A new uncertainty principle from quantum gravity and its implications

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06 January 2011


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Plan:

- A problem with Quantum Gravity
- A new uncertainty principle from Quantum Gravity
- Phenomenology and Predictions: can it be tested in the laboratory? 1 and 3 dimensions
- Discreteness of Space?
- Summary and Outlook

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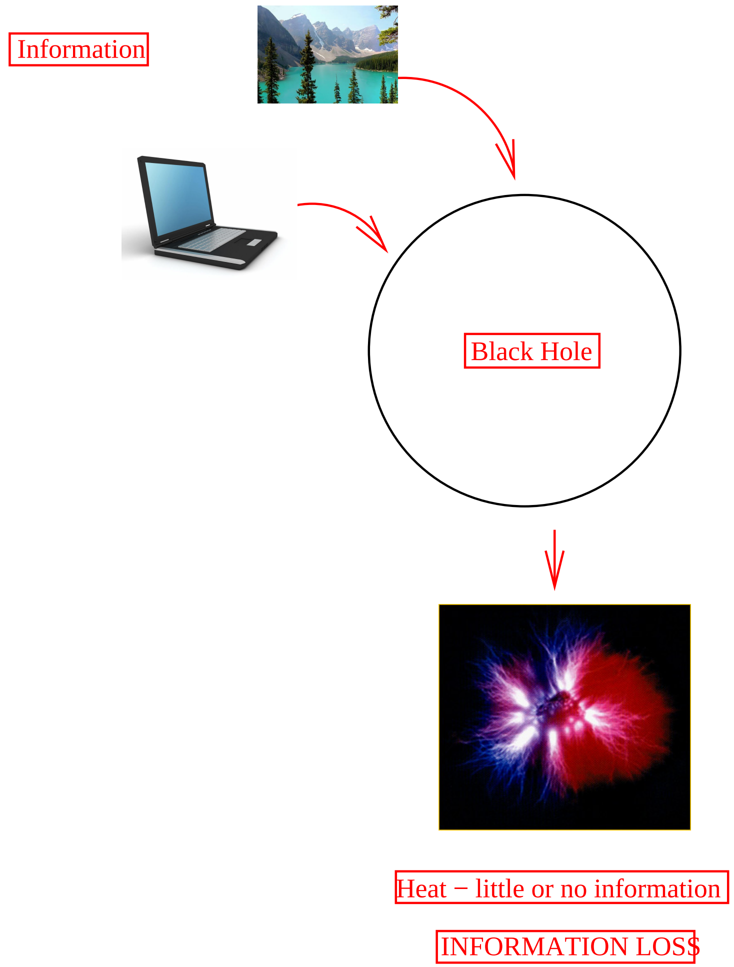
1. S. Das, E. C. Vagenas, Phys. Rev. Lett. **101**, 221301 (2008), arXiv:0810.5333
2. A. Ali, S. Das, E. C. Vagenas, Phys. Lett. **B678**, 497-499 (2009), arXiv:0906.5396
3. S. Das, E. C. Vagenas, A. Ali, Phys. Lett. **B690** (2010) 407, arXiv:1005.3368

Why Quantum Gravity?

- Why not? (3 other forces are quantum)
- Classical Gravity + Quantum Fields $\rightarrow$ Information Loss/Non-unitary QM
- Resolution of singularities
- Interaction of classical gravity wave with a quantum wavefunction $\rightarrow$ energy/momentum non-conservation
- $\cdots\cdots$



$$\rho_t^2 = (U\rho_o \underbrace{U^\dagger)(U}_{I} \rho_0 U^\dagger) = U\rho_0^2 U^\dagger = U\rho_0 U^\dagger = \rho_t$$

Thus:

Pure density matrix $\rho_0^2 = \rho_0$	$\rightarrow$	Non-unitary evolution $U^\dagger \neq U^{-1}$	$\rightarrow$	Mixed density matrix $\rho_t^2 \neq \rho_t$

Problem with Quantum Gravity

Newton's constant $G_d = \frac{1}{M_{Pl}^{d-2}}$

Dimensionless $G_d s^{d-2/2}, G_d t^{d-2/2} \to \infty$ as $s, t \to \infty$

$(s, t \approx \text{ (Energy)}^2)$

Perturbatively, Gravity is Non-Renormalizable

Some candidate theories

String Theory, Loop Quantum Gravity, Path Integrals, Causal Sets, Causal Dynamical Triangulations, Non-Commutative Geometry, Supergravity, . . .

Another problem with Quantum Gravity

- Too many theories: *String Theory, Loop Quantum Gravity, Non-Commutative Field Theory, Dynamical Triangulations, Causal Sets,...*
- Too few experiments $=$ *Zero*
- Why? *Quantum Gravity effects expected at the Planck Scale* $\approx 10^{16}\ TeV$

  Atomic Physics $\approx 10\ eV \approx 10^{-11}\ TeV$. LHC $\approx 10\ TeV$
- Difference of $15 - 27$ orders of magnitude

QG $\rightarrow$ Experimental Signatures?

First: A New Uncertainty Principle in Quantum Gravity

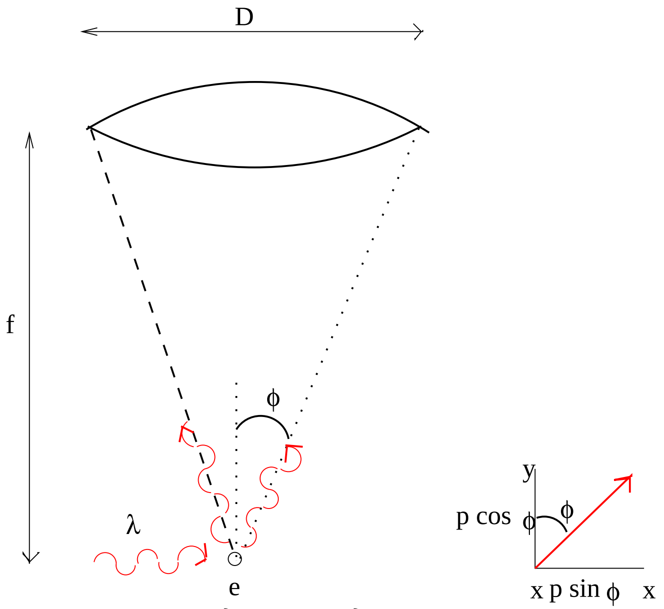
'It is my honest opinion that when – people try to get hold of the laws of nature by thinking alone, the result is pure rubbish' - Max Born to Einstein, 1944

Generalized Uncertainty Principle: why?

- Black Hole Physics
- String Theory
- Loop Quantum Gravity, via Polymer Quantization
- Non-commutative geometry
- $\cdots\cdots$

A meeting ground for various theories?

Heisenberg's Microscope



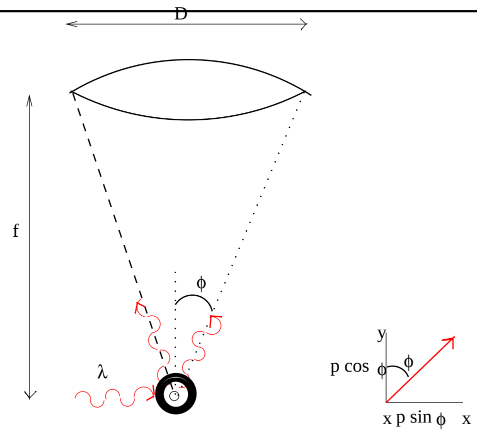
Position uncertainty $\Delta x \geq f\,\frac{\lambda}{D} \approx \frac{\lambda}{2\phi}$ $\left(\textit{Minimum Resolving Power}\right)$

Momentum uncertainty $\Delta p = p \sin\phi = \frac{\hbar}{\lambda}\sin\phi \approx \frac{\hbar\phi}{\lambda}$

$$\Delta p \Delta x \geq \frac{\hbar}{2}$$

Heisenberg Uncertainty Principle

Heisenberg's Microscope with a Black Hole (Extremal RN)



$$r_+ = GM + \sqrt{(GM)^2 - GQ^2}$$

$$\Delta x_{new} = r_+(M + \Delta M) - r_+(M)$$

$$= G\Delta M + \sqrt{(GM + G\Delta M)^2 - GQ^2} - \sqrt{(GM)^2 - GQ^2} \geq 2G\Delta M \sim \frac{\ell_{Pl}^2}{\lambda} (\Delta M = \frac{\hbar}{\lambda}, \Delta p = \frac{\hbar \sin\phi}{\lambda})$$

$$\sim \frac{\ell_{Pl}^2 \Delta p}{\hbar \sin\phi} \geq \frac{\ell_{Pl}^2 \Delta p}{\hbar} \rightarrow \Delta x + \Delta x_{new} \geq \frac{\hbar}{\Delta x} + \beta_0 \frac{\ell_{Pl}^2 \Delta p}{\hbar}$$

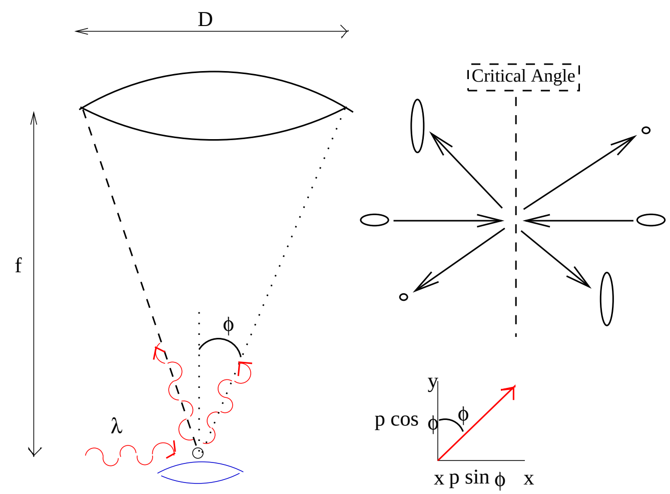
$$\Delta p \Delta x \geq \frac{\hbar}{2} \left[1 + \beta_0 \frac{\ell_{Pl}^2}{\hbar^2} \Delta p^2 \right]$$

Generalized Uncertainty Principle

$\ell_{Pl} = \sqrt{\frac{G\hbar}{c^3}} = 10^{-35} m$. The new term is effective only when $x \approx \ell_{Pl}$ or $p \approx 10^{16} TeV/c$

M. Maggiore, Phys. Lett. **B304**, 65 (1993)

Heisenberg's Microscope with an elementary string



$$\Delta x_{new} \sim \sqrt{E} \sim p \geq \frac{\ell_{Pl}}{\hbar} \Delta p$$

$$\Delta p \Delta x \geq \frac{\hbar}{2} \left[1 + \beta_0 \frac{\ell_{Pl}^2}{\hbar^2} \Delta p^2 \right]$$

Generalized Uncertainty Principle

D. Amati, M. Ciafaloni, G. Veneziano, Phys. Lett. **B216** 41 (1989)

Minimum Observable Length from the GUP

$$\Delta p \Delta x \geq \frac{\hbar}{2}\left[1 + \beta_0 \frac{\ell_{Pl}^2}{\hbar^2} \Delta p^2\right]$$

Invert $\quad \Delta p \leq \frac{\hbar}{\ell_{Pl}}\left[1 \pm \sqrt{\Delta x^2 - \beta_0 \ell_{Pl}^2}\right]$

$$\Delta x \geq \sqrt{\beta_0}\,\ell_{Pl} \equiv \Delta x_{min}$$

(Min length)

Note: $\Delta x_{min} \not\Rightarrow$ Space is discrete

$$\Delta p \Delta x \geq | < [x, p] > |$$

	HUP	GUP
Principle	$\Delta p_i \Delta x_i \geq \frac{\hbar}{2}$	$\Delta p_i \Delta x_i \geq \frac{\hbar}{2}\left[1 + \beta_0 \frac{\ell_{Pl}^2}{\hbar^2} \Delta p^2\right]$
Algebra	$[x_i, p_j] = i\hbar\delta_{ij}$	$[x_i, p_j] = i\hbar[\delta_{ij} + \underbrace{\frac{\beta_0}{\hbar^2} \ell_{Pl}^2 (p^2 \delta_{ij} + 2 p_i p_j)}_{New}]$

A. Kempf, G. Mangano, R. B. Mann, Phys. Rev. **D52** 1108 (1995)

Problems

- $10^{16} Tev$, 10^{-35} m (Planck scale/QG) in 'whose frame'??
  $\rightarrow$ Problem of *Lorentz Covariance of QG*
- Effects $\propto \ell_{Pl}^2 \approx 10^{-70} m^2$ $\leftarrow$ *Bad for phenomenology*

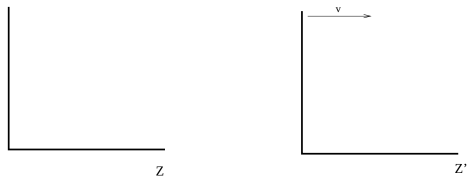
Is there a GUP linear in ℓ_{Pl}?

- Postulate a linear term
- Look for a linear term

New $[x,p]$ algebra from Doubly Special Relativity Theories

(One way to solve the 'QG in whose frame' problem)

$$[J_i, K_j] = \epsilon^{ijk} K_k \ , \ [K^i, K^j] = \epsilon^{ijk} J_k \ \text{But} \ K^i = L_0^i + \ell_{Pl} p^i \ p_a \frac{\partial}{\partial p_a}$$



$$p_0' = \frac{\gamma(p_0 - v p_z)}{1 + \ell_{Pl}(\gamma - 1) p_0 - \ell_{Pl} \gamma v p_z}$$

$$p_z' = \frac{\gamma(p_z - v p_0)}{1 + \ell_{Pl}(\gamma - 1) p_0 - \ell_{Pl} \gamma v p_z}$$

$$p_x' = \frac{\gamma(p_z - v p_0)}{1 + \ell_{Pl}(\gamma - 1) p_0 - \ell_{Pl} \gamma v p_z}$$

$$p_y' = \frac{\gamma(p_z - v p_0)}{1 + \ell_{Pl}(\gamma - 1) p_0 - \ell_{Pl} \gamma v p_z}$$

$$E^2 - \vec{p}^2 \neq m^2$$

$$E^2 f(E, \vec{p}, \ell_{Pl}) - \vec{p}^2 g(E, \vec{p}, \ell_{Pl}) = m^2 \equiv \epsilon^2 - \vec{\pi}^2$$

$$[x_i, p_j] = i\hbar \frac{\partial p_i}{\partial \pi_j}$$

$$f = \frac{|\vec{p}|}{E} \ , \ g = \frac{1}{1 - \ell_{Pl}|\vec{p}|} \ \text{(massless)}$$

$$[x_i, p_j] = i\hbar[(1 - \ell_{Pl}|\vec{p}|)\delta_{ij} + \ell_{Pl}^2 p_i p_j]$$

J. Magueijo, L. Smolin, Phys. Rev. Lett. **88** 190403 (2002),

J. L. Cortes, J. Gamboa, Phys. Rev. **D71** 026010 (2005)

So now we have *two* new algebras/GUPs

$[x_i, p_j] = i\hbar[1 + \frac{\beta_0 \ell_{Pl}^2}{\hbar^2}(p^2 \delta_{ij} + 2p_i p_j)]$ Quadratic & $\Delta x \geq \ell_{Pl}$

$[x_i, p_j] = i\hbar[(1 - \ell_{Pl}|\vec{p}|)\delta_{ij} + \ell_{Pl}^2 p_i p_j]$ Linear & quadratic & $\Delta p \leq M_{Pl} c$

Can we make them compatible?

$$\text{Try } [x_i, p_j] = i\hbar[\delta_{ij} + \delta_{ij}\alpha_1 p + \alpha_2 \frac{p_i p_j}{p} + \beta_1 \delta_{ij} p^2 + \beta_2 p_i p_j]$$

Use Jacobi identity with commuting coordinates and momenta

$$-\Big[[x_i, x_j], p_k\Big] = \Big[[x_j, p_k], x_i\Big] + \Big[[p_k, x_i], x_j\Big] = 0$$

$$\boxed{[x_i, p_j] = i\hbar\left[\delta_{ij} - \alpha\left(p\delta_{ij} + \frac{p_i p_j}{p}\right) + \alpha^2(p^2\delta_{ij} + 3p_i p_j)\right]}$$

$$\Delta x \geq (\Delta x)_{min} \approx \alpha_0 \ell_{Pl} \ , \ \Delta p \leq (\Delta p)_{max} \approx \frac{M_{Pl} c}{\alpha_0}$$

$$\alpha = \frac{\alpha_0}{M_{Pl} c} = \frac{\alpha_0 \ell_{Pl}}{\hbar} \ . \ \alpha_0 = \mathcal{O}(1) \text{ (normally)}$$

Although current experiments $\Rightarrow \alpha_0 \leq 10^{17} \rightarrow \quad \alpha^{-1} \approx 10 \ TeV/c$

Consequences

$$[x_i, p_j] = i\hbar \left[\delta_{ij} - \alpha \left(p\delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 \left(p^2 \delta_{ij} + 3 p_i p_j \right) \right] \Rightarrow \ p_j \neq -i\hbar \frac{\partial}{\partial x_i} \text{ pos'n space}$$

But define:

$$p_j = p_{0j} \left(1 - \alpha p_0 + 2\alpha^2 p_0^2\right) \quad \text{with } [x_i, p_{0j}] = i\hbar \delta_{ij} \ , \ p_{0j} = -i\hbar \frac{\partial}{\partial x_j}$$

$[x_i, p_j] = \ldots$ is satisfied

Consider any Hamiltonian

$$H = \frac{p^2}{2m} + V(\vec{r}) = \frac{1}{2m} \left(p_{0j} \left(1 - \alpha p_0 + 2\alpha^2 p_0^2 \right) \right)^2 + V(r)$$

$$= \underbrace{\frac{p_0^2}{2m} + V(\vec{r})}_{H_0} \underbrace{- \frac{\alpha}{m} p_0^3 + \mathcal{O}(\alpha^2)}_{H_1} = \frac{p_0^2}{2m} + V(\vec{r}) \quad \underbrace{- \frac{i\hbar^3 \alpha}{m} \frac{d^3}{dx^3}}_{\text{position space}}$$

Universal Quantum Gravity Effect!

Schrödinger Equation

$$[H_0 + H_1]\psi = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) - i\frac{\alpha\hbar^3}{m}\frac{d^3}{dx^3}\right]\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Two Consequences

1. New Perturbed Solutions *and* New Conserved Current

$$J = \frac{\hbar}{2mi}\left(\psi^\star\frac{d\psi}{dx} - \psi\frac{d\psi^\star}{dx}\right) + \frac{\alpha\hbar^2}{m}\left(\frac{d^2|\psi|^2}{dx^2} - 3\frac{d\psi}{dx}\frac{d\psi^\star}{dx}\right)$$

$\rho = |\psi|^2$, $\frac{\partial J}{\partial x} + \frac{\partial\rho}{\partial t} = 0 \rightarrow$ *New Reflection/Transmission Currents*

2. New *Non-Perturbative* solution $\sim e^{ix/\ell_{Pl}}$ $\rightarrow$ *Discreteness of space*

Applications to Quantum Mechanical Systems

- Simple Harmonic Oscillator
- Landau Levels
- Lamb Shift
- Scanning Tunneling Microscope
- Particle in a box - *new non-perturbative solution*

Precision required to test GUP: 1 part in $10^{12} - 10^{25}$

S. Das, E. C. Vagenas, Phys. Rev. Lett. **101**, 221301 (2008), arXiv:0810.5333

Simple Harmonic Oscillator)

$$H = \underbrace{\frac{p_0^2}{2m} + \frac{1}{2} m\omega x^2}_{H_0} \quad \underbrace{+ \frac{\alpha}{m} p^3 + \frac{3\alpha^2}{2} p^4}_{H_1}$$

$$\left[\psi_n = \frac{1}{2^n n!} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{\frac{-m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right) \right]$$

$$\Delta E_{GUP} = \langle \psi_n | H_1 | \psi_n \rangle |_{(p^4 \text{ term})} + \sum_{k \neq n} \frac{|<k^0|H_1|n^0>|^2}{E_n^0 - E_k^0} |_{(p^3 \text{ term})}$$

$$\frac{\Delta E_{GUP(0)}}{E_0} = \frac{9}{2} \hbar^2 \omega^2 m \alpha^2 + 4\hbar\omega m \alpha^2$$

$$\Delta E_{GUP(0)} \sim \alpha^2 \ \leftarrow \textit{Not so good}$$

Landau Levels

Particle of mass m, charge e in constant $\vec{B} = B\hat{z}$, i.e.$\vec{A} = Bx\hat{y}$, $\omega_c = eB/m$

$$H_0 = \frac{1}{2m}\left(\vec{p}_0 - e\vec{A}\right)^2 = \frac{p_{0x}^2}{2m} + \underbrace{\frac{p_{0y}^2}{2m}}_{\hbar^2 k^2/2m} - \frac{eB}{m}\, x p_{0y} + \frac{e^2 B^2}{2m}\, x^2 = \frac{p_{0x}^2}{2m} + \frac{1}{2} m\omega_c^2 \left(x - \frac{\hbar k}{m\omega_c}\right)^2$$

$$H = \frac{1}{2m}\left(\vec{p}_0 - e\vec{A}\right)^2 - \frac{\alpha}{m}\left(\vec{p}_0 - e\vec{A}\right)^3 = H_0 - \sqrt{8m}\,\alpha H_0^{\frac{3}{2}}$$

$$\frac{\Delta E_{n(GUP)}}{E_n} = -\sqrt{8m}\alpha(\hbar\omega_c)^{\frac{1}{2}}(n + \tfrac{1}{2})^{\frac{1}{2}} \approx -10^{-27}\alpha \quad (B = 10\ T)$$

Conclude

- $\alpha \sim 1$ and $\frac{\Delta E_{n(GUP)}}{E_n}$ is too small, *or*
- Measurement accuracy of 1 in 10^3 in STM $\rightarrow$ $\alpha_0 < 10^{24}$

Lamb Shift

$$H_0 = \frac{p_0^2}{2m} - \frac{k}{r} \; , H_1 = -\frac{\alpha}{m} p_0^3 = (\alpha\sqrt{8m}) \left[H_0 + \frac{k}{r}\right] \left[H_0 + \frac{k}{r}\right]^{\frac{1}{2}}$$

$$\Delta E_n = \frac{4\alpha^2}{3m^2} \left(\ln \frac{1}{\alpha}\right) |\psi_{nlm}(0)|^2 \quad (\textit{Lamb Shift})$$

$$\frac{\Delta E_{n(GUP)}}{\Delta E_n} = 2 \frac{\Delta|\psi_{nlm(0)}|}{\psi_{nlm}(0)}$$

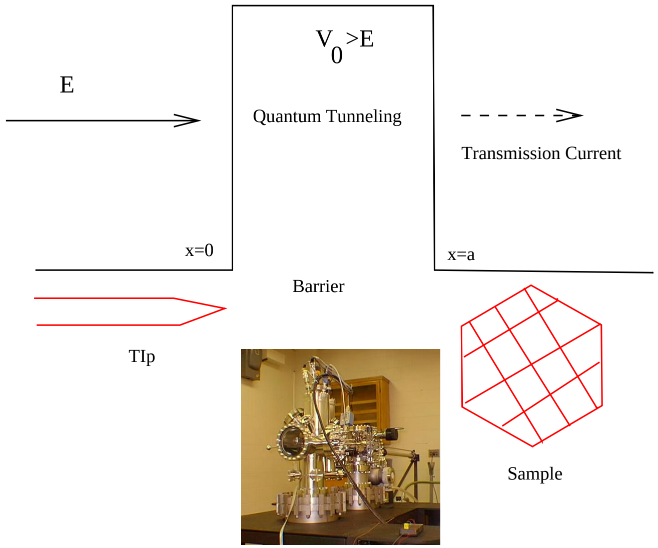
$$\Delta\psi_{100}(\vec{r}) = \sum_{\{n'l'm'\}\neq\{nlm\}} \frac{\langle n'm'l'|H_1|nlm\rangle}{E_n - E_{n'}} \langle \vec{r}|n'l'm'\rangle = 10^3 \alpha E_0 \;\psi_{200}(\vec{r})$$

$$\frac{\Delta E_{n(GUP)}}{\Delta E_n} = 2 \frac{\Delta|\psi_{nlm(0)}|}{\psi_{nlm}(0)} \approx \alpha_0 \frac{4.2\times 10^4 E_0}{27 M_{Pl} c^2} \approx 10^{-24} \alpha_0$$

Conclude

- $\alpha \sim 1$ and $\frac{\Delta E_{n(GUP)}}{E_n}$ is too small, *or*
- Measurement accuracy of 1 in $10^{12} \rightarrow \alpha_0 < 10^{12}$ (*better!*)

Potential Barrier (Scanning Tunneling Microscope)



$$[H_0 + H_1]\psi = E\psi \qquad [H_0 + H_1]\psi = -(V_0 - E)\psi \qquad [H_0 + H_1]\psi = E\psi$$

$$\psi_1 = Ae^{ik'x} + Be^{-ik''x} + Pe^{\frac{ix}{2\alpha\hbar}} \qquad \psi_2 = Fe^{k_1'x} + Ge^{-k_1''x} + Qe^{\frac{ix}{2\alpha\hbar}} \qquad \psi_3 = Ce^{ik'x} + Re^{\frac{ix}{2\alpha\hbar}}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \;,\; k_1 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

$$k' = k(1 + \alpha\hbar k),\; k'' = k(1 - \alpha\hbar k),\; k_1' = k_1(1 - i\alpha\hbar k_1),\; k_1'' = k_1(1 + i\alpha\hbar k_1)$$

Take into account

- Continuity of ψ, ψ', ψ'' at each boundary (cannot set $P, Q, R = 0$)
- New current

Transmission Current

$$T = \frac{J_R}{J_L} = \left|\frac{C}{A}\right|^2 - 2\alpha\hbar k \left|\frac{B}{A}\right|^2 .$$

$$= T_0\left[1 + 2\alpha\hbar k(1 - T_0^{-1})\right], \quad T_0 = \frac{16E(V_0 - E)}{V_0^2} e^{-2k_1 a} = \text{usual}$$

$m = m_e = 0.5\ MeV/c^2\ ,\ E \approx V_0 = 10\ eV a = 10^{-10}\ m\ ,\ I = 10^{-9}\ A\ ,\ \mathcal{G} = 10^9\ ,$

$$I \propto T$$

$$\frac{\delta I_{GUP}}{I_0} = \frac{\delta T_{GUP}}{T_0} = 10^{-26},$$
$$\delta \mathcal{I}_{\mathcal{GUP}} = \mathcal{G}\delta I_{GUP} = 10^{-26} A, \alpha_0 = 1, T_0 = 10^{-3}$$

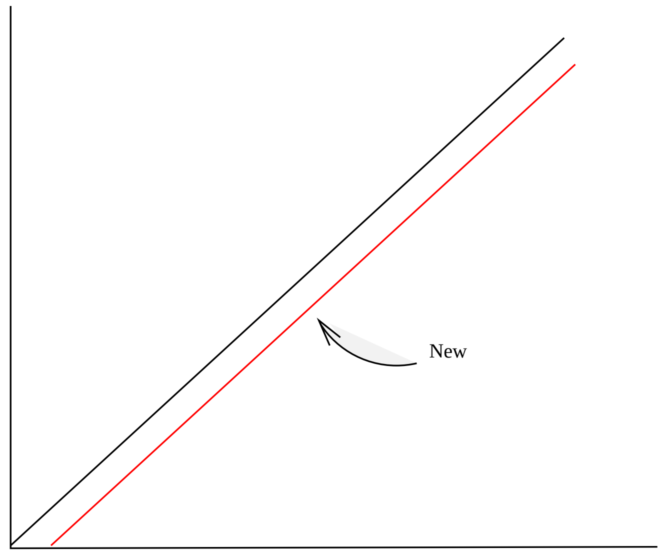
$$\tau = \frac{e}{\delta \mathcal{I}_{\mathcal{GUP}}} = 10^7\ s \approx \text{ a month}$$

Apparent barrier height

$$\Phi_A \equiv V_0 - E$$

$$\sqrt{\Phi_A} = \frac{\hbar}{\sqrt{8m}} \left| \frac{d \ln I}{da} \right| - \frac{\alpha \hbar^2 (k^2 + k_1^2)^2}{8m(kk_1)} e^{2k_1 a}$$

$$\left(T_0 = \frac{16E(V_0 - E)}{V_0^2} e^{-2k_1 a} \right)$$



$$\sqrt{\Phi_A} \text{ vs } \frac{d \ln I}{da}$$

GUP effects on

Atomic/Molecular/Condensed Matter Systems

Stark Effect, Zeeman Effect, Berry's Phase, Bohm-Aharonov effect, Dirac Quantization, Anomalous Magnetic Moment of Electron, Quantum Hall Effect, Anderson Localization, Superconductivity, Coherent States, Lasers,...

Statistical Mechanical Systems

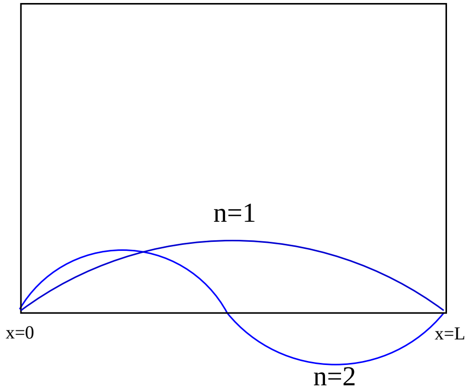
Bose-Einstein Condensation, Fermi Levels, Chandrasekhar Limit,...

Normally forbidden processes

Atomic Transitions

Look at *New Non-perturbative* solution of the *Cubic* Schrödinger Equation

Particle in a Box



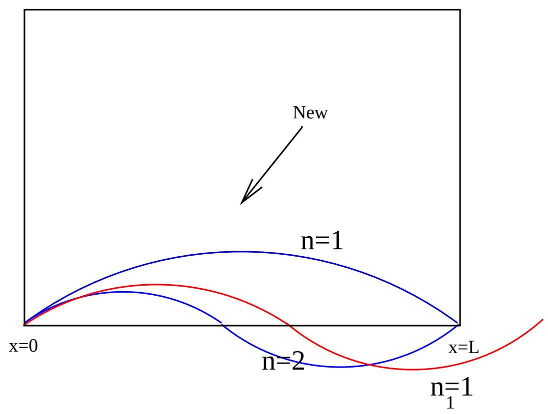
$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad (k \equiv \sqrt{2mE/\hbar^2})$$

$$\psi(0) = 0 \;\rightarrow A + B = 0 \;\rightarrow \psi(x) = 2iA\sin(kx)$$

$$\psi(L) = 0 \;\rightarrow kL = n\pi \;\rightarrow E_n = \frac{k^2\hbar^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

Particle in a Box with GUP



$$\left[-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - i\frac{\alpha\hbar^3}{m}\frac{d^3}{dx^3}\right] = E\psi$$

$$\psi(x) = \underset{\substack{\uparrow \\ \textit{Choose Real}}}{A} e^{ik'x} + Be^{-ik''x} + \underset{\substack{\uparrow \\ \text{New Solution} \\ \textit{(Cannot just set it to zero!)}}}{C} e^{\frac{ix}{2\alpha\hbar}}$$

$$k' = k(1 + k\alpha\hbar)\ ,\ \ k'' = k(1 - k\alpha\hbar)$$

$$\psi(0) = 0 \ \rightarrow A + B + C = 0 \ \rightarrow$$

$$\psi = 2iA\sin(kx) + C\left[-e^{-ikx} + e^{ix/2\alpha\hbar}\right] - \alpha\hbar k^2 x\left[i\,Ce^{-ikx} + 2A\sin(kx)\right]$$

$$\psi(L) = 0 \ \rightarrow$$

$$2iA\sin(kL) = |C|\left[e^{-i(kL+\theta_C)} - e^{i(L/2\alpha\hbar-\theta_C)}\right] + \underbrace{\alpha\hbar k^2 L\left[i\,|C|\,e^{-i(kL+\theta_C)} + 2A\sin(kL)\right]}_{\mathcal{O}(\alpha^{\text{higher power}})}$$

$$= |C|\left[e^{-i(kL+\theta_C)} - e^{i(L/2\alpha\hbar-\theta_C)}\right]$$

$C = |C|e^{-i\theta_C}$ etc

Take real parts of both sides (A = Real)

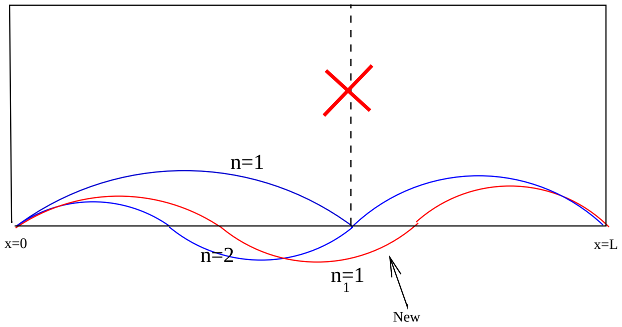
$$\cos\left(\frac{L}{2\alpha\hbar} - \theta_C\right) = \cos(kL + \theta_C) = \cos(n\pi + \theta_C + \epsilon)$$

Solution

$$\frac{L}{2\alpha\hbar} = \frac{L}{2\alpha_0\ell_{Pl}} = n\pi + 2q\pi + 2\theta_C \equiv n_1\pi + 2\theta_C$$

$$\frac{L}{2\alpha\hbar} = \frac{L}{2\alpha_0\ell_{Pl}} = -n\pi + 2q\pi \equiv n_1\pi$$

$$n_1 \equiv 2q \pm n \in N.$$



Only certain Ls can fit both $\sin(kL)$ and $\cos(\frac{L}{2\alpha\hbar})$

Box Length is Quantized!

Need at least one particle for measuring lengths

Perhaps all measured lengths are quantized?

A. Ali, S. Das, E. C. Vagenas, Phys. Lett. **B678**, 497-499 (2009), arXiv:0906.5396

Relativistic Wave Equations

- Klein-Gordon

For Stationary States: $2mE \to E^2 - m^2$, $k \to k\sqrt{\frac{E}{2mc^2} - \frac{mc^2}{2E}}$

L Quantization Unchanged

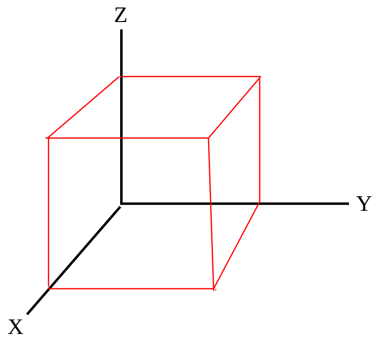
Problems with KG

- Most elementary particles are fermions
- How to generalize to 2 and 3 dimensional box?

$$\vec{p} = \vec{p}_0 - \alpha p_0 \vec{p} = \vec{p}_0 - \alpha \sqrt{p_{0x}^2 + p_{0y}^2 + p_{0z}^2}\ \vec{p}$$

$$\to \vec{p}_0 - \alpha\hbar \underbrace{\sqrt{-\frac{d^2}{dx^2} - \frac{d^2}{dy^2} - \frac{d^2}{dz^2}}}_{\text{Non-local}}\ \vec{p}$$

Dirac Equation



- $p_0 \to \vec{\alpha} \cdot \vec{p}$

$$H\psi = \left(\vec{\alpha} \cdot p + \beta mc^2\right)\psi = \left(\vec{\alpha} \cdot \vec{p}_0 - \alpha(\vec{\alpha} \cdot \vec{p}_0)(\vec{\alpha} \cdot \vec{p}_0) + \beta mc^2\right)\psi = E\psi$$

$$\psi \equiv e^{i\vec{t}\cdot\vec{r}} \begin{pmatrix} \chi \\ \vec{\rho} \cdot \vec{\sigma}\chi \end{pmatrix} \to H\psi = e^{i\vec{t}\cdot\vec{r}} \begin{pmatrix} ((m - \alpha t^2) + \vec{t} \cdot \vec{r} + i\vec{\sigma} \cdot (\vec{t} \times \vec{\rho}))\chi \\ \left(\vec{t} - (m + \alpha^2 t^2)\vec{\rho}\right) \cdot \vec{\sigma}\chi \end{pmatrix} = E\psi \ ,$$

$$\psi = e^{i\vec{k}\cdot\vec{r}} \begin{pmatrix} \chi \\ r\hat{k} \cdot \vec{\sigma}\chi \end{pmatrix} \ , \ \psi = e^{i\frac{\hat{q}\cdot\vec{r}}{\alpha\hbar}} \begin{pmatrix} \chi \\ \hat{q} \cdot \vec{\sigma}\chi \end{pmatrix}$$

Confining Wavefunction

Superposition of $2^d + 1$ eigenfunctions

(d=1,2,3)

$$\psi = \begin{pmatrix} \left[\prod_{i=1}^{d} \left(e^{ik_i x_i} + e^{-i(k_i x_i - \delta_i)} \right) + F e^{i \frac{\hat{q} \cdot \vec{r}}{\alpha \hbar}} \right] \chi \\ \sum_{j=1}^{d} \left[\prod_{i=1}^{d} \left(e^{ik_i x_i} + (-1)^{\delta_{ij}} e^{-i(k_i x_i - \delta_i)} \right) \hat{rk_j} + F e^{i \frac{\hat{q} \cdot \vec{r}}{\alpha \hbar}} \hat{q_j} \right] \sigma_j \chi \end{pmatrix}$$

MIT Bag Boundary Conditions

(Zero flux through boundaries)

$$\bar{\psi} \gamma^{\mu} \psi = 0 \leftrightarrow \pm i \beta \alpha_l \psi = \psi$$

$$e^{i\delta_k}\left(1 + ir\hat{k}_k\right) = \left(ir\hat{k}_k - 1\right) + f_{\bar{k}}^{-1} F'_k e^{-i\theta_k} \quad (x_k = 0)$$

$$e^{i(2k_k L_k - \delta_k)}\left(1 + ir\hat{k}_k\right) = \left(ir\hat{k}_k - 1\right) + f_{\bar{k}}^{-1} F'_k e^{i\left(\frac{\hat{q}_k L_k}{\alpha\hbar} + \theta_k\right)} e^{i(k_k L_k - \delta_k)} \quad (x_k = L_k)$$

$$\left[F'_k \equiv \sqrt{1 + |\hat{q}_k|^2} F \ , \ \ \theta_k \equiv \arctan \hat{q}_k\right]$$

Comparing

$$k_k L_k = \delta_k = \arctan\left(-\frac{\hbar k_k}{mc}\right) + \mathcal{O}(\alpha)$$

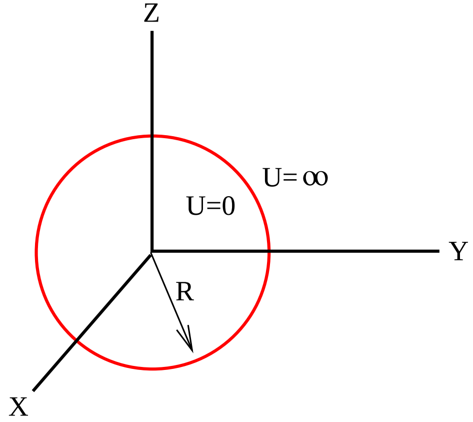
$$\boxed{\frac{L_k}{\alpha_0 \ell_{Pl}} = (2p_k\pi - 2\theta_k)\sqrt{d} \ , \quad p_k \in N}$$

$$\left[|\hat{q}_k| = 1/\sqrt{d}\right]$$

$$A_N \equiv \prod_{k=1}^{N} \frac{L_k}{\alpha_0 \ell_{Pl}} = d^{N/2} \prod_{k=1}^{N} (2p_k\pi - 2\theta_k) \ , \quad p_k \in N \ .$$

Measurable Lengths, Areas and Volumes are Quantized!

Dirac equation: Spherical Cavity



$$(\vec{\sigma}\cdot\vec{p}_0)\,\chi_2 + \left(mc^2 + U\right)\chi_1 - \alpha p_0^2 \chi_1 = E\chi_1$$

$$(\vec{\sigma}\cdot\vec{p}_0)\,\chi_1 - \left(mc^2 + U\right)\chi_1 - \alpha p_0^2 \chi_2 = E\chi_2 \ .$$

$$g_\kappa(r) = N j_\ell(p_0 r) \ , \ f_\kappa(r) = N'' j'_\ell(p_0 r) + \mathcal{O}(a)$$

$$\psi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} g_\kappa(r)\mathcal{Y}^{j_3}_{j\ell}(\hat{r}) \\ i f_\kappa(r)\mathcal{Y}^{j_3}_{j\ell'}(\hat{r}) \end{pmatrix} \ ; \qquad \underbrace{f_\kappa^{\mathcal{N}} = N' e^{ir/\alpha\hbar} \ , \ g_\kappa^{\mathcal{N}} = iN' e^{ir/\alpha\hbar}}_{New} \ ,$$

Boundary Conditions $\Rightarrow j_\ell(pR) = j_{\ell'}(pR) \ , \ \tan(\pm R/\alpha) = 1 \ .$

$$\boxed{\frac{R}{\alpha} = \pm\frac{\pi}{4} + 2p\pi \ , \quad p \in N}$$

Radii, Areas, Volumes discrete

Curved Spacetime: Schrödinger Equation

General Solution for *any* (linear) potential

$$\psi = A\psi_1 + B\psi_2 + C(\alpha) \underbrace{\psi_3}_{e^{ix/2\alpha\hbar}}$$

$$\psi(0) = 0 = \psi(L) \ \rightarrow \ e^{iL/2\alpha\hbar} = 1 + \mathcal{O}(\alpha)$$

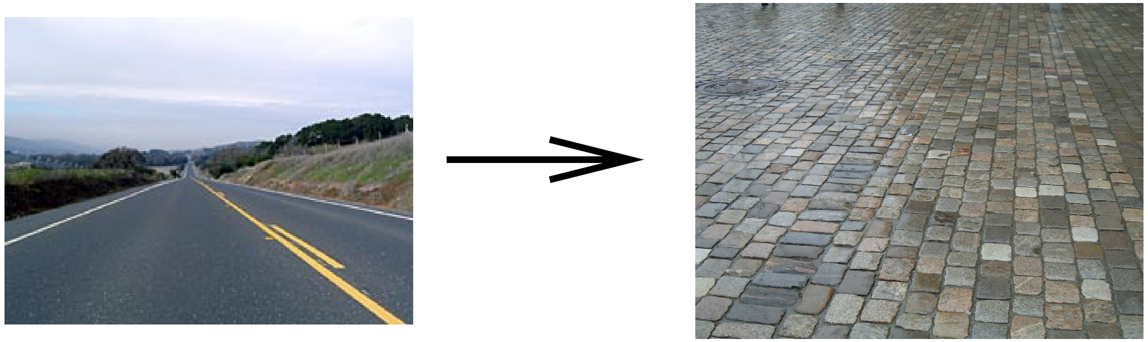
$$\boxed{\frac{L}{2\alpha\hbar} = \text{quantized}}$$

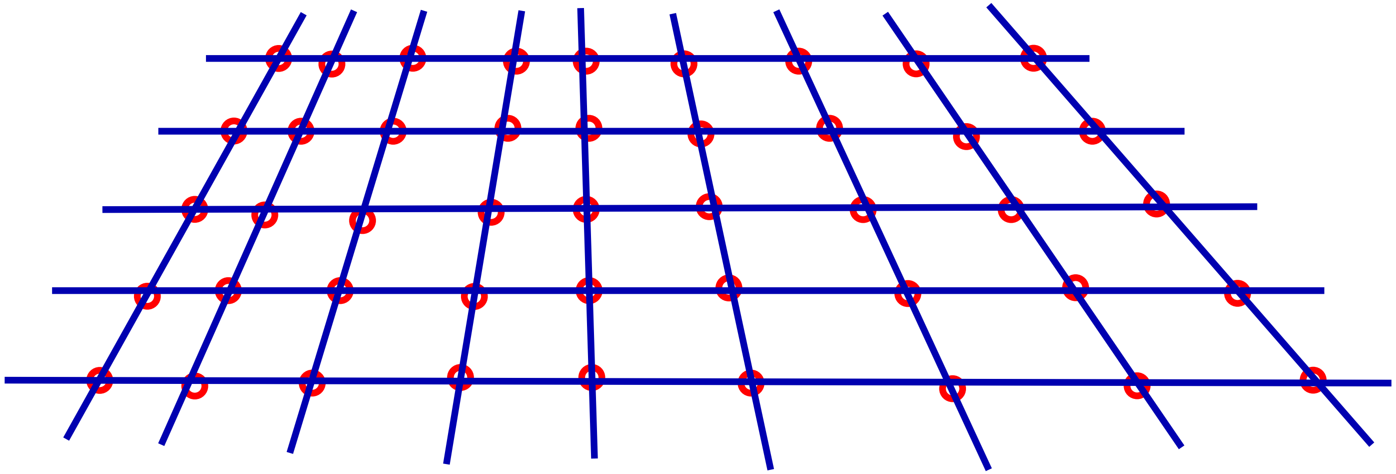
Results hold in Curved Spacetime

(small lengths = linear gravitational potential)

Curved Spacetime: Dirac Equation?

Spacetime near the Planck Scale

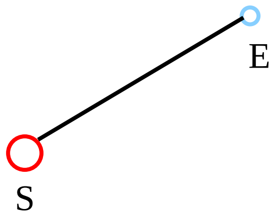




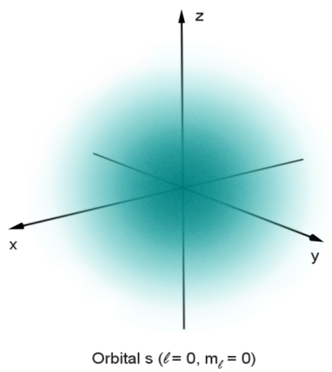
S. Das, E. C. Vagenas, A. Ali, to appear in Phys. Lett. **B** , arXiv:1005.3368

A note on Planck scale and reference frames

Sph. Symm. Hamiltonian $H = \frac{p^2}{2m} + V(r)$



Classical Solution
(Breaks Spherical Symm)



Quantum Mech Solution
(l=m=s=0)

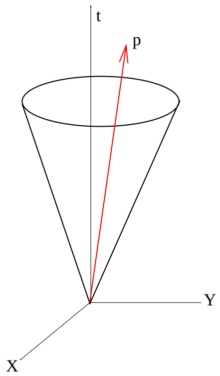
$$\frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$$

(Uncertainty restores Spherical Symm)

A preferred frame/ *'Aether'* $\rightarrow$ would solve the *'QG in whose frame?'* problem - *Planck scale in that frame!*

But aether breaks Lorentz Symm (one p^μ in the light cone)

<u>But</u> *If* $\underbrace{|\Psi\rangle}_{aether} = N \int d\Omega_p \, |\Psi_p\rangle$ $\leftarrow$ Superpos'n of all p^μ within the light cone



(Again, uncertainty restores Lorentz Symmetry)

Preferred frame chosen momentarily when a measurement is made!

$$\langle\Psi|\Psi\rangle = |N|^2 4\pi \int_0^{M_{Pl}} \frac{p^2 dp}{2\sqrt{p^2+m^2}} = |N|^2 \times M_{Pl}^2 \ (also\ normalizable)$$

So, a Lorentz covariant aether may solve the problem

Dirac, Nature 1951

Summary and Conclusions

- One GUP seems to fit Black Holes, String Theory, DSR,...
- Planck Scale in whose frame? *DSR, Quantum aether à la Dirac?*
- GUP affects all QM Hamiltonians. At least 1 part in 10^{12} precision required for measuring effects
- Space Quantized near the Planck scale. *But*, Discreteness at $10^{-35}\ m \ \rightarrow$ observable effects at $10^{-20}\ m$? Gravity waves, photons, LHC
- *And* can do calculus
- Statistical Mechanical Systems, Normally forbidden Processes, $\alpha^{1/n}$ Effects
- Optimistic Scenario: A Low Energy Window to Quantum Gravity Phenomenology?

All our results hold so long as there is a $\mathcal{O}(\alpha)$ term in the GUP