CHAPTER 3

Sequences

1. Regular and automatic sequences

Examples of functions satisfying linear systems of functional equations arise from the so-called *regular sequences*, *see* [**2**, chapter 16] and [**41**, §5.1].

Consider $R \subset \mathbf{C}$ a subring and $u = (u(k))_{k \in \mathbf{N}} \in R^{\mathbf{N}}$ a sequence with values in R.

Definition 1. *Let $q \in \mathbf{N}$, $q > 1$, a sequence $u \in R^{\mathbf{N}}$ is* q-regular *if the set of all sub-sequences $(u(q^e k + a))_{k \in \mathbf{N}}$ for $a, e \in \mathbf{N}$, $0 \le a < q^e$, is contained in a finitely generated R-module. The set of all sub-sequences $(u(q^e k + a))_{k \in \mathbf{N}}$, $a, e \in \mathbf{N}$, $0 \le a < q^e$, is called the* q-kernel *of the sequence.*

For the following definition we refer to [**2**, chapters 4 and 5], especially sections 4.1, 4.3 and 5.1 therein.

Definition 2. *A* deterministic finite q-automaton with output in R *consist in a finite set S of* states *with a distinguished* initial state s_0 *a* transition map $\delta : S \times \{0, \dots, q-1\} \to S$ *and an* ouput function $\tau : S \to R$. *To each* finite word $w_0 \dots w_\ell$ *on the* alphabet $\{0, \dots, q-1\}$ *it associates the element of R defined as*

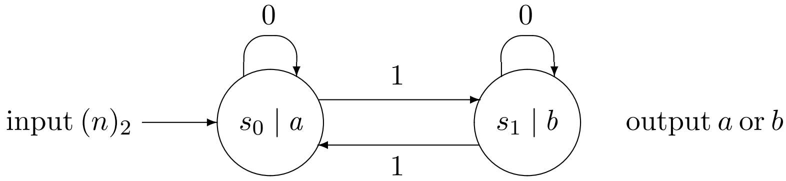
$$\tau \circ \delta(s_0, w_0 \dots w_\ell) = \tau \circ \delta(\delta(\dots \delta(s_0, w_0), \dots), w_\ell).$$

A sequence $u = (u(k))_{k \in \mathbf{N}}$ is q-automatic *if there exists a deterministic finite automaton with output that, for all $k \in \mathbf{N}$, sends the word formed with the digits of k in base q (starting on the left with the most significant digit), to u_k.*

For a finite word $w = w_0 \dots w_\ell$ on the alphabet $\{0, \dots, q-1\}$ we denote $[w]_q$ the number $w_0 + w_1 q + \dots + w_\ell q^\ell$. The above definition entails that some deterministic finite automaton with output produces for any $k \in \mathbf{N}$ the k-th element of the given sequence from any word w satisfying $[w]_q = k$. We may even assume that $\delta(s_0, 0) = s_0$. Also, it implies that there exists some other deterministic finite automaton with output that produces the given sequence from the word formed with the digits of k in base q written in reverse order (*i.e.* starting from the least significant digit).

According to [**2**, Theorem 6.6.2, page 185], q-automatic sequences are characterised as the sequences the q-kernel of which is finite. Furthermore, Theorem 16.1.5 in [**2**, page 441] asserts that q-automatic sequences are precisely the q-regular sequences that take only finitely many values.

Example 11. *The* Morse -Thue automaton *is the following two states* 2-*automaton with output in* $\{a,b\}$



which produces the Morse -Thue sequence *: abbabaabbaababba... From this automatic description it is clear that* $\tau\circ\delta(s_0,(n)_2)$ *is* a *or* b *according to the sum of digits in the expansion* $(n)_2$ *of* n *in base* 2 *being even or odd.*

Given a finite alphabet Σ we denote Σ^q the set of words of length q on Σ. A *q-morphism on* Σ is a map $\sigma:\Sigma\to\Sigma^q$. It extends naturally to words on Σ (finite and infinite) by replacing each letter of the word by its image through σ. For finite words this process multiplies the length of the word by q.

Consider a q-automatic sequence $u=(u(k))_{k\in\mathbf{N}}\in R^{\mathbf{N}}$ given by a deterministic finite q-automaton with output (S,s_0,δ,τ). Viewing S as an alphabet we set $\sigma:S\to S^q$ the map $s\mapsto\delta(s,0)\ldots\delta(s,q-1)$. If the initial state s_0 satisfies $\delta(s_0,0)=s_0$ then u is the image by τ of the unique word invariant by σ and starting with s_0. Reciprocally, a finite alphabet S, a q-morphism σ on S, an infinite word $s_0s_1\ldots$ on S invariant by σ and a function $\tau:S\to R$ defines a q-automaton (S,s_0,δ,τ), where δ is the transition map that sends $(s,w)\in S\times\{0,\ldots,q-1\}$ to the $(w+1)$-th letter in $\sigma(s)$, that produces the q-automatic sequence $\tau(s_0)\tau(s_1)\ldots$.

Example 12. *The Morse -Thue sequence is invariant under the* 2-*morphism on* $\{a,b\}$ *defined by* $\sigma(a)=ab$ *and* $\sigma(b)=ba$.

We now cite two important theorems in the theory of automatic sequences.

Theorem 25 (A.Cobham [**2**, Thm.11.2.2]) 25. *Let* $q,q'>1$ *be two multiplicatively independent integers, a sequence that is both q- and q'-automatic is* ultimately periodic.

Theorem 26 (G.Christol [**2**, Thm.12.2.5]) 26. *Let* p *be a prime number,* q *a power of* p *and* $\mathbf{F}_q$ *the field with* q *elements. A series* $f(z)=\sum_{k\in\mathbf{N}}u(k)z^k\in\mathbf{F}_q[[z]]$ *is algebraic over* $\mathbf{F}_q(z)$ *if and only if the sequence* u *is* p*-automatic.*

2. Generating functions

As in the last theorem of the previous section, to each sequence $u \in R^{\mathbf{N}}$ we associate its *generating function* (or *generating series*) :

$$f_u(z) := \sum_{k \in \mathbf{N}} u(k) z^k \in R[[z]] \ .$$

We deduce from Christol's and Cobham's theorems the following characterisation of algebraic generating series with bounded coefficients.

THEOREM 27. *Let $f(z) \in \mathbf{Z}[[z]]$ be a series with coefficients bounded in $\mathbf{Z}$, then either $f(z)$ is a polynomial or $(1-z^N)f(z)$ is a polynomial, for some $N \in \mathbf{N}^*$, or $f(z)$ is transcendental over $\mathbf{Q}(z)$. In the first case the sequence of coefficients of $f(z)$ is finite and in the second case it is ultimately periodic with length of period dividing N.*

Said differently, the theorem asserts that an algebraic function the Taylor series of which has bounded coefficients in $\mathbf{Z}$, is a rational functional the poles of which are roots of unity.

Proof – Suppose $f(z) = \sum_{k \in \mathbf{N}} u(k) z^k$ algebraic over $\mathbf{Q}(z)$, then according to [**2**, Thm.12.6.1] the same holds over $\mathbf{F}_p(z)$ for all its reductions modulo any prime number p. For any p larger than the largest difference between the coefficients of $f(z)$, Christol's theorem 26 implies that the sequence u is p-automatic. Selecting two distinct such p's, Cobham's theorem 25 shows that the sequence u is ultimately periodic. Say the period starts at index M and is of length N, this means that $u(mN+\ell+M) = u(\ell+M)$ for any $m \in \mathbf{N}$ and $\ell = 0, \ldots, N-1$, therefore

$$\begin{aligned} f(z) &= \sum_{k=0}^{M-1} u(k) z^k + \sum_{m \in \mathbf{N}} \sum_{\ell=0}^{N-1} u(\ell+M) z^{mN+\ell+M} \\ &= \sum_{k=0}^{M-1} u(k) z^k + \sum_{\ell=0}^{N-1} u(\ell+M) z^{\ell+M} \sum_{m \in \mathbf{N}} z^{mN} \\ &= \sum_{k=0}^{M-1} u(k) z^k + \frac{1}{1-z^N} \cdot \sum_{\ell=0}^{N-1} u(\ell+M) z^{\ell+M}. \ \square \end{aligned}$$

Generating functions of regular sequences satisfy functional equations :

THEOREM 28 (K.NISHIOKA [**41**, Thm.5.1.2]) 28. *A sequence $u \in R^{\mathbf{N}}$ is q-regular if and only if there exists an integer $r \in \mathbf{N}^*$ and r formal power series $f_1 = f_u, \ldots, f_r \in R[[z]]$ satisfying*

$$\begin{pmatrix} f_1(z) \\ \vdots \\ f_r(z) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z^q) \\ \vdots \\ f_r(z^q) \end{pmatrix}$$

where $A(z)$ denote an $r \times r$ matrix with entries in $R[z]$ of degree $< q$.

Theorem 28 implies that the generating function f_u of the q-regular sequence u satisfies a functional equation of the type

$$\sum_{i=0}^{\ell} c_i(z) f_u(z^{q^i}) = 0 \tag{17}$$

with $c_i(z) \in R(z)$, $c_\ell(z) \neq 0$ and $\ell \leq r$. P-G.Becker [**9**, Theorem 1] asserts $c_0(z) \neq 0$, but he shows that not all the series satisfying a functional equation of type (17) with $c_0(z) \neq 0$ is the generating function of a q-regular sequence. Indeed, the function $f(z) = \frac{1}{z-\alpha}$, $\alpha^{-1} \in R$ not a root of unity, satisfies $(z^q - \alpha)f(z^q) - (z - \alpha)f(z) = 0$ for any integer $q > 1$, but, in view of theorem 29 below, it is not the generating function of a q-regular sequence. However, P-G.Becker [**9**, Theorem 2] proves that all the series satisfying a functional equation of type (17) with $a_0 \in R \setminus \{0\}$ being a non zero constant, is the generating function of a q-regular sequence.

Given the integer $q > 1$ the set of generating functions of q-regular sequences forms a ring, *see* [**2**, Corollary 16.4.2, page 446]. The following statement put together [**9**, Lemma 5] and [**2**, Theorem 16.4.3, page 446].

Theorem 29. *The generating function of a q-regular sequence is either a rational function or it is transcendental over $\mathbf{C}(z)$. A rational function is q-regular if and only if its poles are roots of unity.*

This theorem is to be compared with theorem 27, the intersection of the two statements asserts that algebraic generating functions of q-automatic sequences are the rational functions the poles of which are roots of unity.

In the case of generating functions of q-automatic sequences the functional equation can be made explicit easily. Let (S, s_0, δ, τ) be a q-automaton that generates a q-automatic sequence $u \in R^{\mathbf{N}}$ and for $s \in S$ put $N(s_0, s) = \{k \in \mathbf{N}; s = \delta(s_0, (k)_q)\}$, where $(k)_q$ stands for the word on the alphabet $\{0, \ldots, q-1\}$ made of the digits of the number $k \in \mathbf{N}$ in base q starting with the most significant. We introduce the following series for $s \in S$

$$f_{s,s_0}(z) = \sum_{k \in N(s_0,s)} z^k$$

so that the generating function of u is $f_u(z) = \sum_{s \in S} \tau(s) f_{s,s_0}(z)$. Fixing some order on S we denote $\mathbf{f}_{s_0}(z)$ the column vector with components the functions $f_{s,s_0}(z)$, $s \in S$. We also introduce the matrix $A(z) = \big(a_{s,s'}(z)\big)_{s,s' \in S}$ where $a_{s,s'}(z) = \sum_{\substack{0 \leq \ell < q \\ s = \delta(s',\ell)}} z^\ell$ (s index the lines and s' the columns of the matrix $A(z)$). Assuming again $s_0 = \delta(s_0, (0)_q)$, we verify

$$f_{s,s_0}(z) = \sum_{k \in N(s_0,s)} z^k = \sum_{s' \in S} \sum_{\substack{0 \leq \ell < q \\ s = \delta(s',\ell)}} z^\ell \sum_{m \in N(s_0,s')} z^{mq} = \sum_{s' \in S} a_{s,s'}(z) f_{s',s_0}(z^q),$$

hence the equation

$$\mathbf{f}_{s_0}(z) = A(z)\mathbf{f}_{s_0}(z^q).$$

We may replace the initial state s_0 by an arbitrary state $s' \in S$ satisfying $s' = \delta(s', (0)_q)$ in order to get further functions $f_{s,s'}(z) = \sum_{k \in N(s',s)} z^k$ and vectors $\mathbf{f}_{s'}(z)$ satisfying $\mathbf{f}_{s'}(z) = A(z)\mathbf{f}_{s'}(z^q)$. In case the map $\delta(\cdot, (0)_q)$ is the identity on S , we get a matrix $G(z)$ the columns of which are the vectors $\mathbf{f}_{s'}(z)$ forming a complete set of solution to the equation $G(z) = A(z)G(z^q)$. We observe that for any $s' \in S$ we have $\sum_{s \in S} f_{s,s'}(z) = \frac{1}{1-z}$ since $\bigcup_{s \in S} N(s', s)$ is a partition of $\mathbf{N}$.

3. Automatic numbers

In the same vein, a (q, b)-*automatic number* is a real number such that the sequence of digits in its expansion in base b is a q-automatic sequence or, equivalently, a q-regular sequence since it takes only finitely many values. A number which is (q, b)-automatic for some q is also said b-*automatic.*

The set of (q, b) automatic numbers is a vector space over the rational numbers but it is not closed by multiplication or inverse. Rational numbers are (q, b) -automatic for all integers $q, b > 1$ and they are the only numbers that are (q, b) -automatic for two distinct values of q , *see* [**2**, chapter 13, sections 1 and 2].

Theorem 30. *Let* $u_1, \ldots, u_\ell$ *be q-automatic sequences such that their generating series are algebraically independent over* $\mathbf{Q}(z)$. *Then for all but finitely many integers* $b > 1$ *the* (q, b)*-automatic numbers* $\sum_{k \in \mathbf{N}} \frac{u_i(k)}{b^k}$ *,* $i = 1, \ldots, \ell$ *are algebraically independent over* $\mathbf{Q}$ *.*

Proof – Let $f_1(z), \ldots, f_\ell(z)$ be the generating series of the q-regular sequences $u_1, \ldots, u_\ell$. Theorem 28 shows that these series satisfy the hypothesis of theorem 15. Therefore, for all but finitely many algebraic numbers α satisfying $|\alpha| \leq \frac{1}{2}$, the values $f_1(\alpha), \ldots, f_\ell(\alpha)$ are algebraically independent over $\mathbf{Q}$. Restricting α to the set of inverse of integers $b \geq 2$ gives the result. □

Sadly enough, given a base b the above theorem does not prove the transcendence of a single automatic number.

Theorem 31 (B.Adamczewski & Y.Bugeaud [**1**, Thm.2]) 31. *Let* $b \geq 2$ *be an integer, a b-automatic number is either rational or transcendental over* $\mathbf{Q}$ *.*

The proof of this result rests on Schmidt subspace theorem which is an higher dimensional extension of Roth theorem. The main observation is that the word of digits in the expansion in base b of an automatic numbers is a *stammering word*, that is a word which contains arbitrary long subwords repeating in fixed proportions. More precisely, an infinite word $w = w_0 w_1 \ldots$ is stammering if there exists a real $\xi > 0$ and sequences $(k_n)_{n \in \mathbf{N}}$,

$(k'_n)_{n\in\mathbf{N}}$ and $(k''_n)_{n\in\mathbf{N}}$ with $\lim_{n\to\infty} k_n = \infty$, $\xi \le k''_n/k'_n \le 1$, k_n/k'_n is bounded above and for any $n \in \mathbf{N}$

$$w = w_0 \dots w_{k_n} w_{k_n+1} \dots w_{k_n+k'_n} w_{k_n+1} \dots w_{k_n+k''_n} \dots$$

The method apply to real numbers which can be expressed as $\sum_{k\in\mathbf{N}} \frac{u(k)}{b^k}$ where $u = (u(k))_{k\in\mathbf{N}}$ is a stammering word on some finite alphabet and $b > 1$ can be any *Pisot-Vijayaraghavan* or *Salem number*. Recall that a Pisot-Vijayaraghavan number (*resp.* a Salem number) is a real algebraic integer > 1 the conjugates of which all lie inside the open unit disc (*resp.* the closed unit disc, with at least one on the unit circle).

4. Pattern's counting

Examples of regular sequences are also given by digits counting. Let $d \ge 2$, $m \ge 1$, and $0 \le \mu < d^m$ be integers, we may consider the digits of μ in base d as a *pattern of length* m (including as many zeros as necessary on the right in order to get m digits). For any $k \in \mathbf{N}$ we denote $e_\mu(k)$ the number of occurrences of the pattern μ in the expansion of k in base d.

Proposition 32. *With the notations above, the sequence* $(e_\mu(k))_{k\in\mathbf{N}}$ *is* d^m *-regular and its generating function* $f_{e_\mu}(z) = \sum_{k\in\mathbf{N}} e_\mu(k) z^k$ *satisfies the equation*

$$f_{e_\mu}(z) = \frac{z^{d^m}-1}{z-1} f_{e_\mu}(z^{d^m}) + \sum_{j=0}^{m-1} \frac{z^{d^j\mu}(z^{d^j}-1)}{z-1} \cdot \frac{1}{1-z^{d^{m+j}}} \quad .$$

The series $f_{e_\mu}(z)$ *converges in the unit disc* $\{z \in \mathbf{C}; |z| < 1\}$ *where it defines a function transcendental over* $\mathbf{C}(z)$ *.*

Proof – Introduce the auxiliary sequences (μ is a pattern of length m, $j \ge 1$ is a non zero integer and $k \in \mathbf{N}$)

$$u_\mu(k) = \begin{cases} 1 & \text{if } k \equiv \mu(d^m) \\ 0 & \text{otherwise} \end{cases}$$

$$v_{j,\mu}(k) = \begin{cases} 1 & \text{if } 0 \le k - \mu d^j < d^j \\ 0 & \text{otherwise} \end{cases}$$

and the indices truncated from $\mu = \mu_0 + \dots + \mu_{m-1} d^{m-1}$:

$$\mu_{(j)} := \mu_0 + \dots + \mu_{m-j-1} d^{m-j-1} \quad \text{and} \quad \mu^{(j)} := \mu_{m-j} + \dots + \mu_{m-1} d^{j-1}$$

for $j = 0, \dots, m$ (setting $\mu_{(m)} = \mu^{(0)} = 0$). For $0 \le a < d^m$ one checks the following identity

$$e_\mu(d^m k + a) = e_\mu(k) + e_\mu(a) + \sum_{j=1}^{m-1} u_{\mu^{(j)}}(k) v_{j,\mu_{(j)}}(a) \quad , \tag{18}$$

where indeed $e_\mu(a) = u_{\mu^{(0)}}(a)v_{0,\mu_{(0)}}(a)$ is equal to 1 if $\mu = a$ and 0 otherwise. The sum on j takes care of the occurrences of μ in $d^m k + a$ which are astride upon a and $d^m k$. Since the sequences $(u_{\mu^{(j)}}(k))_{k\in\mathbf{N}}$ are d^m-regular, we deduce that the same holds for $(e_\mu(k))_{k\in\mathbf{N}}$.

Next, using (18) we write

$$\begin{aligned} f_{e_\mu}(z) &= \sum_{a=0}^{d^m-1} \sum_{\ell\in\mathbf{N}} e_\mu(d^m\ell + a) z^{d^m\ell+a} \\ &= \sum_{a=0}^{d^m-1} z^a \sum_{\ell\in\mathbf{N}} e_\mu(\ell) z^{d^m\ell} + \sum_{a=0}^{d^m-1} z^a \sum_{j=0}^{m-1} v_{j,\mu_{(j)}}(a) \sum_{\ell\in\mathbf{N}} u_{\mu^{(j)}}(\ell) z^{d^m\ell} \\ &= \frac{z^{d^m}-1}{z-1} f_{e_\mu}(z^{d^m}) + \sum_{j=0}^{m-1} \sum_{a=0}^{d^m-1} v_{j,\mu_{(j)}}(a) z^a \sum_{k\in\mathbf{N}} z^{d^{m+j}k + d^m\mu^{(j)}} \quad . \end{aligned}$$

But,

$$\sum_{a=0}^{d^m-1} v_{j,\mu_{(j)}}(a) z^a = \sum_{b=0}^{d^j-1} z^{b+d^j\mu_{(j)}} = \frac{z^{d^j\mu_{(j)}}(z^{d^j}-1)}{z-1}$$

and

$$\sum_{k\in\mathbf{N}} z^{d^{m+j}k+d^m\mu^{(j)}} = \frac{z^{d^m\mu^{(j)}}}{1-z^{d^{m+j}}} \quad .$$

Therefore

$$f_{e_\mu}(z) = \frac{z^{d^m}-1}{z-1} f_{e_\mu}(z^{d^m}) + \sum_{j=0}^{m-1} \frac{z^{d^m\mu^{(j)}+d^j\mu_{(j)}}(z^{d^j}-1)}{z-1} \cdot \frac{1}{1-z^{d^{m+j}}}$$

and the first part of the theorem is proved because $d^m\mu^{(j)} + d^j\mu_{(j)} = d^j\mu$.

The coefficient $e_\mu(k)$ is bounded by the number of digits of the expansion of k in base d and it grows at most logarithmically as k tends to infinity : $e_\mu(k) \le \left[\frac{\log(k)}{\log(d)}\right] + 1$, therefore the series $f_{e_\mu}(z)$ converges in the unit disc $\{z \in \mathbf{C}; |z| < 1\}$.

Now, by theorem 3 we know that $f_{e_\mu}(z)$ is either transcendental over $\mathbf{C}(z)$ or a rational function. But, the coefficients of the series $f_{e_\mu} \in \mathbf{Z}[[z]]$ are not bounded as k tends to infinity (think of $e_\mu(\mu(1+d^m+\cdots+d^{m\ell})) = \ell$) and grows at most logarithmically (*see* above). Lemma 4 implies that $f_{e_\mu}(z)$ cannot be a rational function, therefore it must be transcendental over $\mathbf{C}(z)$. □

Note that the series $f_{e_\mu}(z)$ and $\frac{1}{1-z^{d^j}}$, $j = 0, \ldots, m-1$, satisfy the condition of theorem 28, except that the degrees of the coefficients are not less than d^m.

Combining proposition 32 and theorem 1, we can state :

COROLLARY 33. *With the notations as in proposition 32, for* $\alpha \in \overline{\mathbf{Q}}^*$ *an algebraic number of absolute value* $|\alpha| < 1$ *the number* $f_{e_\mu}(\alpha)$ *is transcendental.*

REMARK 13. *Is the value of the generating function of a sequence at some special algebraic point, meaningful?*

EXAMPLE 14. *In base* 2 *the sequence* $(e_1(k))_{k\in\mathbf{N}}$ *satisfies* $e_1(2^e k + a) = e_1(k) + e_1(a)$ *for* $0 \le a < 2^e$ *and therefore it is* 2 *-regular (the only companion sequence is the sequence constant equal to* 1 *, the generating function of which is* $\frac{1}{1-z}$ *). The generating function* f_{e_1} *of* $(e_1(k))_{k\in\mathbf{N}}$ *satisfies the equation*

$$f_{e_1}(z^2) = \frac{(1-z^2)f_{e_1}(z) - z}{(1+z)(1-z^2)} \quad .$$

The sequence $(a_1(k))_{k\in\mathbf{N}} \in \{\pm 1\}^{\mathbf{N}}$ *defined by* $a_1(k) = (-1)^{e_1(k)}$ *is the* Morse -Thue sequence *(on the alphabet* $\{\pm 1\}$ *), its generating function* f_{a_1} *satisfies the equation*

$$f_{a_1}(z^2) = \frac{f_{a_1}(z)}{1-z} \quad .$$

Similarly, we check $e_{11}(2^e k + a) = e_{11}(k) + e_{11}(a)$ *for* $0 \le a < 2^{e-1}$ *or* $2^{e-1} \le a < 2^e$ *and* k *even and* $e_{11}(2^e k + a) = e_{11}(k) + e_{11}(a) + 1$ *for* $2^{e-1} \le a < 2^e$ *and* k *odd. Thus* $(e_{11}(k))_{k\in\mathbf{N}}$ *is* 2 *-regular (with companion sequence the sequence constant equal to* 1 *, as above) whereas proposition 32 only shows that it is* 4 *-regular. The generating function* $f_{e_{11}}$ *of* $(e_{11}(k))_{k\in\mathbf{N}}$ *satisfies the equation*

$$f_{e_{11}}(z^2) = \frac{(1-z^4)f_{e_{11}}(z) - z^3}{(1+z)(1-z^4)} \quad .$$

The sequence $(a_{11}(k))_{k\in\mathbf{N}} \in \{\pm 1\}^{\mathbf{N}}$ *defined by* $a_{11}(k) = (-1)^{e_{11}(k)}$ *is known as the* Rudin-Shapiro sequence*, its generating function* $f_{a_{11}}$ *satisfies the equations*

$$f_{a_{11}}(z^2) = \frac{f_{a_{11}}(z) + f_{a_{11}}(-z)}{2} \quad f_{a_{11}}(-z^2) = \frac{f_{a_{11}}(z) - f_{a_{11}}(-z)}{2z} \quad .$$

5. Linear recurrences

Linear recurrence sequences also give rise to Mahler type functions, *see* [**11**]. Let's have a look at the famous *Fibonacci sequence* defined by

$$F_0 = 0 \ , \quad F_1 = 1 \ , \quad F_{k+2} = F_{k+1} + F_k \ , \quad k \in \mathbf{N},$$

the terms of which are computed as $F_k = \frac{1}{\sqrt{5}}(\alpha^k - \overline{\alpha}^k)$, $k \in \mathbf{N}$, where $\alpha = \frac{1+\sqrt{5}}{2}$ designate the *golden ratio* and $\overline{\alpha} = \frac{1-\sqrt{5}}{2}$ its conjugate.

For $i \in \mathbf{N}$ introduce the series

$$f_i(z) := \sum_{k\in\mathbf{N}} \frac{1}{\alpha^i z^{-2^k} - \overline{\alpha}^i z^{2^k}} = \sum_{m\in\mathbf{N}^*} (-1)^{i(\ell(m)+1)} \overline{\alpha}^{i(2\ell(m)+1)} z^m \quad ,$$

where $\ell(m) := \frac{1}{2}(m2^{-v_2(m)} - 1)$ (v_2 stands for the 2-adic valuation), which converges on the unit disc in $\mathbf{C}$, it defines a function satisfying the functional equation

$$f_i(z^2) = f_i(z) - \frac{1}{\alpha^i z^{-1} - \overline{\alpha}^i z} \quad .$$

A direct computation shows $f_i(\overline{\alpha}) = \frac{1}{\sqrt{5}} \sum_{k\in\mathbf{N}^*} F^{-1}_{2^k+i} - L^{-1}_{i+1}$ where $L_{i+1} := \alpha^{i+1} + \overline{\alpha}^{i+1}$ is the $(i+1)$-th *Lucas number*.

Beside $f_0(z) = \frac{z}{1-z}$ the function f_i , $i \in \mathbf{N}^*$, are transcendental over $\mathbf{C}(z)$ (by theorem 3 it must be either rational or transcendental). Applying theorem 1 we obtain that for $i \in \mathbf{N}^*$ the numbers

$$\sum_{k\in\mathbf{N}} F^{-1}_{2^k+i}$$

are transcendental over $\mathbf{Q}$. On the other hand a computation gives

$$\sum_{k\in\mathbf{N}} F^{-1}_{2^k} = F_1^{-1} + \sqrt{5}\left(f_0(\overline{\alpha}) + L_1^{-1}\right) = 1 + \sqrt{5}\left(\frac{1-\sqrt{5}}{1+\sqrt{5}} + 1\right) = \frac{7-\sqrt{5}}{2} \quad .$$

Using Kubota's theorem on algebraic independence of functions, one can prove the following algebraic independence result, *see* [**42**] for a much more complete version.

THEOREM 34. *Let* $1 \leq i_1 < \cdots < i_m$ *be integers, then the numbers* $\sum_{k\in\mathbf{N}} F^{-1}_{2^k+i_1}, \ldots, \sum_{k\in\mathbf{N}} F^{-1}_{2^k+i_m}$ *are algebraically independent over* $\mathbf{Q}$.

Proof – Since each number under consideration is expressed as an affine function of the value at $\overline{\alpha}$ of the corresponding function $f_{i_1}(z), \ldots, f_{i_m}(z)$, according to Theorem 13 it suffices to establish the algebraic independence of these function over $\mathbf{C}(z)$ in order to prove the algebraic independence of $f_{i_1}(\overline{\alpha}), \ldots, f_{i_m}(\overline{\alpha})$ over $\mathbf{Q}$ and hence the theorem. Thanks to Kubota's theorem 19 the algebraic independence reduces to the linear independence modulo the rational fractions. Therefore suppose some linear combination

$$r(z) = \sum_{h=1}^{m} c_h f_{i_h}(z)$$

is a rational function $r(z) \in \mathbf{C}(z)$. It satisfies the functional equation $r(z^2) = r(z) - R(z)$ with $R(z) = \sum_{h=1}^{m} \frac{c_h}{\alpha^{i_h} z^{-1} - \overline{\alpha}^{i_h} z}$ and we notice that both sets of poles of $r(z^2)$ and $R(z)$ are invariant by multiplication by -1 , since these functions are even and odd respectively.

Then $r(z)$ can have only one pole of a given absolute value $\rho \neq 0, 1$, otherwise $r(z^2)$ would have at least four poles of absolute value $\rho^{1/2}$ and since $R(z)$ has at most two poles of that absolute value this would implies that $r(z)$ has also at least two poles of absolute value $\rho^{1/2}$ and so on we would produce infinitely many poles of $r(z)$.

Now, the poles of $r(z^2)$ and $R(z)$ of a given absolute value $\rho \neq 0, 1$ must coincide possibly except for one, but then the invariance of the sets of poles by multiplication by -1 entails that the rational fractions $r(z^2)$ and $R(z)$ indeed have the same poles. Let ρ be the largest absolute value of a pole of $R(z)$, then $r(z^2)$ has two poles of absolute value ρ and $r(z)$ must have one pole of absolute value ρ^2, which is impossible by the functional equation if $\rho > 1$. But, in view of its definition, the rational fraction $R(z)$ has poles $\pm \left(\frac{\alpha}{\bar{\alpha}}\right)^{i_h/2}$, $h \in \{1, \ldots, m\}$, of absolute values > 1 as soon as $c_h \neq 0$. This proves that the functions $f_{i_1}(z), \ldots, f_{i_m}(z)$ are linearly independent modulo the rational fractions, hence algebraically independent over $\mathbf{C}(z)$ by Kubota's theorem 19, and it establishes the theorem. □

The above application of Mahler theorem deals with indices in *geometric progression*. Actually, for indices increasing more quickly the general term of the series grows so rapidly that the transcendence of the sum can be obtained from Roth theorem, *see* for example [**43**]. Considering indices in *arithmetic progression*, we have a whole zoo of reciprocal sums of Fibonacci numbers, some of them are algebraic

$$\sum_{k \in \mathbf{N}^*} (F_k F_{k+2})^{-1} = 1$$

$$\sum_{k \in \mathbf{N}^*} (-1)^k (F_k F_{k+1})^{-1} = \frac{1 - \sqrt{5}}{2}$$

$$\sum_{k \in \mathbf{N}^*} (F_{2k-1} + 1)^{-1} = \frac{\sqrt{5}}{2} \ ,$$

others transcendental over $\mathbf{Q}$

$$\sum_{k \in \mathbf{N}^*} F_k^{-2s} \ , \quad \sum_{k \in \mathbf{N}^*} F_{2k-1}^{-s} \ , \quad \sum_{k \in \mathbf{N}^*} (-1)^k F_k^{-2} \ , \quad \sum_{k \in \mathbf{N}^*} k F_{2k}^{-1} \notin \overline{\mathbf{Q}}$$

for any positive integer s, while the series $\sum_{k \in \mathbf{N}^*} F_k^{-1}$ is only known to be irrational. As for the Lucas numbers one knows that for any positive integer s the series $\sum_{k \in \mathbf{N}^*} L_k^{-2s}$ and $\sum_{k \in \mathbf{N}^*} L_{2k}^{-s}$ are transcendental over $\mathbf{Q}$. All these transcendence results are corollaries of Nesterenko's theorem 10, worked out by D.Duverney, Ke. and Ku. Nishioka, I.Shiokawa in [**18**] and [**19**], for the Fibonacci series along with some more general second order linear recurrences.

The proof consists in writing each series as the value at some algebraic point of a non constant function which is algebraic over the field generated by the Ramanujan functions $E_1(z)$, $E_2(z)$ and $E_3(z)$ and then apply corollary 11. In particular, this shows as well that the transcendence degree of the field generated over $\mathbf{Q}$ by all the above series is 3. For example, the following formulas can be proved

$$\sum_{k\in\mathbf{N}^*}\frac{1}{L_{2k}^2}=\frac{1}{2}\frac{\Theta\theta_2(0,\alpha^2)}{\theta_2(0,\alpha^2)}-\frac{1}{8}\ ,\quad \sum_{k\in\mathbf{N}^*}\frac{1}{F_{2k-1}^2}=\frac{5}{2}\frac{\Theta\theta_3(0,\alpha^2)}{\theta_3(0,\alpha^2)}\ ,$$

$$\sum_{k\in\mathbf{N}^*}\frac{1}{L_{2k-1}^2}=-\frac{1}{2}\frac{\Theta\theta_4(0,\alpha^2)}{\theta_4(0,\alpha^2)}\ ,\quad \sum_{k\in\mathbf{N}^*}\frac{1}{F_{2k}^2}=5\left(\frac{1}{24}-\frac{\Theta\eta(\alpha^4)}{\eta(\alpha^4)}\right)\ .$$

Exercise 15. *Try to prove the above formulas and if you encounter difficulties check the appropriate lemma in* [**19**].

By the third question in exercice 6 in Chap.1, Sec.3, any three of the above four sums are algebraically independent over $\mathbf{Q}$, but the four are linked by an algebraic relation. Indeed, by the differential system (10) in *loc. cit.* we have

$$\frac{\Theta\theta_2(0,\alpha^2)}{\theta_2(0,\alpha^2)}+\frac{\Theta\theta_3(0,\alpha^2)}{\theta_3(0,\alpha^2)}+\frac{\Theta\theta_4(0,\alpha^2)}{\theta_4(0,\alpha^2)}=6\frac{\Theta\eta(\alpha^4)}{\eta(\alpha^4)}$$

from which follows a linear relation between the four sums

$$5\sum_{k\in\mathbf{N}^*}\frac{1}{L_{2k}^2}-5\sum_{k\in\mathbf{N}^*}\frac{1}{L_{2k-1}^2}+\sum_{k\in\mathbf{N}^*}\frac{1}{F_{2k-1}^2}+3\sum_{k\in\mathbf{N}^*}\frac{1}{F_{2k}^2}=0\ .$$

We further deduce

$$-5\sum_{k\in\mathbf{N}^*}\frac{(-1)^k}{L_k^2}=\sum_{k\in\mathbf{N}^*}\frac{1}{F_{2k-1}^2}+3\sum_{k\in\mathbf{N}^*}\frac{1}{F_{2k}^2}$$

$$\sum_{k\in\mathbf{N}^*}\frac{1}{F_k^2}=5\sum_{k\in\mathbf{N}^*}\frac{1}{L_{2k-1}^2}-5\sum_{k\in\mathbf{N}^*}\frac{1}{L_{2k}^2}-2\sum_{k\in\mathbf{N}^*}\frac{1}{F_{2k}^2}$$

and since the sums in the right hand side are algebraically independent over $\mathbf{Q}$ we conclude that we can add those on the left hand side to the list of transcendental sums.

Question 35. *Can one devise a method to determine whether a given sum is algebraic? Or even find the algebraic relations between several sums? Which type of sums can one hope to deal with?*

Remark 16. *In another direction, using Mahler's method for functions in several complex variables, Becker and Töpfer* [**11**] *have proved the transcendency of numbers*

$$\sum_{k\in\mathbf{N}}b_h(a_1\alpha_1^{d^k}+\cdots+a_m\alpha_m^{d^k})^{-1}$$

where $m,d\in\mathbf{N}^*$, $d\geq 2$, $a_1,\ldots,a_m\in\overline{\mathbf{Q}}^*$, $(b_k)_{k\in\mathbf{N}}\in\overline{\mathbf{Q}}^{\mathbf{N}}$ *is a periodic sequence not identically zero and* $\alpha_1,\ldots,\alpha_m$ *are* multiplicatively independent *algebraic numbers satisfying* $|\alpha_1|>\max(1;|\alpha_2|;\ldots;|\alpha_m|)$ *and*

$$a_1\alpha_1^{d^k}+\cdots+a_m\alpha_m^{d^k}\neq 0\ ,\quad k\in\mathbf{N}\ .$$

In the case of the inverse Fibonacci series the numbers α *and* $\overline{\alpha}$ *are* not *multiplicatively independent (i.e. a product of powers of these numbers is equal to* 1 *).*

The *Rogers-Ramanujan continued fraction*

$$\mathrm{RR}(z) = 1 + \frac{z\,|}{|\,1} + \frac{z^2\,|}{|\,1} + \frac{z^3\,|}{|\,1} + \cdots = \prod_{k\in\mathbf{N}} \frac{(1-z^{5k+3})(1-z^{5k+3})}{(1-z^{5k+1})(1-z^{5k+4})}$$

enters in the circle of transcendental functions that takes transcendental values at algebraic points, since it can be expressed in terms of the η Dedekind function, *see* [**19**]. Namely, $\mathrm{RR}(z)$ satisfies the equation

$$\frac{\mathrm{RR}(z)}{z^{1/5}} - \frac{z^{1/5}}{\mathrm{RR}(z)} = 1 + z^{2/5}\frac{\eta(z^{1/5})}{\eta(z^5)} \quad .$$

Therefore, for any $\alpha \in \overline{\mathbf{Q}}$, $0 < |\alpha| < 1$, the value $\mathrm{RR}(\alpha)$ is transcendental over $\mathbf{Q}$.