CHAPTER 2

Algebraic independence

1. Numbers

We now turn to algebraic independence results. Restricting the type of functional equations, Mahler proved a result on algebraic independence of values of several functions.

THEOREM 13 (K.MAHLER [**29**]) 13. *Let* $f_1(z), \ldots, f_m(z)$ *be functions admitting Taylor expansions at the origin, with coefficients in a given number field and which converge in the open unit disc. We assume that for some integer* $d > 1$ *the* $f_i(z)$ *'s satisfy the system of functional equations*

$$f_i(z^d) = a_i f_i(z) + b_i(z) \ , \quad 1 \leq i \leq m \tag{11}$$

with a_i *algebraic numbers and* $b_i(z)$ *rational fractions with algebraic coefficients.*

If α *is a non zero algebraic number of absolute value* < 1 *then the transcendence degree over* $\mathbf{Q}$ *of the field generated by* $f_1(\alpha), \ldots, f_m(\alpha)$ *is equal to the transcendence degree over* $\mathbf{Q}(z)$ *of the field generated by the functions* $f_1(z), \ldots, f_m(z)$ *.*

This result was extended by K.Kubota [**25**] to the case of function satisfying functional equations of type (11) with $a_i = a_i(z)$ being also rational fractions with rational coefficients, the denominator of which does not vanish at any α^{d^i} , $i \in \mathbf{N}$.

EXAMPLE 7. *Let* $a \geq 2$ *be an integer, the series* $f(z) := \sum_{k\in\mathbf{N}} z^{a^k}$ *defines a function in the unit disc and the functions* $f_i(z) := f(z^i)$ *satisfy the functional equations* $f_i(z^a) = f_i(z) - z^i$ *. The functions* $f_1, \ldots, f_{a-1}$ *are algebraically independent over* $\mathbf{Q}(z)$ *and therefore we derive from theorem 13 that for all* $\alpha \in \overline{\mathbf{Q}}$ *, non zero, of modulus* < 1 *, the numbers* $f(\alpha), \ldots, f(\alpha^{a-1})$ *are algebraically independent over* $\mathbf{Q}$ *.*

Extension of Mahler's theorem 13 above has led to the following result which deals with general *linear systems of functional equations.*

THEOREM 14 (K.NISHIOKA [**41**, Thm.4.2.1]) 14. *In the setting of the previous theorem 13 assume that the* $f_i(z)$ *'s satisfy the system of linear functional equations*

$$f_i(z^d) = \sum_{j=1}^{m} a_{i,j}(z) f_j(z) + b_i(z) \ , \quad 1 \leq i \leq m \tag{12}$$

with $a_{i,j}(z)$ *and* $b_i(z)$ *rational fractions with algebraic coefficients.*

If α *is a non zero algebraic number of absolute value* < 1 *such that none of the numbers* α^{d^k} , $k \in \mathbf{N}$, *is a pole of any* $a_{i,j}(z)$ *or* $b_i(z)$, *then the transcendence degree over* $\mathbf{Q}$ *of the field generated by* $f_1(\alpha), \ldots, f_m(\alpha)$ *is equal to the transcendence degree over* $\mathbf{Q}(z)$ *of the field generated by the functions* $f_1(z), \ldots, f_m(z)$.

We have already remarked that the assumption on the successive d-th powers of α not being pole of the $a_{i,j}(z)$'s and $b_i(z)$'s is necessary, *see* remark 1. It is worth observing that in case the transcendence degree over $\mathbf{Q}(z)$ of the field $\mathbf{Q}(z, f_1(z), \ldots, f_m(z))$ is not m , selecting a base of transcendence of this field does not insure that the values at α of the elements of this base form a transcendence basis of $\mathbf{Q}(f_1(\alpha), \ldots, f_m(\alpha))$ over $\mathbf{Q}$. Let $g_1(z), \ldots, g_t(z)$ be a transcendence basis of $\mathbf{Q}(z, f_1(z), \ldots, f_m(z))$, the caveat comes from the fact that all the coefficients of an algebraic equation of a series over $\mathbf{Q}(g_1(z), \ldots, g_t(z))$ may well vanish at some special point α , but this can only occurs with the values $g_1(\alpha), \ldots, g_t(\alpha)$ not being algebraically independent over $\mathbf{Q}$.

Example 8. *As in example 7 let* $f(z) = \sum_{k \in \mathbf{N}} z^{a^k}$ *and consider the functions* $f_1(z) = \left(z - \frac{1}{2}\right) f(z)$ *and* $f_2(z) = zf(z)$, *which satisfies the system of functional equations*

$$\begin{aligned} f_1(z^a) &= f_1(z) + (z^{a-1} - 1) f_2(z) - z \left(z^a - \frac{1}{2} \right) \\ f_2(z^a) &= z^{a-1} f_2(z) - z^{a+1} \quad . \end{aligned}$$

Each of the functions $f_1(z)$ *and* $f_2(z)$ *is transcendental and both functions are related by the linear equation* $zf_1(z) = \left(z - \frac{1}{2}\right) f_2(z)$. *But, the value at* $z = \frac{1}{2}$ *of the function* $f_1(z)$ *is zero while the value at* $z = 0$ *of* $f_2(z)$ *is zero.*

However, such a collapse can only occurs at finitely many points and we can state, *see* [**9**] for an example of result of this type.

Corollary 15. *Let* K *be a number field and* $f_1(z), \ldots, f_\ell(z) \in K[[z]]$ *be series algebraically independent over* $K(z)$. *We assume that there exists series* $f_{\ell+1}(z), \ldots, f_m(z) \in K[[z]]$ *such that for some integer* $d > 1$ *the* $f_i(z)$ *'s converge in the open unit disc and satisfy the system of linear functional equations*

$$f_i(z^d) = \sum_{j=1}^{m} a_{i,j}(z) f_j(z) + b_i(z) \ , \quad 1 \leq i \leq m$$

with $a_{i,j}(z)$ *and* $b_i(z)$ *rational fractions with algebraic coefficients.*

Let $0 < \rho < 1$ *be a real number, then for all but finitely many algebraic numbers* α *of absolute value* $< \rho$, *the numbers* $f_1(\alpha), \ldots, f_\ell(\alpha)$ *are algebraically independent over* $\mathbf{Q}$.

Proof – There are only finitely many numbers α of absolute value $< \rho$ such that α^{d^k} is a pole of some $a_{i,j}(z)$ or $b_i(z)$ for some $k \in \mathbf{N}$, because the modules of the d^k-th roots of the pole of these functions, tends to 1 as k tends to infinity. Suppose α distinct from these finitely many numbers, theorem 14 implies

$$\text{tr.deg}_{\mathbf{Q}}\mathbf{Q}(f_1(\alpha), \ldots, f_m(\alpha)) = \text{tr.deg}_{\mathbf{Q}(z)}\mathbf{Q}(z, f_1(z), \ldots, f_m(z)). \tag{13}$$

Up to a reordering of $f_{\ell+1}(z), \ldots, f_m(z)$ we may assume that a transcendence basis of $\mathbf{Q}(z, f_1(z), \ldots, f_m(z))$ over $\mathbf{Q}(z)$ is given by the series $f_1(z), \ldots, f_{\ell_1}(z)$ for some $\ell \leq \ell_1 \leq m$. For each $\ell_1 < m_1 \leq m$ write a relation of algebraic dependence of $f_{m_1}(z)$ over $\mathbf{Q}(z, f_1(z), \ldots, f_{\ell_1}(z))$:

$$\sum_{i=0}^{\delta_{m_1}} A_{m_1,i}(z, f_1(z), \ldots, f_{\ell_1}(z)) f_{m_1}(z)^i = 0$$

where $A_{m_1,i} \in \mathbf{Q}[z, X_1, \ldots, X_{\ell_1}]$, $i = 0, \ldots, \delta_{m_1}$, and $A_{m_1,\delta_{m_1}} \neq 0$. Since $f_1(z), \ldots, f_{\ell_1}(z)$ are algebraically independent over $\mathbf{Q}(z)$ the function

$$\prod_{m_1=\ell_1+1}^{m} A_{m_1,\delta_{m_1}}(z, f_1(z), \ldots, f_{\ell_1}(z))$$

is non zero and analytic inside the unit disk. Therefore it has only finitely many zeros in the disk of radius ρ centred at the origin. Suppose now that α is also distinct from these finitely many zeros, the algebraic relation above specialises to a non trivial relation of algebraic dependence of $f_{m_1}(\alpha)$ over $\overline{\mathbf{Q}}(f_1(\alpha), \ldots, f_{\ell_1}(\alpha))$:

$$\sum_{i=0}^{\delta_{m_1}} A_{m_1,i}(\alpha, f_1(\alpha), \ldots, f_{\ell_1}(\alpha)) f_{m_1}(\alpha)^i = 0$$

with $A_{m_1,\delta_{m_1}}(\alpha, f_1(\alpha), \ldots, f_{\ell_1}(\alpha)) \neq 0$, for $m_1 = \ell_1 + 1, \ldots, m$. Now, if the values $f_1(\alpha), \ldots, f_{\ell_1}(\alpha)$ were not algebraically independent over $\mathbf{Q}$ the transcendence degree of the field $\mathbf{Q}(f_1(\alpha), \ldots, f_m(\alpha))$ would be strictly less than ℓ_1, which is the transcendence degree of $\mathbf{Q}(z, f_1(z), \ldots, f_m(z))$ over $\mathbf{Q}(z)$ and this would contradict (13). Thus the numbers $f_1(\alpha), \ldots, f_\ell(\alpha)$ are part of a transcendence basis and are therefore algebraically independent, as to be proved. □

THEOREM ([**49**, Thm.4]) 16. *Let $f_1(z), \ldots, f_m(z)$ be functions admitting Taylor expansions at the origin, with coefficients in a given number field and which converges in the unit disk. We assume that for some integer $d > 1$ the $f_i(z)$'s satisfy the system of linear functional equations*

$$f_i(z) = \sum_{j=1}^{m} a_{i,j}(z) f_j(z^d) + b_i(z), \quad 1 \leq i \leq m \tag{14}$$

with $a_{i,j}(z)$ and $b_i(z)$ polynomials with algebraic coefficients.

If α *is a non zero complex number of absolute value* < 1 *such that none of the numbers* α^{d^k} *,* $k \in \mathbf{N}$ *, is a zero of* $\operatorname{Det}(a_{i,j}(z))_{1\le i,j\le m}$ *, then the transcendence degree over* $\mathbf{Q}$ *of the field generated by* $\alpha, f_1(\alpha), \dots, f_m(\alpha)$ *is at least the transcendence degree over* $\mathbf{Q}(z)$ *of the field generated by the functions* $f_1(z), \dots, f_m(z)$ *.*

Example 7 (continued). *With the function* $f(z)$ *of example 7 for any* $\alpha \in \mathbf{C}$ *, non zero, of modulus* < 1 *, at least* a *of the* $a+1$ *numbers* $\alpha, f(\alpha), \dots, f(\alpha^{a-1})$ *are algebraically independent over* $\mathbf{Q}$ *.*

Of course, one cannot expect more in theorem 16 since one can always take α an algebraic number or a number such that some $f_i(\alpha)$ is algebraic, for example. However, for transcendental numbers α which are very well approximated by algebraic numbers one has the following result. Recall that a *Liouville number* is a real number α such that for any real $\kappa > 0$ there exists a reduced fraction $\frac{p}{q}$ satisfying $0 < \left|\alpha - \frac{p}{q}\right| < \exp\left(-\kappa \log(|q|+1)\right)$, such numbers are transcendental over $\mathbf{Q}$.

Theorem 17 (T.Töpfer [**54**], [**41**, Thm.4.5.4]) 17. *In the setting of the previous theorem 16, assume the number* α *is a non zero real number of absolute value* < 1 *such that for some reals* $\tau > 2m+1$ *and* $c > 0$ *there exists an infinite sequence of distinct reduced fractions* $\left(\frac{p_k}{q_k}\right)_{k\in\mathbf{N}}$ *satisfying*

$$0 < \left|\alpha - \frac{p_k}{q_k}\right| < \exp\left(-c(\log|q_k|)^\tau\right).$$

Then the numbers $f_1(\alpha), \dots, f_m(\alpha)$ *are algebraically independent over* $\mathbf{Q}$ *.*

Since α is transcendental (being obviously a Liouville number) one would like to add it to the list in the conclusion of theorem 17, thus obtaining the algebraic independence of the $m+1$ numbers $\alpha, f_1(\alpha), \dots, f_m(\alpha)$.

2. Functions

As for transcendence results, the first issue in the context of algebraic independence of values of functions is the algebraic independence of the functions themselves. We have seen in corollary 15 that the occurrence of algebraic relations among a complete set of functions satisfying a system of functional equations of type (12) may impair the transcendence of the value of one of its transcendental members at some algebraic points.

However, if functions $f_1(z), \dots, f_m(z)$ satisfy a system of functional equations of type (12) with the $a_{i,j}$ constant, their algebraic independence over the field of rational fractions is equivalent to their linear independence over the field of constants modulo the rational fractions.

Theorem 18 (K.Nishioka [**41**, Thm.3.2.2]) 18. *In the setting of theorem 13 assume that the* $f_i(z)$ *'s satisfy the system of linear functional equations*

$$f_i(z) = \sum_{j=1}^{m} a_{i,j} f_j(z^d) + b_i(z) \ , \quad 1 \le i \le m \tag{15}$$

with $a_{i,j}$ *complex numbers and* $b_i(z)$ *rational fractions with complex coefficients.*

The functions $f_1(z), \dots, f_m(z)$ *are algebraically independent over* $\mathbf{C}(z)$ *if and only if no non trivial linear combination of them with coefficients in* $\mathbf{C}$ *is a rational function in* $\mathbf{C}(z)$ *(*i.e. *for all* $c_1, \dots, c_m \in \mathbf{C}$ *, not all zero, one has* $c_1 f_1(z) + \cdots + c_m f_m(z) \notin \mathbf{C}(z)$ *).*

Note that when the determinant of the matrix $(a_{i,j})_{1\le i,j\le m}$ is non zero, the system of functional equations (15) is equivalent to the system (12). However, when this determinant is zero the system (15) would only deliver the existence of numbers $c_1, \dots, c_m \in \mathbf{C}$ such that $c_1 f_1(z^d) + \cdots + c_m f_m(z^d) \in \mathbf{C}(z)$ whereas system (12) gives $c_1 f_1(z) + \cdots + c_m f_m(z) \in \mathbf{C}(z)$.

The algebraic independence of solutions of another type of functional equations is given in [**25**]. In the following theorem we let H denote the multiplicative subgroup of non zero rational fractions of the form $r(z)/r(z^d)$, $r(z) \in \mathbf{C}(z) \setminus \{0\}$.

Theorem 19 (K.K.Kubota [**25**]) 19. *In the setting of theorem 13 assume that the* $f_i(z)$ *'s satisfy the system of linear functional equations*

$$f_i(z^d) = a_i(z) f_i(z) + b_i(z) \ , \quad 1 \le i \le m$$

with $a_i(z)$ *and* $b_i(z)$ *rational fractions with complex coefficients satisfying* $a_1(z) \dots a_m(z) \ne 0$ *. For* $1 \le i, j \le m$ *such that* $a_j \in a_i H$ *we fix a rational fraction* $r_{i,j}(z)$ *such that* $a_j(z) = a_i(z) r_{i,j}(z)/r_{i,j}(z^d)$ *and otherwise we set* $r_{i,j}(z) = 0$ *.*

Consider a maximal subset I *of indices* $i \in \{1, \dots, m\}$ *such that the* $a_i(z)$ *'s,* $i \in I$ *, are distinct modulo* H *. For* $i \in I$ *we set* V_i *the* $\mathbf{C}$*-vector space* $V_i = \sum_{j=1}^{m} \mathbf{C} r_{i,j}(z) f_j(z)$ *and* W_i *the* $\mathbf{C}$*-vector space of solutions of the functional equation* $g(z^d) = a_i(z) g(z)$ *. We also introduce the multiplicative subgroup* G *generated by the* $a_i(z)$ *'s for* $i \in I$ *such that* $W_i \cap (V_i + \mathbf{C}(z)) \ne 0$ *.*

Then the transcendence degree of the field generated by $f_1(z), \dots, f_m(z)$ *over* $\mathbf{C}(z)$ *is equal to*

$$\sum_{i \in I} \dim_{\mathbf{C}} \left(V_i / V_i \cap (W_i + \mathbf{C}(z)) \right) + \mathrm{rank}_{\mathbf{Z}} \left(GH/H \right).$$

Furthermore, the ideal of algebraic dependence relations is generated by relations of the following form :

- $\sum_{j=1}^{m} c_j r_{i,j}(z) f_j(z) \in \mathbf{C}(z)$ *with* $c_1, \dots, c_m \in \mathbf{C}$ *;*

- $\prod_{i\in I}\left(\sum_{j=1}^{m} c_{i,j} r_{i,j}(z) f_j(z) - r_i(z)\right)^{m_i} \in \mathbf{C}(z)^{\times}$ *with* $c_{i,j} \in \mathbf{C}$, $j = 1, \ldots, m$, $r_i(z) \in \mathbf{C}(z)$ *and* $m_i \in \mathbf{Z}$, *for* $i \in I$.

This theorem is in fact valid for functions of several variables satisfying a system of linear equations of Mahler type for some monomial transformation of the variables, *see* [**41**, Thm.3.5]. The theorem is also valid replacing the transformation $z \mapsto z^d$ of the variable by a rational transformation $z \mapsto r(z)$ with $r(z) \in \mathbf{C}(z)$, as long as the constants are the only functions, meromorphic in a neighbourhood of the origin, satisfying $g \circ r = g$.

Proof – For each $i \in I$ one selects indices j such that $r_{i,j}(z) \neq 0$ and the corresponding functions $r_{i,j}(z) f_j(z)$ induce a basis of $V_i/V_i \cap (W_i + \mathbf{C}(z))$. Letting δ_i denote the dimension of this vector space, this gives sets of functions $g_{i,h}(z) = r_{i,j_h}(z) f_{j_h}(z)$, $h = 1, \ldots, \delta_i$, $i \in I$, satisfying functional equations

$$g_{i,h}(z^d) = a_i(z) g_{i,h}(z) + b_{i,h}(z).$$

Furthermore, for $i \in I$ and any reals $c_1, \ldots, c_{\delta_i}$ there is no rational solution to the functional equation $g(z^d) = a_i(z) g(z) + \sum_{h=1}^{\delta_i} c_h b_{i,h}(z)$, since such a solution $r(z) \in \mathbf{C}(z)$ would lead to a non trivial linear relation between the images of $g_{i,1}(z), \ldots, g_{i,\delta_i}(z)$ in $V_i/V_i \cap (W_i + \mathbf{C}(z))$, namely $\sum_{h=1}^{\delta_i} c_h g_{i,h}(z) - r(z) \in W_i$ with $r(z) \in \mathbf{C}(z)$.

Then, one chooses a maximal subset of $J \subset I$ so that $0 \neq W_i \subset V_i + \mathbf{C}(z)$ and the corresponding generators $a_i(z)$ of G are free modulo H. This gives functions satisfying $g_i(z^d) = a_i(z) g_i(z)$, $i \in J$.

Using the dependence relations described in the statement of the theorem, one shows that the functions $f_1(z), \ldots, f_m(z)$ belong to the algebraic closure of the field generated over $\mathbf{C}(z)$ by the functions $g_{i,j}(z)$ and $g_i(z)$. However, these latter functions satisfy the hypothesis of [**41**, Thm.3.5], which implies that they are algebraically independent over $\mathbf{C}(z)$. The transcendence degree of the field generated by $f_1(z), \ldots, f_m(z)$ over $\mathbf{C}(z)$ is therefore equal to $\sum_{i\in I} \delta_i + \mathrm{card}(J)$, which is just the statement of the theorem. □

QUESTION 20. *Can one extend theorem 19 to more general system of functional equations of type*

$$f_i(z^d) = \sum_{j=1}^{m} a_{i,j}(z) f_j(z) + b_i(z) \ , \quad 1 \leq i \leq m?$$

One can try to answer this question using *Galois theory for* σ*-fields*, that is fields endowed with an endomorphism. In our context one would consider the field of meromorphic functions, for example, together with the *Mahler endomorphism* induced by the change of variable $z \mapsto z^d$. Such an approach is studied by P.Nguyen in [**36**] who recovers in this way the main features of theorem 19. It is worth noting that the same question arises in the theory of E-functions where it is handled with the help of *differemtial Galois theory*.

3. Rational transformation of the variable

Just as in transcendence theory, one may be interested in transformation of the variable z that are more general than the power transformation $z \mapsto z^d$ of the previous sections. In theorem 2, P-G.Becker treats algebraic transformations of the variable. Up to now, Mahler's method for algebraic independence (of values of functions of one variable) allows only to deal with rational transformations. Let $r(z) \in \mathbf{Q}(z)$, we assume that the origin is a *super-attracting* (or *critical*) fixed point of r, that is $\mathrm{ord}_0(r) \geq 2$. Denote $U_r(0)$ the *basin of attraction* of 0 for r, that is the set of points $z \in \mathbf{C}$ such that $r^{\circ k}(z)$ tends to 0 as k tends to ∞ and $r^{\circ k}(z) \neq 0$ for all $k \in \mathbf{N}$. The *degree of* r is the maximum of the degrees of its numerator and denominators in reduced form.

Theorem 21 (E.Zorin) 21. *Let $f_1(z), \ldots, f_m(z)$ be functions admitting Taylor expansion at the origin, with coefficients in a given number field and which converges in a neighbourhood of the origin. We assume these functions algebraically independent over $\mathbf{C}(z)$ and satisfy a system of functional equations*

$$f_i(z) = \sum_{j=1}^{m} a_{i,j}(z) f_j(r(z)) + b_i(z) \tag{16}$$

with $a_{i,j}(z)$ and $b_i(z)$ rational functions with algebraic coefficients. Let $\alpha \in U_r(0) \cap \overline{\mathbf{Q}}$ such that for any $k \in \mathbf{N}$ none of the numbers $r^{\circ k}(\alpha)$ is not a pole of any $a_{i,j}(z)$ or $b_i(z)$, nor a zero of $\mathrm{Det}(a_{i,j}(z))$.

Then the transcendence degree over $\mathbf{Q}$ of the field generated by the numbers $f_1(\alpha), \ldots, f_m(\alpha)$ is at least

$$2m + 1 - (m+1)\frac{\log(\deg(r))}{\log(\mathrm{ord}_0(r))}.$$

This statement improves upon previous results by T.Töpfer [**55**], but it becomes weaker as the degree of r is larger than its order at 0. The optimal result (m numbers algebraically independent over $\mathbf{Q}$) is obtained when $r(z)$ has the shape $r(z) = \frac{z^d}{p(z)}$ where $d \geq 2$ is an integer and $p(z)$ is a polynomial of degree $\leq d$ satisfying $p(0) \neq 0$.

Example 9. *Consider non constant polynomials $q_1(z), \ldots, q_m(z)$ with algebraic coefficients.*

1. *Assuming $q_i(0) = 0$, the series $f_i(z) = \sum_{k \in \mathbf{N}} q_i \circ r^{\circ k}(z)$ define functions satisfying $f_i \circ r(z) = f_i(z) - q_i(z)$, $i = 1, \ldots, m$.*
2. *Assuming $|q_i(0)| > 1$, the series $f_i(z) = \sum_{k \in \mathbf{N}} \prod_{h=0}^{k} (q_i \circ r^{\circ h}(z))^{-1}$ define functions satisfying $f_i \circ r(z) = q_i(z) f_i(z) - 1$, $i = 1, \ldots, m$.*

In both cases, one has to find conditions on the $q_i(z)$ *'s so that the set of functions is algebraically independent over* $\mathbf{C}(z)$ *. Compare these functions with those of example 2.*

With the help of theorem 18 and 19 (extended to rational transformations), one can give in each case a criterion for the algebraic independence of the function $f_1(z), \dots, f_m(z)$ *over* $\mathbf{C}(z)$ *.*

1. *If* $f_1(z), \dots, f_m(z)$ *are* $\mathbf{C}$*-linearly independent modulo* $\mathbf{C}(z)$ *then they are algebraically independent over* $\mathbf{C}(z)$ *. This occurs when* $q_1(z), \dots, q_m(z)$ *are* $\mathbf{C}$*-linearly independent modulo the additive subgroup of rational fractions of the form* $t(z) - t \circ r(z)$ *,* $t(z) \in \mathbf{C}(z)$ *.*
2. *If* $q_1(z), \dots, q_m(z)$ *are pairwise distinct modulo the multiplicative subgroup of rational fractions of the form* $t(z)/t \circ r(z)$ *,* $t(z) \in \mathbf{C}(z) \setminus \{0\}$ *and there is no rational fraction solution to the functional equations* $g \circ r(z) = q_i(z)g(z) - 1$ *for* $i = 1, \dots, m$ *, then the functions* $f_1(z), \dots, f_m(z)$ *are algebraically independent over* $\mathbf{C}(z)$ *.*

Assume now that $r(z)$ is a polynomial with algebraic coefficients, we have a sharper lower bound, again improving on previous results established by T.Töpfer [**55**].

Theorem 22 (E.Zorin) 22. *The setting is the same as in theorem 21, but now* $r(z)$ *is a polynomial with algebraic coefficients. Let* $\alpha \in U_r(0)$ *such that for any* $k \in \mathbf{N}$ *none of the numbers* $r^{\circ k}(\alpha)$ *is not a pole of any* $a_{i,j}(z)$ *or* $b_i(z)$ *, nor a zero of* $\mathrm{Det}(a_{i,j}(z))$ *.*

Then the transcendence degree over $\mathbf{Q}$ *of the field generated by the numbers* $\alpha, f_1(\alpha), \dots, f_m(\alpha)$ *is at least*

$$m + 1 - \left[2 \frac{\log(\deg(r))}{\log(\mathrm{ord}_0(r))} \right].$$

If α *is algebraic then the transcendence degree over* $\mathbf{Q}$ *of the field generated by the numbers* $f_1(\alpha), \dots, f_m(\alpha)$ *is at least*

$$m + 1 - \left[\frac{\log(\deg(r))}{\log(\mathrm{ord}_0(r))} \right].$$

We observe that in the two above theorems we assume, in contrast with the results of section 1, that there is no relation of algebraic dependence among the whole set of functions forming a solution of the functional equation.

In case $r(z) = z^d$ theorem 22 gives the algebraic independence of the m numbers $f_1(\alpha), \dots, f_m(\alpha)$ when α is algebraic. However, when α is not algebraic it only gives the lower bound $m - 1$, rather than the expected m, for the transcendence degree of the field generated by the numbers $\alpha, f_1(\alpha), \dots, f_m(\alpha)$.

For the proof of the two theorems of this section, when $\mathrm{ord}_0(r) < \deg(r)$ one needs a multiplicity estimate. In constrast, when $\mathrm{ord}_0(r) = \deg(r)$ such a multiplicity estimate is only needed for establishing measures of algebraic independence.

Also, the arithmetic growth of the Taylor coefficients of the functions involved is crucial. Here are general estimates for solutions of functional equations of type (16), taken from [**55**, lemma 7 & 12].

LEMMA 23. *The setting being the same as in theorem 21, where we assume* $\mathrm{Det}(a_{i,j}(z)) \neq 0$ *, write* $f_i(z) = \sum_{k\in\mathbf{N}} f_{i,k} z^k$ *for* $i = 1, \ldots, m$ *. Then there exists a number field* K *, a real* $C \geq 1$ *and a positive integer* D *such that* $|\sigma(f_{i,k})| \leq C^{k+1}$ *and* $D^{k+1} f_{i,k}$ *is an algebraic integer for* $i = 1, \ldots, m$ *and* σ *any embedding of* K *in* $\mathbf{C}$ *.*

LEMMA 24. *The setting being the same as in theorem 22, where we assume* $r(z) \in \overline{\mathbf{Q}}[z]$ *and* $\mathrm{Det}(a_{i,j}(z)) \neq 0$ *, write* $f_i(z) = \sum_{k\in\mathbf{N}} f_{i,k} z^k$ *for* $i = 1, \ldots, m$ *. Then there exists a number field* K *, a real* $C \geq 1$ *and a positive integer* D *such that* $|\sigma(f_{i,k})| \leq C^{\log(k+2)}$ *and* $D^{[\log(k+2)]} f_{i,k}$ *is an algebraic integer for* $i = 1, \ldots, m$ *and* σ *any embedding of* K *in* $\mathbf{C}$ *.*

EXAMPLE 10. *1) The function* $f(z) = \sum_{k\in\mathbf{N}} \prod_{j=0}^{k} \frac{1}{2(1-3z^{d^j})}$ *satisfies the functional equation* $f(z) = \frac{1}{2(1-3z)}\left(f(z^d)+1\right)$ *. It has Taylor expansion around the origin* $f(z) = \sum_{\ell\in\mathbf{N}} f_\ell z^\ell$ *with*

$$f_\ell = \sum_{k\in\mathbf{N}} 2^{-k} \sum_{\substack{h_0,\ldots,h_k\in\mathbf{N}\\ h_0+\cdots+h_k d^k=\ell}} 3^{h_0+\cdots+h_k} .$$

Considering the term for $k = 0$ *in the expression of* f_ℓ *one checks* $|f_\ell| \geq 3^\ell$ *and, taking into account the number of possible* $(h_0, \ldots, h_k)$ *for each* $k \in \mathbf{N}$ *, one has* $|f_\ell| \leq 3^{\ell(1+o_\ell(1))}$ *.*

2) The function $g(z) = \sum_{k\in\mathbf{N}} \prod_{j=0}^{k} \frac{1}{2}\left(1+3z^{d^j}\right)$ *satisfies the functional equation* $g(z) = \frac{1}{2}(1+3z)\left(g(z^d)+1\right)$ *. It has Taylor expansion around the origin* $g(z) = \sum_{\ell\in\mathbf{N}} g_\ell z^\ell$ *with*

$$g_\ell = \sum_{k\in\mathbf{N}} 2^{-k} \sum_{\substack{h_0,\ldots,h_k\in\{0,1\}\\ h_0+\cdots+h_k d^k=\ell}} 3^{h_0+\cdots+h_k} .$$

One then checks that g_ℓ *is zero except when the expansion of* ℓ *in base* d *shows only the digits* 0 *and* 1 *. And then* $g_\ell = 2^{2-\nu(\ell,d)} 3^{\sigma(\ell,d)}$ *, where* $\nu(\ell,d)$ *(*resp. $\sigma(\ell,d)$ *) designates the number of digits (*resp. *the sum of digits) in the expansion of* ℓ *in base* d *. Therefore one has* $|g_\ell| \leq 4\left(\frac{3}{2}\right)^{\nu(\ell,d)}$ *with equality if* $\ell = 1 + \cdots + d^k$ *for some* k *and in this latter case* $\nu(\ell,d) = \frac{\log(\ell(d-1)+1)}{\log(d)} + 1$ *and* $|g_\ell| = \ell^{\frac{\log(3/2)}{\log(d)} + o_\ell(1)}$ *.*