CHAPTER 1

Introduction to Mahler theory

The purpose of transcendence theory is to prove that numbers of all kind are transcendental. Numbers of interest may be given by means of series, infinite products, integrals or continued fractions, for example. It is fair to say that such numbers can likely be also produced (possibly in many ways) as values of transcendental functions which are analytic in some domain. However, whereas the numbers are going to be proved transcendental over some base field (*e.g.* $\mathbf{Q}$), the functions have to be transcendental over the field of rational fractions with coefficients in this base field and the argument should be algebraic over this same base field.

Proving that transcendental functions takes transcendental values may look like working hard to check what everybody expects and it may seem more exciting to find unexpected algebraic relations between remarkable numbers. On an other hand, too much exotic relations can also bring bewilderment and then one can reformulate the task of transcendence and algebraic independence theory as showing that algebraic relations between numbers comes from algebraic relations between functions.

Concerning the values of the exponential function, a beautiful *conjecture due to S.Schanuel* is, according to S.Lang, supposed to contain all the reasonable statements one could wish to formulate on their algebraic relations. It states that, given a collection of complex numbers linearly independent over the field of rational numbers, then the transcendence degree of the field generated by these numbers and their images by the exponential function should be at least equal to the numbers of members of the given collection. As a consequence, it entails a positive answer to the celebrated *four exponential conjecture* asserting that given to couples of non zero complex numbers, say (x_1, x_2) and (y_1, y_2), the ratio of which are irrational, *i.e.* $x_1/x_2 \notin \mathbf{Q}$ and $y_1/y_2 \notin \mathbf{Q}$, then at least one of the four exponentials $\mathrm{e}^{x_1y_1}, \mathrm{e}^{x_2y_1}, \mathrm{e}^{x_1y_2}, \mathrm{e}^{x_2y_2}$, is transcendental over $\mathbf{Q}$. Although this conjecture is still open, we can mention the famous *six exponential theorem*, due to K.Ramachandra and S.Lang : given (x_1, x_2, x_3) and (y_1, y_2) collections of complex numbers linearly independent over $\mathbf{Q}$, at least one of the six exponentials $\mathrm{e}^{x_1y_1}, \mathrm{e}^{x_2y_1}, \mathrm{e}^{x_3y_1}, \mathrm{e}^{x_1y_2}, \mathrm{e}^{x_2y_2}, \mathrm{e}^{x_3y_2}$, is transcendental over $\mathbf{Q}$.

In this chapter we present some transcendence results that have been obtained through the so-called Mahler method, which applies around an attractive fixed point of a dynamical transformation leaving unchanged a function field.

1. Transcendence

1.1. Numbers. The method that K.Mahler devised in his seminal papers [**28, 29, 30**] for proving the transcendence and algebraic independence of values of functions analytic in the neighbourhood of some point, rests on the property that one could produce other points where these functions take values which can be expressed algebraically in terms of the value at the initial point.

In [**28**] this property took the form of the functional equation :

$$f(z^d) = R(z, f(z)) \tag{1}$$

where $d \geq 2$ is an integer and R is a rational function in z and $f(z)$, with coefficients in some number field. Mahler then considers the solutions of (1) admitting a Taylor expansion at the origin the coefficients of which belong to a given number field, he proves :

Theorem 1 (K.Mahler [**28**]) 1. *In the above setting assume f is transcendental and the degrees in $f(z)$ of the numerator and denominator of $R(z, f(z))$ are $< d$. Consider α a non zero algebraic number of absolute value < 1 such that none of the numbers α^{d^k}, $k \in \mathbf{N}$, is a zero of the denominator of $R(z, f(z))$, $f(\alpha)$ is defined and (1) holds at $z = \alpha^{d^k}$, $k \in \mathbf{N}$, then $f(\alpha)$ is transcendental.*

Remark 1. *The assumption on the successive d-th powers of α not being zero of the denominator of R is necessary. P-G.Becker* [**9**, Remark 1] *(*see *also* [**42**]*) mention the case of the function*

$$f(z) = \prod_{i \in \mathbf{N}} (1 - 2z^{d^i}) = 1 + \sum_{\substack{k \in \mathbf{N} \\ (\varepsilon_0, \ldots, \varepsilon_{k-1}) \in \{0,1\}^k}} (-2)^{1+\varepsilon_{k-1}+\cdots+\varepsilon_0} . z^{d^k + \varepsilon_{k-1} d^{k-1} + \cdots + \varepsilon_0}$$

satisfies the functional equation

$$f(z^d) = \frac{f(z)}{1 - 2z}.$$

It vanishes at every α satisfying $\alpha^{d^i} = 1/2$ for some $i \in \mathbf{N}$ and it is transcendental, since it has infinitely many zeros.

K.Nishioka [**40**, Thm.1.5.1] has relaxed the condition on the degrees in $f(z)$ of the numerator and denominators of R, requiring only that they are $< d^2$ for the same conclusion, *see* also Theorem 2 below. Actually, K.Nishioka deals with more general functional equations of the type :

$$E(f(z^d), f(z), z) = 0 \ , \quad E \in \overline{\mathbf{Q}}[X, Y, Z] \setminus \overline{\mathbf{Q}}[Y, Z] \ . \tag{2}$$

P-G.Becker has further generalised the method replacing the power transformation $z \mapsto z^d$ by a rational one [**7**] (*see* also [**36**] for polynomial transformations), then by an algebraic one [**8**] $z \mapsto t(z)$ where t is non constant, algebraic over $\mathbf{Q}(z)$ and meromorphic in a neighbourhood U of one of its attractive fixed points $\omega \in \mathbf{C} \cup \{\infty\}$. In the next statement, we will restrict U to be contained in the basin of attraction of ω and we denote $T \in \overline{\mathbf{Q}}[X,Y]$ a minimal equation of t over $\mathbf{Q}(z)$: $T(t(z),z) \equiv 0$.

THEOREM 2 (P-G.BECKER [**8**]) 2. *In the above setting assume f is holomorphic in U with algebraic Taylor coefficients at ω, satisfies the functional equation*

$$E(f(t(z)), f(z), z) \equiv 0 \ , \quad E \in \overline{\mathbf{Q}}[X,Y,Z] \tag{3}$$

and is transcendental over $\mathbf{C}(z)$. Further assume

$$\deg_Y(T) \max(\deg_Y(T); \deg_Y(E))(\deg_X(T)\deg_X(E))^2 < \operatorname{ord}_\omega(t)^3 \ , \tag{4}$$

then for α an algebraic number in U such that $E(X, f(t^{\circ k}(\alpha)), t^{\circ k}(\alpha)) \not\equiv 0$ and $t^{\circ k}(\alpha) \neq \omega$ for $k \in \mathbf{N}$, the number $f(\alpha)$ is transcendental.

Condition (4) is technical, it is a problem to find an optimal condition that could replace it. In case $t(z) = z^d$ and equation (3) is linear in X, then f satisfies an equation of type (1) and condition (4) reduces to $\max(d; \deg_f(R)) < d^2$.

The simplest form of functional equation is for $R(z, f(z)) = a(z)f(z) + b(z)$ with $a, b \in \mathbf{Q}(z)$. The most classical example then comes for $a(z) = 1$, $b(z) = -z$ and $t(z) = z^d$, that gives $f(z) = \sum_{k\in\mathbf{N}} z^{d^k}$.

EXAMPLE 2. *Let $m \in \mathbf{N}^*$, $q, r, s \in \overline{\mathbf{Q}}(z) \setminus \overline{\mathbf{Q}}$, the formal sum and product*

$$f(z) := \sum_{k\in\mathbf{N}} q \circ r^{\circ k}(z) \prod_{j=0}^{k-1} s \circ r^{\circ j}(z) \ , \quad g(z) := \prod_{k\in\mathbf{N}} \left(s \circ r^{\circ k}(z)\right)^{m^k} \ ,$$

where $r^{\circ k}$ denote $r \circ \cdots \circ r$ iterated k times, define holomorphic functions $f(z)$ (resp. $g(z)$) in the neighbourhoods of any point ω in $s^{-1}(0) \subset \mathbf{C} \cup \{\infty\}$ (resp. $s^{-1}(1) \subset \mathbf{C} \cup \{\infty\}$) which is an attracting fixed point for r. *These functions satisfy the functional equations*

$$f(z) = s(z)f(r(z)) + q(z) \ , \quad g(z) = s(z)g(r(z))^m \ .$$

We assume $0 < \deg(r) < \operatorname{ord}_\omega(r)^{3/2}$ in the first case (functions f) and $0 < m\deg(r) < \operatorname{ord}_\omega(r)^{3/2}$ in the second case (functions g).

We now apply theorem 2. Let α be an algebraic number satisfying $\lim_{k\to\infty} r^{\circ k}(\alpha) = \omega$ but $r^{\circ k}(\alpha) \neq \omega$, $k \in \mathbf{N}$. Suppose the function $f(z)$ is transcendental over $\mathbf{C}(z)$ and $s \circ r^{\circ k}(\alpha) \neq \infty$, $k \in \mathbf{N}$, then the number $f(\alpha)$ is trancendental. Suppose the function $g(z)$ is transcendental over $\mathbf{C}(z)$ and $s \circ r^{\circ k}(\alpha) \neq 0, \infty$, $k \in \mathbf{N}$, then the number $g(\alpha)$ is transcendental.

1.2. Functions. The most obvious property the considered function must satisfy in theorem 1 and 2 is that it has to be transcendental. However usually it is not the most obvious task to check this (*see* example above). But it turns out that in case of equations of type (1) solutions which are algebraic power series must be in fact rational functions (mind that the author of the next theorem is Keiji Nishioka although the reference points to Kumiko Nishioka's Lecture Notes ... a family affair!).

THEOREM 3 (K.NISHIOKA [**40**, thm.1.3]) 3. *Let* $d > 1$ *be an integer,* R *be a rational function in* $\mathbf{C}(X,Y)$ *and* $f \in \mathbf{C}[[z]]$ *satisfying either equation*

$$f(z^d) = R(z, f(z)) \quad or \quad f(z) = R(z, f(z^d)) \ .$$

Then $f \in \mathbf{C}(z)$ *or* f *is transcendental over* $\mathbf{C}(z)$ *.*

A series $f(z) = \sum_{k\geq 0} f_k z^k$, with coefficients in some subfield K of $\mathbf{C}$, is a rational function if and only if the sequence $(f_k)_{k\in\mathbf{N}} \in K^{\mathbf{N}}$ is *ultimately linear recursive*, that is satisfies for k larger than the degree of the numerator of f a recurrence relation

$$f_k = c_1 f_{k-1} + \cdots + c_n f_{k-n} \ , \quad f_k = P_1(k)\alpha_1^k + \cdots + P_s(k)\alpha_s^k \ ,$$

where $1, -c_1, \ldots, -c_n \in K$, $c_n \neq 0$ are the coefficients of the denominator of f (of degree n and normalised so that its constant coefficient is 1), $\alpha_1^{-1}, \ldots, \alpha_s^{-1}$ are the distinct roots of this denominator and $P_i \in K(\alpha_i)[z]$ has degree equal to the multiplicity of α_i in the denominator of f (*i.e.* as a pole of f) minus 1, for $i = 1, \ldots, s$. That is

$$\operatorname{den}(f) = 1 - c_1 z - \cdots - c_n z^n = \prod_{i=1}^{s} (1 - z\alpha_i)^{\deg(P_i)+1} \ .$$

We denote by I the set of indices $i \in \{1, \ldots, s\}$ such that $|\alpha_i|$ is maximal equal to $A = \max_{1\leq j\leq s}(|\alpha_j|)$ and D the maximum of $\deg(P_i)$ for $i \in I$. For k large enough we have $|f_k| \leq \gamma_2' k^D A^k$ for some real $\gamma_2' > 0$ independent of k.

Let's put $\delta = \min_{i,j\in I; i\neq j}(|\alpha_i - \alpha_j|) \cdot \min_{i\in I}(|\alpha_i|^{-1}) > 0$, thanks to a *theorem of Túran*, *cf.* [**40**, p.59], for any $k \in \mathbf{N}$ there exists $k+1 \leq \kappa \leq k+\nu$ such that

$$\left| \sum_{i\in I} p_{i,D} \alpha_i^\kappa \right| \geq \frac{1}{\nu} \left(\frac{\delta}{2}\right)^{\nu-1} \sum_{i\in I} |p_{i,D}| |\alpha_i|^\kappa = \frac{1}{\nu} \left(\frac{\delta}{2}\right)^{\nu-1} \sum_{i\in I} |p_{i,D}| A^\kappa$$

where $p_{i,D}$ is the coefficient (possibly 0) of z^D in P_i and $\nu \leq n$ is the cardinality of I. Since $\sum_{i\in I} |p_{i,D}| > 0$ by the definition of D, we deduce that there exists $\gamma_2 > \gamma_1 > 0$ satisfying for k large enough

$$\gamma_1 k^D A^k < \max(|f_{k+1}|, \ldots, |f_{k+\nu}|) < \gamma_2 k^D A^k \ . \tag{5}$$

LEMMA 4. *Let* $K \subset \mathbf{C}$ *be a subfield and* $f(z) = \sum_{k\in\mathbf{N}} f_k z^k \in K[[z]]$ *a series which is a rational function. Then there exists* $\nu \in \mathbf{N}^*$ *such that the* *sequence* $(\max(|f_{k+1}|,\dots,|f_{k+\nu}|))_{k\in\mathbf{N}}$ *is either bounded or grows at least linearly as* k *tends to infinity.*

If $f(z)\in\mathbf{Z}[[z]]$ *is a rational function the denominator of which is not a product of cyclotomic polynomials, then there exists* $\nu\in\mathbf{N}^*$ *such that the sequence* $(\max(|f_{k+1}|,\dots,|f_{k+\nu}|))_{k\in\mathbf{N}}$ *grows exponentially as* k *tends to infinity.*

Proof – The first part follows directly from (5). For the second, lemma 2.6.1 in [**40**] shows that the α_i 's compose complete sets of conjugates of algebraic integers. Therefore, if they are not all roots of unity one of them has absolute value $A>1$, hence the conclusion. □

REMARK 3. *Theorem 3 cannot be immediately extended around a fixed point* ω *of a rational substitution* $z\mapsto t(z)$ *in place of* $z\mapsto z^d$ *, with* $t\in\mathbf{C}(z)$ *and* $\mathrm{ord}_\omega(t)>1$ *.*

For example, the truly algebraic function $\varphi(z)=\frac{2z}{1+\sqrt{1-4z^2}}$ *satisfies the equation* $\varphi\left(\frac{z^2}{1-2z^2}\right)=\varphi(z)^2$ *and around the origin it has a Taylor expansion in* $\mathbf{Q}[[z]]$ *starting as* $z+z^3+O(z^5)$ *.*

EXAMPLE 4. *Coming back to example 2, theorem 3 implies that a function* f *or* g *, associated to* $r(z)=z^d$ *,* $d>1$ *, and* $s\in\overline{\mathbf{Q}}(z)\setminus\overline{\mathbf{Q}}$ *, is either rational or transcendental. In some special circumstances they can be rational, for* $m=1$ *,* $r(z)=z^d$ *and* $s(z)=1+z+\cdots+z^{d-1}$ *we have :*

$$g(z)=\frac{1}{1-z}\quad .$$

In fact, for $r(z)=z^d$ *,* $0<s_1<\cdots<s_\ell<d$ *and* $s(z)=1+z^{s_1}+\cdots+z^{s_\ell}$ *and* $m=1$ *the corresponding function* $g(z)$ *has the Taylor expansion* $\sum_{k\in\mathbf{N}}w_s(k)z^k$ *where* $w_s(k)$ *is* 1 *if the only digits appearing in the expansion of* k *in base* d *are* $0,s_1,\dots,s_\ell$ *and* 0 *otherwise.*

When $r(z)$ *is not a monomial can such a function as in example 2 be algebraic?*

On an other hand, given a functional equation of type (3) there exists a power series solution.

THEOREM 5 [**40**, Thm.1.7.2] 5. *Let* $d\in\mathbf{N}^*$ *,* $K\subset\mathbf{C}$ *be a subfield and* $E\in K[X,Y,Z]$ *such that* $E(0,0,0)=0$ *and* $E'_Y(0,0,0)\neq0$ *. Then there exists a power series* $f\in zK[[z]]$ *satisfying*

$$E(f(z^d),f(z),z)\equiv0$$

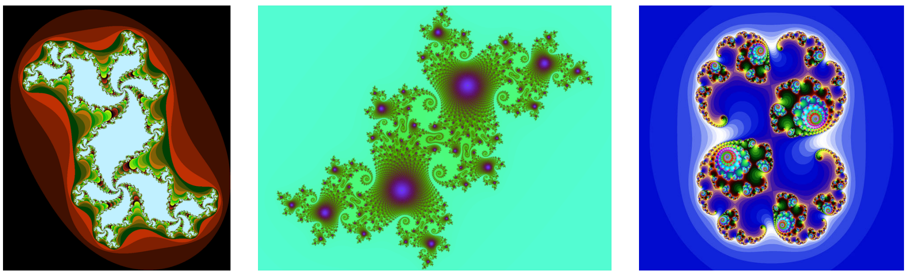
and converging on some disc of positive (effectively computable) radius, centred at the origin.

2. A dynamical point of view

2.1. Julia sets. One context where Mahler functions beautifully occurs has been pointed by P-G.Becker and W.Bergweiler [**10**], it is in connexion with the dynamic of polynomials.

Let's recall that the *Julia set* of a rational function r (in one complex variable) is the complement of the set of point where the family of iterates $(r^{\circ k})_{k\in\mathbf{N}^*}$ is *normal* (here $r^{\circ k}$ stands for $r\circ\cdots\circ r$ iterated k times). That is one can extract from any infinite sequence of iterates a sub-sequence which converges uniformly (to an analytic function or infinity) on some compact neighbourhood of the given point. Alternatively, the Julia set of r can be described as the closure of the set of *periodic points* z that are *repelling*, *i.e.* such that $r^{\circ k}(z)=z$ and $|(r^{\circ k})'(z)|=\prod_{\ell=0}^{k-1}|r'\circ r^{\circ\ell}(z)|>1$, for some $k\in\mathbf{N}^*$. It is invariant under direct and inverse image by r, in fact the set of pre-images by $r^{\circ k}$, $k\in\mathbf{N}$, of any point of the Julia set is dense in it. In almost all cases this set has a fractal dimension in $]0,2[$, but it may be either totally disconnected or arcwise connected with empty interior or equal to $\mathbf{C}$, *see* [**24**].

If r is a polynomial the Julia set is also the boundary of the set of points $z\in\mathbf{C}$ such that $r^{\circ k}(z)$ remains bounded in $\mathbf{C}$ as k tends to ∞. This latter set is called the *full Julia set* of the polynomial r.



c=0.3+0.5i c=−(0.1568···+0.6497...i) c=0.285+0.01i

FIGURE 1. Three Julia sets for $r(z)=z^2+c$ ...

EXAMPLE 2 (CONTINUED). *Let $s\in\overline{\mathbf{Q}}(z)\setminus\overline{\mathbf{Q}}$ and fix $\omega\in s^{-1}(0)$ (respectively $\omega\in s^{-1}(1)$). For $p\in\overline{\mathbf{Q}}[z]\setminus\overline{\mathbf{Q}}$ a non constant polynomial, set $r_{\omega,p}(z)=\frac{1}{p(1/z-\omega)}+\omega$ and denote by K_p the full Julia set of p.*

The condition on the algebraic number α in example 2 now reduces to $(\alpha-\omega)^{-1}\notin K_p$. Indeed, we have $r_{\omega,p}^{\circ k}(z)=\frac{1}{p^{\circ k}(1/z-\omega)}+\omega$ for all $k\in\mathbf{N}^*$ and $\lim_{k\to\infty}r_{\omega,p}^{\circ k}(\alpha)=\omega$ if and only if $\lim_{k\to\infty}p^{\circ k}(1/(\alpha-\omega))=\infty$, that is precisely $1/(\alpha-\omega)\notin K_p$. The condition $r_{\omega,p}^{\circ k}(\alpha)\neq\infty$, $k\in\mathbf{N}^*$, amounts simply to $\alpha\neq\omega$ in this case.

2.2. Quadratic polynomials and the Mandelbrot set. For the epitomical example $r_c(z) = z^2 + c$, $c \in \mathbf{C}$, B.Mandelbrot introduced his famous set M composed of the parameters c for which the full Julia set K_c of r_c is connected. It turns out that the complement of M in $\mathbf{C}$ can also be described as

$$M = \left\{ c \in \mathbf{C};\ r_c^{\circ k}(0) \text{ remains bounded as } k \text{ tends to } \infty \right\} .$$

In fact, P.Fatou and G.Julia proved that the full Julia set K_c is connected if and only if 0 belongs to K_c. And if not, then K_c is homeomorphic to the Cantor set. On an other hand, A.Douady and J.H.Hubbard have proved [**17**, Thm.5] that the Mandelbrot set itself is connected.

QUESTION 6. *Is the Mandelbrot set M locally connected?*

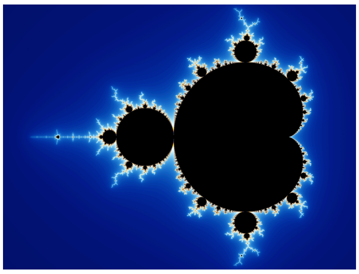


FIGURE 2. ... and the Mandelbrot set

For $c \in \mathbf{C}$ there exists a solution to the functional equation

$$\varphi_c(z^2 + c) = \varphi_c(r_c(z)) = r_0(\varphi_c(z)) = \varphi_c(z)^2 \tag{6}$$

that is defined and analytic near the point at infinity and leaves this point fixed. Since $\varphi_c(z)$ is analytic and non vanishing near the point at infinity, the function $\log|\varphi_c(z)|$ is *harmonic* (*i.e.* solution to Laplace's equation) for $|z|$ large enough. It has a continuous extension $\eta_c(z)$ to $\mathbf{C}$ that is identically zero on K_c, which is harmonic on the whole $\mathbf{C}\setminus K_c$ and satisfies $\eta_c(z^2+c) = 2\eta_c(z)$. The boundary of $\mathbf{C}\setminus K_c$ is $J_c \cup \{\infty\}$ (where J_c denotes the Julia set of r_c) and near the point at infinity $\eta_c(z)$ behaves like $\log|z|$ while $\eta_c(z)$ vanishes on $J_c \subset K_c$. The *minimum principle* for harmonic functions implies that $\eta_c(z) > 0$ for all $z \in \mathbf{C}\setminus K_c$. In particular, if $c \notin M$ then $0 \notin K_c$ and therefore $\eta_c(0) > 0$, while if $c \in M$ then $0 \in K_c$ and $\eta_c(0) = 0$.

Setting $L_c := \{z \in \mathbf{C}; \eta_c(z) \leq \eta_c(0)\} \supset K_c$, Douady and Hubbard showed that the function φ_c can be itself extended to a *conformal map* (*i.e.* that preserves angles and dilates length equally in any direction) from $\mathbf{C}\setminus L_c$ to $\mathbf{C}\setminus D_c$, where D_c is the closed disc of radius $\mathrm{e}^{\eta_c(0)} \geq 1$. Considering the value of $\eta_c(0)$ we check that if $c \notin M$ then $\eta_c(c) = 2\eta_c(0) > \eta_c(0)$ and therefore $c \notin L_c$, while if $c \in M$ then $L_c = K_c$ and D_c is the closed unit disc in $\mathbf{C}$.

Douady and Hubbard deduce from the above considerations the existence of a map $\Phi : \mathbf{C}\backslash M \to \mathbf{C}\backslash D_0$, $c \mapsto \varphi_c(c)$, that defines an isomorphism between $\mathbf{C} \setminus M$ and $\mathbf{C} \setminus D_0$ (D_0 the closed unit disc in $\mathbf{C}$).

The conformal map $\varphi_c : \mathbf{C} \setminus L_c \to \mathbf{C} \setminus D_c$ satisfies the functional equation (6) it has an inverse function ψ_c which satisfies the functional equation

$$\psi_c(z^2) = \psi_c(r_0(z)) = r_c(\psi_c(z)) = \psi_c(z)^2 + c \ . \tag{7}$$

The functions φ_c and ψ_c are tangent to the identity at infinity where they have Laurent expansion in $\mathbf{Q}(c)((1/z))$ starting as $z + O(1/z)$.

LEMMA 7. *With the above notations, the function* ψ_c *is rational for* $c = -2, 0$ *and transcendental otherwise.*

Proof – Theorem 3 implies that the function ψ_c is either rational or transcendental. But, the only possible pole of a rational solution to (7) is the origin and the desired solution must expand at infinity as $z + O(1/z)$. Therefore we are reduced to find a polynomial solution $a(z) \in \mathbf{C}[z]$ of the equation : $a(z^2) = a(z)^2 + cz^{2(n-1)}$, of degree $n \geq 1$, prime to z^{n-1}. A quick analysis shows that it can exists only when $c = 0$ (then $n = 1$ and $a(z) = z$) or $c = -2$ (then necessarily $a(z) = z^n + z^{n-2}$ and $n = 2$). □

REMARK 5. *For* $c \in \{-2, 0\}$ *one has :* $\psi_{-2}(z) = z + z^{-1}$, $\varphi_{-2}(z) = \frac{z}{2}\left(1 + \sqrt{1 - \frac{4}{z^2}}\right)$ *(the square root is defined with a cut along* $]-\infty, 0]$ *so that* $\sqrt{1} = 1$ *) and* $\psi_0(z) = z$, $\varphi_0(z) = z$.

Note that $J_{-2} = K_{-2} = [-2, 2] \subset \mathbf{R}$ *while* K_0 *is the closed unit disc in* $\mathbf{C}$ *and* J_0 *is the unit circle. In particular,* -2 *and* 0 *belongs to* M, *since* $0 \in K_{-2} \cap K_0$.

In this context, we can show that for $c \neq -2, 0$ the functions φ_c and ψ_c take algebraic points to transcendental ones. In order to apply theorem 2 we have to exchange the point at infinity with 0, this is done by considering the function $f(z) := 1/\psi_c(1/z)$, which satisfies the functional equation $f(z^2) = \frac{f(z)^2}{cf(z)^2+1}$ of type (1), and has a Taylor expansion at 0 with coefficients in $\mathbf{Q}(c)$. For $\alpha \in \overline{\mathbf{Q}}$ of modulus $> \mathrm{e}^{\eta_c(0)}$ and $i \in \mathbf{N}$ we have $\psi_c(\alpha^{2^i}) \in \psi_c(\mathbf{C} \setminus D_c) = \mathbf{C} \setminus L_c$ and therefore $\psi_c(\alpha^{2^i}) \neq 0$ since $0 \in L_c$. By the functional equation we deduce $\psi_c(\alpha^{2^i})^2 + c \neq 0$, and $cf(1/\alpha^{2^i})^2 + 1 \neq 0$ for all $i \in \mathbf{N}$. We can state :

THEOREM 8. *For* $c \in \overline{\mathbf{Q}} \setminus \{-2, 0\}$ *and* $\alpha \in \overline{\mathbf{Q}}$ *of modulus* $> \mathrm{e}^{\eta_c(0)}$, *the number* $\psi_c(\alpha)$ *is transcendental.*

On the other side, for c *as above and* $\beta \in \overline{\mathbf{Q}} \setminus (L_c \cap \overline{\mathbf{Q}})$, *then* $\varphi_c(\beta)$ *is transcendental.*

In particular, if $c \in \overline{\mathbf{Q}} \setminus (M \cap \overline{\mathbf{Q}})$ *then* $\Phi(c)$ *is transcendental.*

QUESTION 9. *What conditions on parameters* $c_1, \ldots, c_q \in \overline{\mathbf{Q}} \setminus \{-2, 0\}$ *force the functions* $\varphi_{c_1}, \ldots, \varphi_{c_q}$ *to be algebraically independent over* $\mathbf{Q}(z)$ *? What is the transcendence degree of the field generated over* $\mathbf{Q}$ *by their values at a point in* $\overline{\mathbf{Q}} \setminus (L_{c_1} \cup \cdots \cup L_{c_q}) \cap \overline{\mathbf{Q}}$ *?*

For $c, u \in \mathbf{C}$, $u \notin K_c$, the sequence $(r_c^{\circ k}(u))_{k \in \mathbf{N}}$ tends to infinity as k tends to infinity. The asymptotic behaviour is $r_c^{\circ k}(u) \sim \Theta_c(u)^{2^k}$ as $k \to \infty$, where $\log(\Theta_c(u)) = \lim_{k\to\infty} 2^{-k} \log(r_c^{\circ k}(u))$. Since $\varphi_c(z) = z + O(1/z)$, one has

$$\begin{aligned}
\log(\Theta_c(u)) &= \lim_{k\to\infty} 2^{-k} \log(\varphi_c(r_c^{\circ k}(u))) \\
&= \lim_{k\to\infty} 2^{-k} \log(r_0^{\circ(k-\ell)}(\varphi_c(r_c^{\circ \ell}(u)))) \\
&= 2^{-\ell} \log(\varphi_c(r_c^{\circ \ell}(u)))
\end{aligned}$$

for any $\ell \in \mathbf{N}$ large enough so that $r_c^{\circ \ell}(u) \notin L_c$. Hence $\Theta_c(u)$ is transcendental for $c \in \overline{\mathbf{Q}} \setminus \{-2, 0\}$ and $u \in \overline{\mathbf{Q}} \setminus (K_c \cap \overline{\mathbf{Q}})$. We note that $\Theta_0(u) = u$ and $\Theta_{-2}(u) = \frac{u}{2}\left(1 + \sqrt{1 - \frac{4}{u^2}}\right)$. This gives in particular the answer to a question of J.N.Franklin and S.W.Golomb, *see* [**20**], [**10**] and [**40**, §1.6].

3. A non example and a true extension

Let's consider for $k = 1, 2, 3$ the *Ramanujan* (or *Eisenstein*) *series* E_k, the Taylor expansion of which is

$$E_k(z) = 1 + \gamma_k . \sum_{i \geq 1} \sigma_{2k-1}(i) z^i$$

with $\sigma_k(i) := \sum_{\ell | i} \ell^k$, $\gamma_1 = -24$, $\gamma_2 = 240$, $\gamma_3 = -504$. We know the upper bound $\sigma_k(i) \leq \zeta(k).i^k$ for $k > 1$ (*cf.* [**50**] , chap.7, §4.3). Therefore $\sigma_1(i) \leq \sigma_3(i) \leq \sigma_5(i) \leq \zeta(5).i^5$ and the series above define analytic functions in the (open) unit disc in $\mathbf{C}$ (or in its p-adic analogue $\mathbf{C}_p$).

The *modular invariant* $J(z)$ is defined by

$$J(z) = 1728. \frac{E_2(z)^3}{E_2(z)^3 - E_3(z)^2} = \frac{1}{z} + 744 + O(z) \quad . \tag{8}$$

It satisfies a functional equation of type (3), more precisely for any integer $n > 1$ there exists an irreducible polynomial $\Phi_n \in \mathbf{Z}[X, Y]$, called the *modular polynomial of order* n, such that

$$\Phi_n(J(z^n), J(z)) = 0 \quad . \tag{9}$$

The modular polynomial Φ_n is symmetric in X, Y, its degree in each variable is equal to $n \prod_{p|n} \frac{p+1}{p}$ and its *length* (sum of the absolute values of its coefficients) is bounded above by $\exp\left(Cn \log(n) \prod_{p|n} \frac{p+1}{p}\right)$, where the products run over all the primes p dividing n and $C > 0$ is a real independent of n (any $C > 6$ would do for n large enough), *see* [**16**].

Trying to apply theorem 2 with $t(z) = z^n$ and $E(X, Y, Z) = \Phi_n(X, Y)$, we check that condition (4) is *not* fulfilled since $\deg_X(E) > n$.

However, K.Barré-Siriex, G.Diaz, F.Gramain and G.Philibert [**5**] succeeded to overcome this difficulty and proved that for $\alpha \in \mathbf{C}$ (*resp.* $\alpha \in \mathbf{C}_p$), $0 < |\alpha| < 1$, at least one of the numbers α or $J(\alpha)$ must be transcendental over $\mathbf{Q}$, solving the so-called *Mahler-Manin conjecture*. This approach dates back to 1972, when it was clarified by D.Bertrand, *see* [**12, 13, 14**].

Later on the method was extended in order to prove algebraic independence results, thanks to the crucial zero estimate for Ramanujan series proved by Y.Nesterenko.

Theorem 10 (Y.Nesterenko [**34**]) 10. *Let* $\alpha \in \mathbf{C}$ *(or* $\mathbf{C}_p$*) with* $0 < |\alpha| < 1$*, then at least three of the four numbers* $\alpha, E_1(\alpha), E_2(\alpha), E_3(\alpha)$ *are algebraically independent over* $\mathbf{Q}$*.*

Thanks to relation (8) one recovers easily the results of K.Barré-Siriex, G.Diaz, F.Gramain and G.Philibert [**5**].

This result answered several conjectures in [**13**] (*see* also [**14**] for other applications), where one can find equivalent formulations. In particular, for the modular invariant J and its iterated derivates through the *Ramanujan derivation* $\Theta = z\frac{d}{dz}$, one has

$$J = 1728.\frac{E_2^3}{E_2^3 - E_3^2} \quad , \quad \frac{\Theta J}{J} = -\frac{E_3}{E_2} \quad , \quad 6.\frac{\Theta^2 J}{\Theta J} = E_1 - 4.\frac{E_3}{E_2} - 3.\frac{E_2^2}{E_3} \quad .$$

Recall that the *Dedekind* η *function* is defined by the infinite product $\eta(z) = z^{\frac{1}{24}} \prod_{k \in \mathbf{N}^*}(1 - z^k)$ and satisfies $1728\eta(z)^{24} = E_2(z)^3 - E_3(z)^2$ and $24\frac{\Theta\eta(z)}{\eta(z)} = E_1(z)$.

We will say that two families of functions (*resp.* numbers) *resp. algebraically equivalent over* $\mathbf{Q}$ if the algebraic closure of the field they generate over $\mathbf{Q}$ coincide. The above formulas show that $(E_1(z), E_2(z), E_3(z))$ and $(J(z), \Theta J(z), \Theta^2 J(z))$ are triples of functions algebraically equivalent over $\mathbf{Q}$. We can substitute any triple of functions algebraically equivalent to $(E_1(z), E_2(z), E_3(z))$ in Theorem 10 and keep the same conclusion.

Since $J(z^n)$ and $J(z)$ are algebraically equivalent over $\mathbf{Q}$ by formula (9), the algebraic closures of the fields $\mathbf{Q}(J(z^n))$ and $\mathbf{Q}(J(z))$ coincide for any integer $n \geq 1$, denote it F. By derivation we deduce from (9) that $F(\Theta J(z^n), \Theta^2 J(z^n)) = F(\Theta J(z), \Theta^2 J(z))$, hence the triples of functions $(J(z^n), \Theta J(z^n), \Theta^2 J(z^n))$ and $(J(z), \Theta J(z), \Theta^2 J(z))$ are algebraically equivalent over $\mathbf{Q}$. But, since $(J(z^n), \Theta J(z^n), \Theta^2 J(z^n))$ is algebraically equivalent over $\mathbf{Q}$ to the triple $(E_1(z^n), E_2(z^n), E_3(z^n))$ for any integer $n \geq 1$ we further get that the triples $(E_1(z^n), E_2(z^n), E_3(z^n))$ and

$(E_1(z), E_2(z), E_3(z))$ are algebraically equivalent over $\mathbf{Q}$. In particular, the algebraic closure of the field generated over $\mathbf{Q}$ by the functions $E_1(z)$, $E_2(z)$ and $E_3(z)$ is stable under all the transformations $z \mapsto z^n$, $n \in \mathbf{N}^*$, of the variable z.

The Ramanujan series also satisfies the system of *Ramanujan differential equations*

$$\Theta E_1 = \frac{1}{12}.(E_1^2 - E_2) \ , \quad \Theta E_2 = \frac{1}{3}(E_1 E_2 - E_3) \ , \quad \Theta E_3 = \frac{1}{2}(E_1 E_3 - E_2^2) \ ,$$

which shows that the field generated over $\mathbf{Q}$ by the functions $E_1(z)$, $E_2(z)$ and $E_3(z)$ is stable under the Ramanujan derivation. The same holds for its algebraic closure.

We will see that Theorem 10 implies the transcendence of various numbers related to linear recurrence sequences, for example. Many of these corollaries can be obtained with the following one.

Corollary (D.Duverney, Ke. & Ku.Nishioka, I.Shiokawa [**18**]) 11. *Let α an algebraic number in $\mathbf{C}$ or $\mathbf{C}_p$ satisfying $0 < |\alpha| < 1$, then for any non constant function $f(z)$ which is algebraic over the function field $\mathbf{Q}(E_1(z), E_2(z), E_3(z))$ and defined at α, the value $f(\alpha)$ is transcendental.*

More generally, if $f_1(z), \dots, f_m(z)$ are algebraic over the function field $\mathbf{Q}(E_1(z), E_2(z), E_3(z))$ and defined at α, then the ideal of relations between the functions $f_i(z)$, $i = 1, \dots, m$, coincide with that of relations between their values $f_i(\alpha)$, $i = 1, \dots, m$.

Proof – Since $f(z)$ is not constant its monic minimal equation over the function field $\mathbf{Q}(E_1(z), E_2(z), E_3(z))$ is an irreducible polynomial depending on at least one of the functions $E_i(z)$, this implies that $f(z)$ is a transcendental function.

Now, the values $E_1(\alpha), E_2(\alpha), E_3(\alpha)$ are algebraically independent over $\mathbf{Q}$ by Theorem 10, hence the specialisation at α of the monic minimal equation of $f(z)$ over $\mathbf{Q}(E_1(z), E_2(z), E_3(z))$ is the monic minimal equation of $f(\alpha)$ over $\mathbf{Q}(E_1(\alpha), E_2(\alpha), E_3(\alpha))$. It also depends on at least one of the numbers $E_i(z)$ and this implies that $f(\alpha)$ is transcendental over $\mathbf{Q}$.

More generally, let $I \subset \mathbf{Q}[X_1, X_2, X_3, Y_1 \dots, Y_m]$ be the prime ideal of relations between the functions $E_1(z)$, $E_2(z)$, $E_3(z)$ and $f_1(z), \dots, f_m(z)$, by the hypothesis it is of rank m. If there exists a polynomial relation over $\mathbf{Q}$ between $f_1(\alpha), \dots, f_m(\alpha)$ which does not belong to $I \cap \mathbf{Q}[Y_1, \dots, Y_m]$, adding this polynomial to I gives an ideal of rank higher than m which must therefore contain a non zero element in $\mathbf{Q}[X_1, X_2, X_3]$, but this would lead to a non trivial algebraic relation between $E_1(\alpha), E_2(\alpha), E_3(\alpha)$ over $\mathbf{Q}$ contrary to Theorem 10. □

For example, introduce the *Jacobi theta functions* [**57**, Chap.XXI]

$$\begin{aligned}\theta_1(u,z) &= 2z^{1/4}\sum_{k\geq 1}(-1)^{k+1}z^{k(k-1)}\sin((2k-1)u)\\ &= 2z^{1/4}\sin(u)\prod_{k\geq 1}(1-z^{2k})(1-2z^{2k}\cos(2u)+z^{4k})\end{aligned}$$

$$\begin{aligned}\theta_2(u,z) &= 2z^{1/4}\sum_{k\geq 1}z^{k(k-1)}\cos((2k-1)u)\\ &= 2z^{1/4}\cos(u)\prod_{k\geq 1}(1-z^{2k})(1+2z^{2k}\cos(2u)+z^{4k})\end{aligned}$$

$$\begin{aligned}\theta_3(u,z) &= 1+2\sum_{k\geq 1}z^{k^2}\cos(2ku)\\ &= \prod_{k\geq 1}(1-z^{2k})(1+2z^{2k-1}\cos(2u)+z^{4k-2})\end{aligned}$$

$$\begin{aligned}\theta_4(u,z) &= 1+2\sum_{k\geq 1}(-1)^k z^{k^2}\cos(2ku)\\ &= \prod_{k\geq 1}(1-z^{2k})(1-2z^{2k-1}\cos(2u)+z^{4k-2})\ .\end{aligned}$$

Writing $z=\mathrm{e}^{\mathrm{i}\pi\tau}$, these theta functions have period π and quasi-period $\pi\tau$ in the variable u : $\theta_4(u+\pi\tau,z) = -z^{-1}\mathrm{e}^{-2\mathrm{i}z}\theta_4(u,z)$. Their *zero argument theta values* $\theta_i(0,z)$, $i=1,2,3,4$, satisfy the relations

$$\theta_1(0,z)=0\ ,\quad \theta_2(0,z)^4+\theta_4(0,z)^4=\theta_3(0,z)^4$$

and furthermore

$$E_2(z^2)^3-E_3(z^2)^2=\tfrac{27}{4}\left(\theta_2(0,z)\theta_3(0,z)\theta_4(0,z)\right)^8$$

$$E_2(z^2)=\tfrac{1}{2}\left(\theta_2(0,z)^8+\theta_3(0,z)^8+\theta_4(0,z)^8\right)$$

which show that the functions $\theta_i(0,z)^8$, $i=2,3,4$, are the roots of the equation

$$X^3-2E_2(z^2)X^2+E_2(z^2)^2X-\frac{4}{27}(E_2(z^2)^3-E_3(z^2)^2)=0\ ,$$

hence belong to the algebraic closure of the field generated over $\mathbf{Q}$ by the functions $E_1(z)$, $E_2(z)$ and $E_3(z)$. Applying power transformations of the variable and Ramanujan derivation we conclude that all the following functions

$$\Theta^\ell E_i(z^n)\ ,\quad \Theta^\ell\theta_{i+1}(0,z^n)\ ,\quad \Theta^\ell J(z^n)\ ,\quad i=1,2,3\ ,\quad n\in\mathbf{N}^*\ ,\quad \ell\in\mathbf{N}$$

are algebraic over the field $\mathbf{Q}(E_1(z),E_2(z),E_3(z))$, since this latter field is stable under these operations. The same formulas show that $E_2(z^2)$ and $E_3(z^2)$ are algebraic over $\mathbf{Q}(\theta_i(0,z),\theta_j(0,z))$ for any $2\leq i\neq j\leq 4$.

The zero argument theta values $\theta_i(0,z)$ satisfy the following differential equations

$$(10)\quad 12\frac{\Theta\theta_2}{\theta_2}-\theta_3^4-\theta_4^4=12\frac{\Theta\theta_3}{\theta_3}-\theta_2^4+\theta_4^4=12\frac{\Theta\theta_4}{\theta_4}+\theta_2^4+\theta_3^4=E_1(z^2)\ ,$$

which show that $E_1(z^2)$ is algebraic over $\mathbf{Q}(\Theta\theta_h(0,z),\theta_i(0,z),\theta_j(0,z))$ for any $2\le h\le 4$ and $2\le i\ne j\le 4$. Therefore each triple of functions $(\Theta\theta_h(0,z),\theta_i(0,z),\theta_j(0,z))$ is algebraically equivalent to the triples $(E_1(z^2),E_2(z^2),E_3(z^2))$ and $(E_1(z),E_2(z),E_3(z))$.

Now, for s any permutation of $\{2,3,4\}$ the quadratic equation

$$X^2-\theta_{s(4)}^4(0,z)X+8\left(\frac{\Theta^2\theta_{s(4)}(0,z)}{\theta_{s(4)}(0,z)}-3\left(\frac{\Theta\theta_{s(4)}(0,z)}{\theta_{s(4)}(0,z)}\right)^2\right)=0$$

has roots $(-1)^{s(2)}\theta_{s(3)}(0,z)$ and $(-1)^{s(3)}\theta_{s(2)}(0,z)$. This shows that the triples $(\theta_{s(2)}(0,z),\theta_{s(3)}(0,z),\Theta_{s(4)}(0,z))$ and $(E_1(z),E_2(z),E_3(z))$ are algebraically equivalent to the triple $(\theta_h(0,z),\Theta\theta_h(0,z),\Theta^2\theta_h(0,z))$ for any $h\in\{2,3,4\}$.

Exercises 6. *1) Prove that for any permutation s of $\{2,3,4\}$ one has*

$$8\left(\frac{\Theta^2\theta_{s(4)}(0,z)}{\theta_{s(4)}(0,z)}-3\left(\frac{\Theta\theta_{s(4)}(0,z)}{\theta_{s(4)}(0,z)}\right)^2\right)=(-1)^{s(2)+s(3)}\theta_{s(2)}(0,z)\theta_{s(3)}(0,z)\ .$$

2) Prove that the triple of functions $(\eta(z),\Theta\eta(z),\Theta^2\eta(z))$ is algebraically equivalent over $\mathbf{Q}$ to the triple of functions $(E_1(z),E_2(z),E_3(z))$.

3) Prove the same for the triple of functions $\left(\frac{\Theta\theta_2(0,z)}{\theta_2(0,z)},\frac{\Theta\theta_3(0,z)}{\theta_3(0,z)},\frac{\Theta\theta_4(0,z)}{\theta_4(0,z)}\right)$.

We end this section with a conjecture proposed by D.Bertrand :

Conjecture (D.Bertrand [**14**]) 12. *Let $\alpha_1,\dots,\alpha_n$ be algebraic numbers satisfying $0<|\alpha_i|<1$, $i=1,\dots,n$, such that the $3n$ numbers $J(\alpha_i)$, $\Theta J(\alpha_i)$, $\Theta^2 J(\alpha_i)$, $i=1,\dots,n$, are algebraically dependent over $\mathbf{Q}$. Then there exists $1\le i\ne j\le n$ such that α_i and α_j are multiplicatively dependent.*