For 5-d, large ℓ, S_R diverges for $q \leq 1$

S_R in higher dimensions

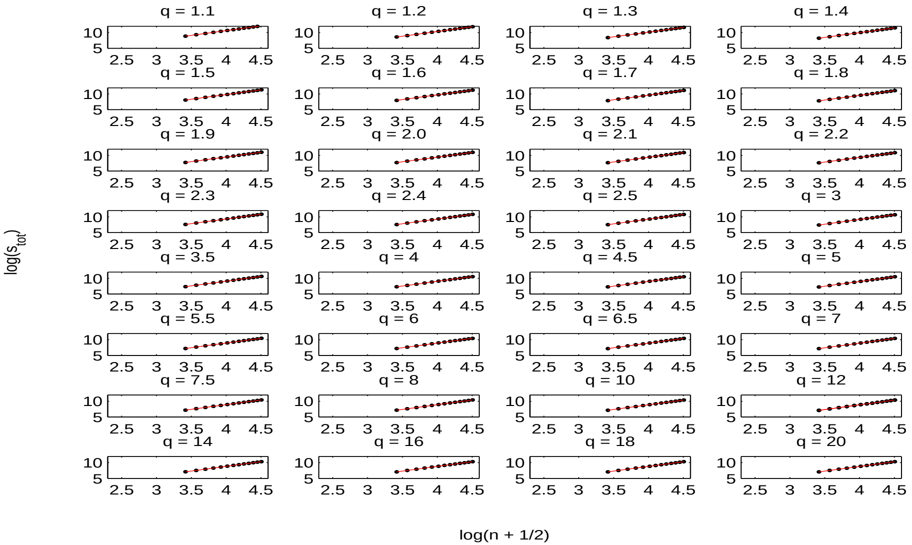
Braunstein, Das & SS '11



For 6-d, large ℓ, S_R diverges for $q \leq 3/2$

S_R in higher dimensions

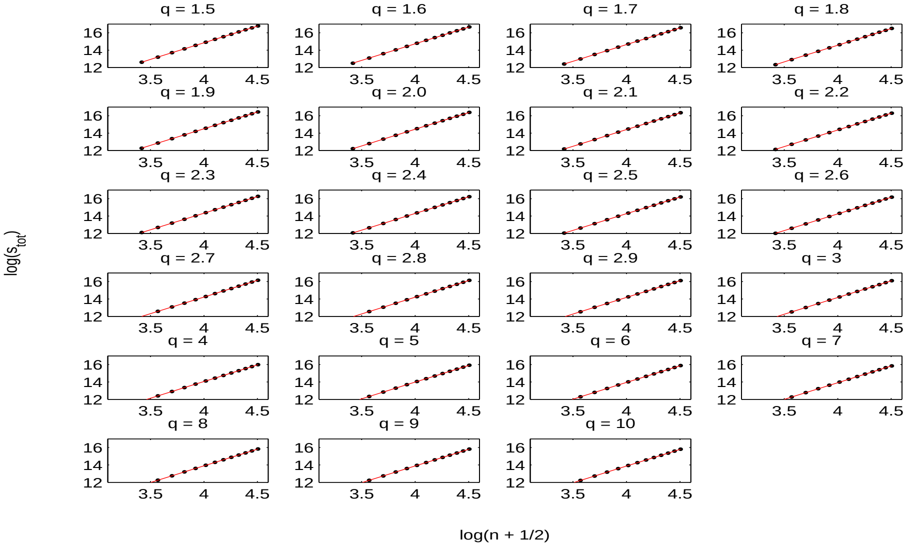
Braunstein, Das & SS '11



For all values of q, $S_R \propto n^3$

S_R in higher dimensions

Braunstein, Das & SS '11



For all values of q, $S_R \propto n^4$

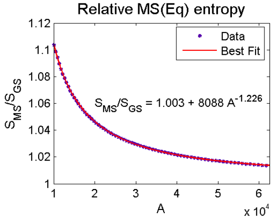
Discussions

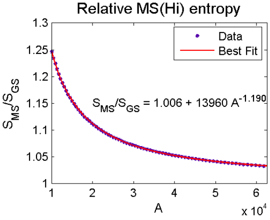
- We have stressed:
  1. the need for any approach to go beyond $S_{\rm BH}$ and provide generic subleading corrections
  2. importance to identify the degrees of freedom that contribute to $S_{\rm BH}$ and its subleading corrections.
- Using entanglement entropy, we have shown
  1. Degrees of freedom contribute that lead to $S_{\rm BH}$ and power law corrections are different; different from Noether charge ▸ Noether
  2. Gives area proportionality in higher dimensions
- The above results are valid for massive fields. The entropy behaves as

$$S_m = S_0 \exp\left[-a_0 m^2\right]$$

▸ Plots

- Need to extend for other topological black-holes in higher dimensions





$$S_{\rm MC} = S_{\rm BH}\left(1 + \frac{a_1}{A^\beta}\right)$$

$$\beta > 0, \quad a_1 \propto |c_1|$$

Back to results

Technical details

- Reduced density matrix is

$$\rho(t;t') = \tilde{\kappa}\, F(t;t')\,\rho_0(t;t') \qquad \tilde{\kappa} = c_0^2 + c_1^2\kappa; \quad \kappa_1 = \frac{c_1^2}{\tilde{\kappa}}; \quad \kappa_2 = \frac{c_0\, c_1}{\tilde{\kappa}};$$

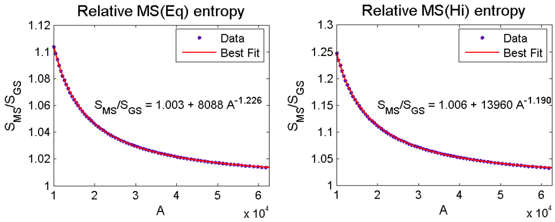
$$F(t;t') = 1 + \kappa_1 w(t;t') + \kappa_2 v(t;t') + \frac{\kappa_2^2}{2} v^2(t;t'); \qquad \kappa_0 = \frac{c_0^2}{\tilde{\kappa}}$$

$$w(t;t') = -\frac{t^T\Lambda_{\gamma'}t + t'^T\Lambda_{\gamma'}t'}{2} + t^T\Lambda_{\beta'}t'; \quad \Lambda_{\beta'} = \Lambda_\beta - 2\kappa_0\left(\Lambda_\beta - \frac{\Lambda_C}{\kappa}\right)$$

$$\Lambda_{\gamma'} = \Lambda_\gamma + 2\kappa_0\left(\Lambda_\beta - \frac{\Lambda_C}{\kappa}\right)$$

4-d results – technical details

Das, SS & Sur '07



$$S_{\rm MC} = S_{\rm BH}\left(1 + \frac{a_1}{A^\beta}\right)$$

$$\beta > 0, \quad a_1 \propto |c_1|$$

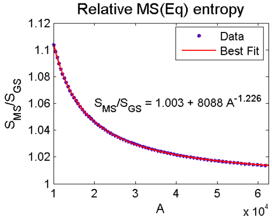
Back to results

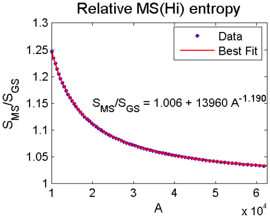
Technical details

- For $N > 50$, S is independent of N while depends on n
- Large values of l do not contribute to entropy $\quad S_l \propto l^{-4}$

4-d results – technical details

Das, SS & Sur '07





$$S_{\rm MC} = S_{\rm BH}\left(1 + \frac{a_1}{A^\beta}\right)$$

$$\beta > 0, \quad a_1 \propto |c_1|$$

Back to results

Technical details

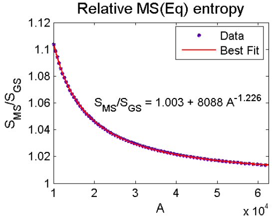
- If the vector t^T is outside the maximum

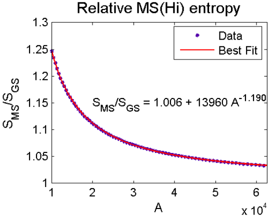
$$t^T_{max} = \left(\frac{3(N-n)}{\sqrt{2Tr(\gamma-\beta)}}\right)(1,1,\dots) \qquad \text{corresponding to } 3\sigma \text{ limits}$$

$$\epsilon_1 \equiv t^T_{max}\,\Lambda_\beta\, t_{max} \ll 1 \quad \text{and} \quad \epsilon_2 \equiv t^T_{max}\,\Lambda_\gamma\, t_{max} \ll 1$$

- Under this condition

$$1 - \frac{1}{2}\left(t^T\Lambda_\gamma t + t'^T\Lambda_\gamma t'\right) + t^T\Lambda_\beta t' \simeq \exp\left[-\frac{1}{2}\left(t^T\Lambda_\gamma t + t'^T\Lambda_\gamma t'\right) + t^T\Lambda_\beta t'\right]$$





$$S_{\rm MC} = S_{\rm BH}\left(1 + \frac{a_1}{A^\beta}\right)$$

$$\beta > 0, \quad a_1 \propto |c_1|$$

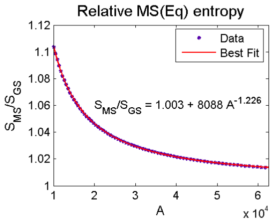
Back to results

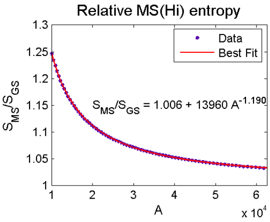
Technical details

- We use MATLAB to test the validity.
- The approximation is valid for large values of N (> 60)
- Error in the approximation is less than 0.01%

4-d results – technical details

Das, SS & Sur '07





$$S_{\rm MC} = S_{\rm BH}\left(1 + \frac{a_1}{A^\beta}\right)$$

$$\beta > 0, \quad a_1 \propto |c_1|$$

Back to results

Technical details

- We used a variety of fitting functions:

$$\begin{aligned}
\frac{S}{S_{\rm BH}} &= a_0 + a_1\left(\frac{\mathcal{A}_{\rm H}}{a^2}\right)^\nu \quad \text{least } \chi^2 \quad a_1 \propto 10^3; a_0 \sim 1 \\
&= a_0 + a_1\left(\frac{\mathcal{A}_{\rm H}}{a^2}\right)^\nu + a_2 \ln\left(\frac{\mathcal{A}_{\rm H}}{a^2}\right) \quad a_1 \propto 10^3; a_2 \sim \mathcal{O}(1) \\
&= a_0 + a_1\left(\frac{\mathcal{A}_{\rm H}}{a^2}\right)^\nu \ln\left(\frac{\mathcal{A}_{\rm H}}{a^2}\right) \quad a_1 \propto 10^3
\end{aligned}$$

Ultra-violet divergences

Demers et al '95

Procedure

- $\Gamma(E)$ diverges close to the horizon
- Pauli-Villars regularization:

  1 scalar field   5 Auxillary fields

  2 scalars with mass M_k; 3 fermions with mass M'_k

- To eliminate the divergences the masses of the auxiliary fields must obey the two restrictions

$$f(1) = f(2) = 0 \qquad f(p) = m^{2p} + \sum_k M_k^{2p} - \sum_r (M'_r)^{2p} = 0$$

- Solving the constraints lead to

$$M_{1,2} = \sqrt{3\mu^2 + m^2} M'_{1,2} = \sqrt{\mu^2 + m^2}, M'_3 = \sqrt{4\mu^2 + m^2}$$

- leading to a regularized density of states

$$\Gamma(E|\mu) \equiv \Gamma(E, m) + \sum_k \Gamma(E, M_k) - \sum_r \Gamma(E, M'_r)$$

Conclusions

Noether charge and scalar fields

Frolov & Fursaev '98

◂ Conclusions

- Consider non-minimally coupled scalar field

$$I[\phi, g] = -\frac{1}{2}\int(\varphi^{,\mu}\varphi_{,\mu} + m^2\varphi^2 + \xi R\varphi^2)\sqrt{-g}\; d^4x$$

Noether charge and scalar fields

Frolov & Fursaev '98

Conclusions

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$$I[\phi, g] = -\frac{1}{2}\int (\varphi^{,\mu}\varphi_{,\mu} + m^2\varphi^2 + \xi R\varphi^2)\sqrt{-g}\, d^4x$$

- Two ways of defining stress-tensor
  1. As variation of the action

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta I[g]}{\delta g^{\mu\nu}} \qquad E = \int_{\mathcal{B}} T_{\mu\nu}\zeta^\mu d\sigma^\nu$$

  2. Define canonical stress-tensor

$$(T^C)_{\mu\nu} = \varphi_{,\mu}\frac{\partial\mathcal{L}}{\partial\varphi^{,\nu}} - g_{\mu\nu}\mathcal{L} \qquad H = \int_{\mathcal{B}} T^C_{\mu\nu}\zeta^\mu d\sigma^\nu$$

Noether charge and scalar fields

Frolov & Fursaev '98

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- In a static spacetime
  E is conserved on the equations of motion of the field φ
  H plays the role of a generator of the evolution of the system along the Killing time
- In general, $T_{\mu\nu}$ and $(T^C)_{\mu\nu}$ do not coincide and their difference yields the Noether current

$$J_{\mu} = \frac{2\pi}{\kappa} \left((T^C)_{\mu\nu} - T_{\mu\nu} \right) \zeta^{\nu}$$

Noether charge and scalar fields

Frolov & Fursaev '98

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- For non-minimal coupling

$$J_\mu = -\xi \frac{2\pi}{\kappa} \left(R_{\mu\nu}\phi^2 + g_{\mu\nu}(\phi^2)^{;\rho}_{\ ;\rho} - (\phi^2)_{;\mu\nu} \right) \zeta^\nu$$

$$H - E = \xi \int_{\partial\mathcal{B}} ds^k \ |g_{00}|^{1/2} \left[(\phi^2)_{,k} - \omega_k \, \phi^2 \right]$$

Noether charge and scalar fields

Frolov & Fursaev '98

Conclusions

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$$H - E = \xi \int_{\partial \mathcal{B}} ds^k \; |g_{00}|^{1/2} \left[(\phi^2)_{,k} - \omega_k \, \phi^2 \right]$$

- For a field falling off at infinity ($E = H$), Noether charge gives the contribution from the horizon

$$H - E = \frac{\kappa}{2\pi} Q \qquad Q = 2\pi\xi \int_\Sigma \sqrt{\sigma} d^2\theta \; \phi^2$$

Why scalar fields?

Set up

- Consider a general $4-d$ spherically symmetric BH space-time

$$ds^2 = \overline{g_{\mu\nu}} dx^\mu dx^\nu = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 = f(r)\left[-dt^2 + dx^2\right] + r^2 d\Omega^2$$

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- Perturbation about BH background

$$g_{\mu\nu} = \overline{g_{\mu\nu}} + h_{\mu\nu}$$

lead to two kind of perturbations

Axial Perturbations

Polar Perturbations

Why scalar fields?

◂ Set up

- The two perturbations satisfy [Chandrasekhar, Mathematical Theory of BHs]

$$\frac{d^2 R_x(r)}{dx^2} + \left[\omega^2 - V_x(r)\right] R_x(r) = 0 \qquad \text{x} = \text{Axial, Polar perturbations}$$

$R_{Polar}(r), R_{Axial}(r)$ related to each other by unitary transformation.

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$R_{Polar}(r), R_{Axial}(r)$ related to each other by unitary transformation.

- V_{Axial} is same as that for scalar field propagating in the above BH background

$$V(r) = \frac{l(l+1)}{r^2}\, f(r) + \frac{f(r)}{r}\frac{d}{dr}(f(r))$$

☞ By calculating entropy of scalar field, we obtain entropy of metric perturbations of BH