# Lectures on geometric transcendence theory

C. Gasbarri

14 December 2010

## 1 Introduction.

In these lectures we will describe the analogy between the theory of rational points on varieties over number fields, the theory of curves over projective varieties and the theory of analytic map of affine curves on projective varieties.

This analogy is explained by showing the similarities between height theory, the intersection theory and Nevanlinna theory. In particular a detailed description of Poincaré Lelong formula in this contest is given.

In the last lectures we well show how these three theories may interact within them. So we will not have a simple analogy but a richer interaction. This is done to prove some generalization of the classical Schneider Lang criterion over affine curves.

The reader is supposed to know only standard definitions of schemes and basic facts about commutative algebra and complex analysis (in one variable). We tried to make fast overview of the tools in algebraic geometry and analytic geometry used.

These are the notes of a series of lectures I gave at the IMSc of Chennai (india). I would like to thank the IMSc, in particular Prof. Balusubramanian and Prof Gun, for the opportunity they gave to me and the warm and friendly hospitality.

## 2 Lecture I.

One of the leading problems in mathematics is the search of methods to solve diophantine equations:

For instance, given an equation (monic to simplify)

$$X^n + a_{n-1}X^{n-1} + \ldots + a_0 = 0$$

with $a_i \in \mathbb{Z}$, a very easy (even if not the more convenient in computational terms) way to find all its eventual solutions *in* $\mathbb{Z}$ is the following: Factorize $a_0$, a possible solution will be a divisor of $a_0$.

In two or more variables, the situation is more difficult; even the case of linear equations in two variables is part of a course in a first year elementary algebra. In the higher degree case, the problem is still more difficult and many questions are still open.

There is another problem which is the other side of question above: suppose that we have a complex number, defined by some properties (for instance differential), is it defined over $\mathbb{Q}$? or just defined over $\overline{\mathbb{Q}}$? One can also ask if, given two, or more, complex numbers $a_1$ and $a_2$, there exist a polynomial $P(X, Y)$ with coefficients in $\mathbb{Q}$ such that $P(a_1, a_2) = 0$.

The two problems are specular one of the other: in the first one we want to find the eventual solutions of an equation; in the second one, we want the find the eventual equation solved by a "solution".

**2.1 Remark.** Observe that there is a theorem by Matiyesevich which tells us that there is no algorithm which can tell us if a diophantine equation has a solution over $\mathbb{Z}$.

When one wants to study a problem, often a good idea is to find a similar problem and understand similarities and analogies; so that we can deduce what is formal and what is strictly related to the situation.

The analogy we have in mind is:

*The analogy between number fields and functions fields*

Consider the two rings $\mathbb{Z}$ and $\mathbb{C}[z]$. They have many properties in common:

– They are both P.I.D.;
– Every prime ideal is either (0) or maximal;
– We can perform euclidean division.

And we will see some other common properties.

Consequently one can think to study the diophantine equations over $\mathbb{C}[z]$ and see if the results we find and the methods developed, have some meaning over the ring $\mathbb{Z}$.

Over the two rings $\mathbb{Z}$ and $\mathbb{C}[z]$ we can develop the four operations, but over $\mathbb{C}[t]$ there is another operation which do not exists over $\mathbb{Z}$: *the derivation.*

The *derivation* is a formal operation $(\cdot)' : \mathbb{C}[z] \to \mathbb{C}[z]$ which is $\mathbb{C}$ linear and verifies the Liebnitz formula: $(fg)' = fg' + f'g$; if we also impose that $(z)' = 1$ the derivation is what we imagine: $(z^n)' = nz^{n-1}$.

**2.2 Remark.** Observe that, if we do not impose that $(z)' = 1$ there are many derivations over $\mathbb{C}[z]$.

It is easy to prove that the only derivation on $\mathbb{Z}$ is $(n)' = 0$.

So $\mathbb{C}[z]$ is a euclidean domain which is equipped with a non trivial operation which satisfy Liebnitz rule. We will see that this new tool is incredibly powerful in the study of diophantine equations: The analogue over $\mathbb{C}[z]$ of the famous Fermat Last Theorem become almost an exercise:

**2.3 Theorem.** *Let $n \geq 3$. If $(f(z), g(z), h(z)) \in (\mathbb{C}[z])^3$ is a non trivial (which means $fgh \neq 0$) solution of the equation*

$$X^n + Y^n = Z^n$$

*then $(f(z), g(z), h(z)) \in \mathbb{C}^3$: $f$, $g$ and $h$ are constants.*

Before we start the proof we give the following definition:

**2.4 Definition.** *Let $f(z) \in \mathbb{C}[z]$. We denote by $N(f)$ the number of zeros of $f(z)$ counted* without *multiplicity and we call it* the conductor of $f$.

For instance $N((z-2)^4(z-3)) = 2$.
*Proof:* The theorem will be consequence of the following very important statement:

**2.5 Theorem.** *Let $f(z)$, $g(z)$ and $h(z)$ three coprime polynomials such that*

$$f(z) + g(z) = h(z).$$

*Then the following inequality holds:*

$$\max\{\deg(f(z), \deg(g(z)), \deg(h(z))\} \leq N(fgh) - 1.$$

Let's show how the Theorem 2.5 implies Theorem 2.3: Suppose that

$$f(z)^n + g(z)^n = h(z)^n$$

and suppose that $d = \deg(f) \geq \deg(g) \geq \deg(h)$ (we can always suppose it). Theorem 2.5 implies that $nd \leq 3d - 1$, consequently

$$(n-3)d \leq -1.$$

The conclusion easily follows.

Let's now prove Theorem 2.5:
*Proof: (of 2.5)* Write

$$f(z) = a_0 \prod_{i=1}^{N(f)} (z - a_i)^{n_i}, \qquad g(z) = b_0 \prod_{j=1}^{N(g)} (z - b_j)^{m_j}, \qquad h(z) = c_0 \prod_{k=1}^{N(h)} (z - c_k)^{\ell_k}.$$

Let $\mathbb{C}(z) := Frac(\mathbb{C}[z])$ be the field of rational functions. The derivation extends to a derivation on $\mathbb{C}(z)$. We introduce the following morphism

$$d\log(\cdot) : \mathbb{C}(z)^* \longrightarrow \mathbb{C}(z)$$
$$F(z) \longrightarrow d\log(F) := \frac{F'(z)}{F(z)}.$$

It is easy to verify that $d\log(FG) = d\log(F) + d\log(G)$.

Denote $F := \frac{f(z)}{h(z)}$ and $G := \frac{g(z)}{h(z)}$. From the hypothesis we have $F + G = 1$, thus

$$Fd\log(F) + Gd\log(G) = 0.$$

From this we obtain

$$\frac{f(z)}{g(z)} = -\frac{d\log(G)}{d\log(F)}.$$

Consider the polynomial

$$N_0(z) = \prod_{i=1}^{N(f)}(z - a_i)\prod_{j=1}^{N(g)}(z - b_j)\prod_{k=1}^{N(h)}(z - c_k).$$

The degree of $N_0(z)$ is $N(fgh)$.

Observe that, since $d\log(F) = d\log(f(z)) - d\log(h(z))$ we have that $N_0(z)d\log(F)$ is a polynomial of degree at most $N(fgh) - 1$. Similarly the degree of the polynomial $N_0(z)d\log(G)$ is at most $N(fgh) - 1$. We have that

$$\frac{f(z)}{g(z)} = -\frac{N_0(z)d\log(G)}{N_0(z)d\log(F)};$$

Since $f(z)$ and $g(z)$ are coprime, the maximum between the degree of $f(z)$ and the degree of $g(z)$ is at most $N(fgh) - 1$. The conclusion follows.

**2.6 Remark.** Theorem 2.5 is called *the abc conjecture over functions fields* and in this form it has been proved by Mason. This proof is taken from [La].

One could find an easy proof of the Fermat Last Theorem if one could prove an analogue of Theorem 2.5 in the arithmetic contest. A conjectural form of it is called *the abc conjecture.*

We would like to explain another important feature that $\mathbb{Z}$ and $\mathbb{C}[z]$ have in common: the *Product Formula.*

It is well known, after the Hilbert Nullstellensatz, that the set of maximal ideals of $\mathbb{C}[t]$ is in bijection with the set of points of the affine line $\mathbb{A}^1(\mathbb{C})$.

The affine line can be compactified to the projective line $\mathbb{A}^1 \hookrightarrow \mathbb{P}_1$. The field $\mathbb{C}(t)$ is the field of meromorphic functions of $\mathbb{P}_1$ over $\mathbb{C}$.

Let $p \in \mathbb{P}_1$ be a closed point. We will denote by $\mathcal{O}_p$ the local ring of $p$ in $\mathbb{P}_1$. The ring $\mathcal{O}_p$ is a discrete valuation ring and we denote then by $t_p$ an uniformizer of it.

**2.7 Example.** Suppose that $p = \infty$. Then we define $t_\infty = \frac{1}{z}$ and $\mathcal{O}_\infty$ is defined in the following way: Let $F(z) \in \mathbb{C}(t)$; We can write in a unique way $F(z) = t_\infty^{v_\infty(F)}G(t_\infty)$ with $G(0) \neq 0$ and $v_\infty(F) \in \mathbb{Z}$. We define then $\mathcal{O}_\infty := \{F(z) \ / \ v_\infty(F) \geq 0\}$. The maximal ideal of $\mathcal{O}_\infty$ is $\{F(z) \ / \ v_\infty(F) > 0\}$.

Given an element $F(z) \in \mathbb{C}(z)^*$, we can write it as $t_p^{v_p(F)}u(z)$ where $v_p(z) \in \mathbb{Z}$ and $u(z)$ is a unit in $\mathcal{O}_p$. By definition we say that $v_p(0) = \infty$

It is important to observe the following properties:

– $v_p(FG) = v_P(F) + v_p(G)$.

– $v_p(F+G) \geq \max\{v_p(F), v_p(G)\}$

– If $p \neq \infty$ and $F \in \mathbb{C}[z]$ then $v_p(F) \geq 0$ and $(z-p)^{v_p(F)}$ divides $F$ while $(z-p)^{v_p(F)+1}$ do not divide $F$.

– It is easy to see that if $F \in \mathbb{C}[z]$ then $v_\infty(F) = -\deg(F)$

From the properties above we deduce

**2.8 Theorem.** *If $F \in \mathbb{C}(z)$ then*

$$\sum_{p \in \mathbb{P}_1} v_p(F) = 0.$$

To prove it, one reduces to the case when $F$ is a polynomial and then one apply the properties above.

If $p \in \mathbb{P}_1$; we can associate to it a *non archimedean norm* $\|\cdot\|_p$ on $\mathbb{C}(t)$ in the following way:

$$\|F(z)\|_p := \exp(-v_p(F)).$$

Theorem 2.8 becomes then

**2.9 Theorem.** *For every $F \in \mathbb{C}(t)^*$ we have*

$$\prod_{p \in \mathbb{P}_1} \|F\|_p = 1.$$

The formula above is called *the product formula for the projective line.* Often one quote it by saying "a meromorphic function on the projective line has the same number of zeros and poles, counting multiplicity".

The Product formula has an important analogue over $\mathbb{Z}$ which we will now briefly describe.

By analogy with the geometric case, the set *Specmax*$(\mathbb{Z})$ of maximal ideals of $\mathbb{Z}$ can be interpreted as an *arithmetic line.* The fraction field of $\mathbb{Z}$ which is $\mathbb{Q}$, can be interpreted as the field of "meromorphic functions on the arithmetic line". Let $p$ be a prime number; it corresponds to a point of the arithmetic line. The local ring $\mathbb{Z}_p$ of $p$ is a discrete valuation ring with uniformizer $p$.

**2.10 Remark.** Observe that this is another feature that $\mathbb{Z}$ and $\mathbb{C}[z]$ have in common: for every maximal ideal of them, the associated local ring is a discrete valuation ring.

Every $m \in \mathbb{Q}$ can be uniquely written as $m = p^{v_p(m)}u$ where $v_p(m) \in \mathbb{Z}$ and $u$ is a unit in $\mathbb{Z}_p$. We can associate to $p$ a *non archimedean norm* $\|\cdot\|_p$ defined as

$$\|m\|_p := p^{-v_p(m)}.$$

Thus, we have norms associated to every maximal ideal of $\mathbb{Z}$. There is another natural norm on $\mathbb{Q}$: the one induced by the inclusion in $\mathbb{R}$:

$$\|m\|_\infty := |m|.$$

This norm is *euclidean*.

Consequently we have the *Product formula for the arithmetic line*:

**2.11 Theorem.** *Let $m \in \mathbb{Q}^*$. Then*

$$\|m\|_\infty \cdot \prod_{p \in Specmax(\mathbb{Z})} \|m\|_p = 1.$$

*Proof:* One easily sees that, of $p$ is a prime number, then

$$\|p\|_\infty \cdot \|p\|_p = 1.$$

By the unique factorization property, we have that if $p \neq q$ are two different primes, then $\|p\|_q = \|q\|_p = 1$. Since the norms are multiplicative, the conclusion easily follows.

Formulas of Theorems 2.9 and 2.11 are formally the same and provide another important common feature betwen $\mathbb{Z}$ and $\mathbb{C}[z]$. We observe the important fact that, the product formula over the projective line requires the point at infinity and the formula on the arithmetic line requires the euclidean norm. Consequently we are induced to see the euclidean norm as a point at infinity of the arithmetic line: the arithmetic line and the point at infinity may be seen as a sort of projective arithmetic line.

Most of the things proved in this lecture generalize to a general curve and to a general ring of integers of a number field.

## 3 Lecture II.

In the last lecture we shown many interesting features in common between $\mathbb{Z}$ and $\mathbb{C}[z]$. In order to explain the product formula, in both cases we had to "compactify" the situation:

– For $\mathbb{C}[z]$ we introduced the projective line as a projective variety naturally compactifying the affine line.

– For $\mathbb{Z}$ : each non archimedean absolute value over $\mathbb{Q}$ corresponds to a closed point of $Spec(\mathbb{Z})$. In order to obtain the product formula, we considered the usual archimedean absolute value as another point: the point at infinity.

As one can imagine, the analogue of the product formula is true over an arbitrary projective curve and over an arbitrary ring of integers of a number field.

Suppose that $X$ is a smooth projective curve over $\mathbb{C}$ and $\mathbb{C}(X)$ is the field of rational functions over it. and $f \in \mathbb{C}(X)$. For every closed poin $p$ t of $X$, denote by $\mathcal{O}_{X,p}$ the local ring of $p$ in $X$ and by $t_p$ an uniformizer of it. Every element $f \in \mathbb{C}(X)$ can be written as $f = t_p^{v_p(f)} \cdot u$ with $u \in \mathcal{O}_{X,p}^{\times}$ and $v_p(f) \in \mathbb{Z}$. We define then

$$\|f\|_p := \exp(-v_p(f)).$$

If $v_p(f) > 0$ then we will say that $f$ *has a zero of order* $v_p(f)$ *at* $p$; if $v_p(f) < 0$ we will say that $f$ as a pole of order $-v_p(f)$ at $p$.

**3.1 Example.** Let $f(z) := \frac{(z-1)^2(z-2)}{z^3(z-3)^5}$ then: $f$ has a zero of order 2 at 1, a zero of order 1 at 2 *a zero of order* 5 *at* $\infty$, a pole of order 3 at 0 and a pole of order 5 at 3.

**3.2 Theorem.** *For every* $f \in \mathbb{C}(X)^{\times}$ *we have that*

$$\prod_{p \in \mathbb{C}(X)} \|f\|_p = 1.$$

Similarly for number fields, even if here we need to be careful about normalizations: Let $K$ be a number field and $O_K$ be its ring of integers. For every maximal ideal $\mathfrak{p}$ of $O_K$, Let $O_{K,\mathfrak{p}}$ and $\pi_p$ be the local ring of it be an uniformizer of it respectively. Let $(p)$ be the ideal $\mathfrak{p} \cap \mathbb{Z}$ of $\mathbb{Z}$. The cardinality of $O_K/\mathfrak{p}$ is $p^{d_\mathfrak{p}}$ for a suitable positive integer $d_\mathfrak{p}$ (which is usually called the *residual degree of* $\mathfrak{p}$). Every $f \in K$ may be written as $\pi_p^{v_\mathfrak{p}(f)} \times u$ with $u \in \mathcal{O}_{K,\mathfrak{p}}^{\times}$ and $v_\mathfrak{p}(f) \in \mathbb{Z}$. in particular $v_\mathfrak{p}(p) = a > 0$. We define then

$$\|f\|_\mathfrak{p} := p^{-v_\mathfrak{p}(f) d_\mathfrak{p}}.$$

Observe that

$$\log(Card(O_K/\mathfrak{p})) = -\log \|\pi_p\|_\mathfrak{p}$$

And more generally, if $f \in O_K$ then

$$\log(Card(O_K/(f)) = -\sum_{\mathfrak{p} \in M_{fin}} \log \|f\|_p.$$

For every complex embedding $\sigma : K \hookrightarrow \mathbb{C}$ we define

$$\|f\|_\sigma := |\sigma(f)|.$$

Denote by $M_\infty$ the set of complex embedding of $K$.

**3.3 Theorem.** *For every* $f \in K^{\times}$ *we have that*

$$\prod_{\sigma \in M_\infty} \|f\|_\sigma \cdot \prod_{\mathfrak{p} \in Specmax(O_K)} \|f\|_\mathfrak{p} = 1.$$

The product formula is a cornerstone (and if you want the main one) for the theory of divisors and line bundles on curves. We recall here the main facts of this theory:

Let $X$ be a smooth projective curve as above.

**3.4 Definition.** *The free abelian group generated by the points of $X$ is called the group of divisors of $X$ and denoted by $Div(X)$. Each element of $Div(X)$ is a formal sum*

$$D = \sum_{p \in X} n_p[p]$$

*with the $n_p$'s all but finitely many zero.*

– Given $D \in Div(X)$ we define *the degree* of $D$ the natural number $\deg(D) := \sum_{p \in x} n_p$.

– Let $f \in \mathbb{C}(X)$ a non zero element. For every point $p \in X$ we associated a number $v_p(f)$. For almost all the points $p$ the number $v_p(f)$ is zero. Thus we can associate to $f$ the divisor $div(f) := \sum_{p \in X} v_p(f)[p]$. The degree of $div(s)$ is zero because the product formula 3.2.

In particular one of the main consequences of the product formula over is the following:

Suppose that we fix two sets of points $Z = \{p_1, \ldots .p_r\}$. For every $p_i$ we fix integers $m_i$ . Then a necessary condition the existence of a function $f \in \mathbb{C}(X)$ such that $div(f) = \sum_{i=1}^{r} m_i[p_i]$ is that $\sum_i m_i = 0$. Or equivalently, that the degree of the divisor $\sum_i m_i[p_i]$ is zero.

If $X = \mathbb{P}^1$ the condition above is also sufficient: we take the function $f(z) := \prod_i (z - i)^{m_i}$. Remark that if we make the same construction when $\sum_i m_i \neq 0$ then we will have troubles with the point $\infty$.

**3.5 Definition.** *A divisor $D$ on a smooth projective curve is said to be* principal *if there exists a function $f \in \mathbb{C}(X)$ such that $D = div(f)$. The set of principal divisors of $X$ is a subgroup of $Div(X)$ and it is denoted by $P(X)$.*

On an arbitrary curve $X$, there exist divisors of degree zero which are not principal.

**3.6 Definition.** *Let $X$ as above. The Picard group of $X$ is the group*

$$Pic(X) := Div(X)/P(X).$$

Observe that the product formula tells us that the degree maps factors

$$\begin{aligned} \deg : Pic(X) &\longrightarrow \mathbb{Z} \\ [D] &\longrightarrow \deg(D) \end{aligned}.$$

The kernel of this map is denoted by $Pic^0(X)$. If we denote by $Div^0(X)$ the subgroup of elements of degree zero of $Div(X)$, then $Pic^0(X) = Div^0(X)/P(X)$.

We can resume what we said by the following commutative diagram of abelian groups:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to \mathbb{C}^{\times} & \longrightarrow & \mathbb{C}(X)^{\times} & \longrightarrow & Div^{0}(X) & \longrightarrow & Pic^{0}(X) \to 0 \\
 & = & & = & & \downarrow & & \downarrow & \\
0 & \to \mathbb{C}^{\times} & \longrightarrow & \mathbb{C}(X)^{\times} & \longrightarrow & Div(X) & \longrightarrow & Pic(X) \to 0 \\
 & & & & & {\scriptstyle \deg}\downarrow & & {\scriptstyle \deg}\downarrow & \\
 & & & & & \mathbb{Z} & = & \mathbb{Z} & \\
 & & & & & \downarrow & & \downarrow & \\
 & & & & & 0 & & 0 &
\end{array}$$

One can prove that the group $Pic^0(X)$ has a natural structure of a smooth projective algebraic group of dimension $g$, the genus of $X$.

Looking carefully to the constructions above, one checks that the main tool was the product formula, consequently one can transport everything in the arithmetic world:

Let $K$ be a number field and $O_K$ be its ring of integers. We denote by $M_\infty$ the set of complex embeddings of $K$ and by $M_{K,f}$ the set of maximal ideals of $O_K$. The set $M_\infty$ corresponds to the classes of archimedean absolute values of $K$, also called *infinite places*; while the set $M_{K,f}$ corresponds to the classes of non archimedean absolute values of $K$, usually called *finite places of* $K$.

**3.7 Definition.** *A* compactified *divisor over $O_K$ is a formal sum*

$$D = \sum_{\sigma \in M_\infty} \lambda_\sigma[\sigma] + \sum_{\mathfrak{p} \in M_{K,f}} n_{\mathfrak{p}}[\mathfrak{p}]$$

*where $\lambda_\sigma \in \mathbb{R}$ and $n_{\mathfrak{p}} \in \mathbb{Z}$ almost all zero. The set of compactified divisors over $O_K$ is naturally a group denoted $\widehat{Div}(O_K)$.*

we define an *arithmetic degree map*, using the notation above:

$$\begin{aligned}
&\widehat{\deg} : \widehat{Div}(O_K) \longrightarrow \mathbb{R} \\
&\quad D \longrightarrow \widehat{\deg}(D) := \sum_{\sigma \in M_\infty} \lambda_\sigma + \sum_{\mathfrak{p} \in M_{K,f}} n_{\mathfrak{p}} \cdot \log(Card(O_K/_{\mathfrak{p}}) \cdot
\end{aligned}$$

This is formally the same definition given over a curve. Indeed, the main difference is that, while in the geometric case the degree of a point is always one, in the arithmetic case the degree of a point is the logarithm of the norm of it. One can remember that in order to obtain the product formulas, in the geometric case we introduced an exponential, while in the arithmetic case we had a natural candidate to pass from the valuations to the norms.

Given an element $f \in K$, we can associate to it a natural compactified divisor:

$$div(f) := \sum_{\mathfrak{p} \in M_{K,f}} v_{\mathfrak{p}}(f)[\mathfrak{p}] - \sum_{\sigma \in M_\infty} \|f\|_\sigma [\sigma].$$

Remark the minus sign in front of infinite places. The product formula can be restated by saying:

**3.8 Proposition.** *For every $f \in K^\times$ we have that*

$$\widehat{\deg}(div(f)) = 0.$$

The set of compactified divisors which are $div(f)$ for some $f \in K^\times$ is a subgroup of the group of compactified divisors of $O_K$ called *the group of principal compactified divisors* and denoted by $\widehat{P}(O_K)$.

And eventually we can define

**3.9 Definition.** *We define the compactified Picard group of a number field as the group*

$$\widehat{Pic}(O_K) : \widehat{Div}(O_K)/\widehat{P}(O_K).$$

By construction we have an arithmetic degree map:

$$\widehat{\deg} : \widehat{Pic}(O_K) \longrightarrow \mathbb{R}.$$

Remark that, given a compactified divisor $D = \sum_{\sigma \in M_\infty} \lambda_\sigma [\sigma] + \sum_{\mathfrak{p} \in M_{K,f}} n_{\mathfrak{p}} [\mathfrak{p}]$, we can associate to it the fractional ideal $\prod \mathfrak{p}^{n_{\mathfrak{p}}}$ of $K$. we leave as an exercise to the reader the fact that this induce a surjective map

$$\widehat{Pic}(O_K) \longrightarrow Cl(O_K)$$

Where $Cl(O_K)$ is the ideal class group of $O_K$.

One of the main theorems of the theory is:

**3.10 Theorem.** *we have a canonical isomorphism*

$$Pic(X) \simeq H^1(X, \mathcal{O}^\times).$$

In order to prove the theorem above we need to introduce the group of *Cartier Divisors of a scheme*:

Let $X$ be a scheme, which we suppose reduced and irreducible (a curve or the spectrum of the ring of integers of a number field if you prefer). We introduce two sheaves in abelian groups on it:

– The *total quotient sheaf of* $X$: it is the sheaf $\mathcal{K}(X)^\times$ defined in the following way: Let $U$ be an open set of $X$, let $S$ be the multiplicatively closed set of elements of $\Gamma(U, \mathcal{O}_X)$ which are *not* zero divisors of it; then $\Gamma(U, \mathcal{K}(X)^\times) = S^{-1}\Gamma(U, O_X)$.

– The *sheaf of units of* $X$: It is the sheaf $\mathcal{O}_X^\times$ defined in the following way: Let $U$ be an open set of $X$, then $\Gamma(U;\mathcal{O}_X^\times) = \Gamma(U,\mathcal{O}_X)^\times$ (the group of units of it).

Remark that if $X$ is reduced and irreductible, then we can speak about its quotient field $K(X)$: it is the local ring of its generic point and, for every $U$ open set of $X$ we have that $\Gamma(U,\mathcal{K}(X)^\times) = K(X)^\times$.

Observe also that there is a natural inclusion of sheaves

$$\mathcal{O}_X^\times \hookrightarrow \mathcal{K}(X)^\times.$$

**3.11 Definition.** *A Cartier divisor on $X$ is a global section of the sheaf $\mathcal{K}(X)^\times/\mathcal{O}_X^\times$. The group of Cartier divisors is denoted by $CaDiv(X)$.*

A more explicit description of a Cartier divisor is the following: A Cartier divisor is the given by a covering $\{U_i\}i \in I$ of $X$ and non zero elements $f_i \in \Gamma(U_i, \mathcal{K})X)^\times)$ such that on $U_i \cap U_j$ we have $f_i/f_j \in \Gamma(U_i \cap U_j;\mathcal{O}_X^\times)$.

**3.12 Definition.** *A Cartier divisor is said to be effective if $f_i \in \Gamma(U_i,\mathcal{O}_X)$.*

**3.13 Example.** Consider the curve $\mathbb{P}^1_\mathbb{C}$. We can describe it in the following way: two open sets $U_1$ and $U_2$ each of them isomorphic to $\mathbb{C}$, the first with variable $z$ and the second with variable $w$; the intersection of them is $U_{12}$ which is $\mathbb{C}\setminus\{0\}$ in $U_i$ and the variable $z$ becomes $1/w$. A Cartier divisor on $\mathbb{P}^1$ is for instance:

– a polynomial of degree $n$ in $z$: $f(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0$;

– the polynomial in $w$: $g(w) = a_0 w^n + \ldots + a_n$.

One easily checks that over $U_{12}$ we have $f(z)/g(w) = z^n$.

**3.14 Definition.** *A Cartier divisor is principal if it is image of an element of $\mathcal{K}(X)^\times$ via the natural map $\mathcal{K}(X)^\times \to CaDiv(X)$.*

It is not difficult to see, via the exact sequence of sheaves of abelian groups

$$0 \to \mathcal{O}_X^\times \longrightarrow \mathcal{K}(X)^\times \longrightarrow CaDiv(X) \to 0$$

that the group $CaDiv(X)/\mathcal{K}(X)^\times$ is isomorphic to $H^1(X,\mathcal{O}_X^\times)$.

If $X$ is s a smooth projective curve, and $D$ is a Cartier divisor on it, we can associate to it a divisor in the following way:

Let $(U_i, f_i)$ be a Cartier divisor on $X$, We associate to it the divisor $D' = \sum_i m_p[p]$ in the following way: suppose that $p \in U_i$ then $m_p = v_p(f_i)$; one easily checks that this is independent on the choice of $i$ or on the representative of the Cartier divisor.

Similarly for the ring of integers of a number field.

Since a smooth projective curve and the ring of integers of a number fields are locally factorial it is not difficult to prove:

**3.15 Theorem.** *Let $X$ be either a smooth projective curve or the spectrum of the ring of integers of a number field then:*

$$Pic(X) \simeq H^1(X, \mathcal{O}_X^\times).$$

*Where, in the arithmetic case, $Pic(X) = Cl(O_K)$.*

We can rewrite the theory of degree of divisors on curves in terms of line bundles and Cartier divisors:

**3.16 Definition.** *A line bundle $L$ on $X$ is a locally free sheaf of rank one.*

We can explicitly describe a line bundle on $X$ in the following way: A line bundle $L$ on $X$ is a sheaf $L$ on $X$ for which there exists a covering $\mathcal{U} = \{U_i\}$ of $X$ by open sets such that there exists an isomorphism $\varphi_i : \mathcal{O}_{U_i} \overset{\simeq}{\to} L|_{U_i}$. Over $U_i \cap U_j$ the map $\varphi_j^{-1} \circ \varphi_i : \mathcal{O}_{U_i \cap U_j} \to \mathcal{O}_{U_i \cap U_j}$ defines an element $g_{ij} \in \Gamma(U_i \cap U_j; \mathcal{O}^\times)$. If $L_1$ and $L_2$ are two line bundles, then $L_1 \otimes L_2$ is again a line bundle on $X$; thus the set of line bundles (up to isomorphism) is an abelian group.

One easily verify that $\{g_{ij}\}$ is a cocycle so it defines an element of $H^1(X, \mathcal{O}^\times)$. Two lines bundles define the same cocycle if and only if they are isomorphic. Thus the set of line bundles on $X$ is an abelian group isomorphic to the group of line bundles on $X$.

Consequently, we see that there is an isomorphism between the group of Cartier divisors up to principal Cartier divisors and the group of line bundles up to isomorphism.

**3.17 Definition.** *The Line bundle associated to a divisor $D = (U_i, f_i)$ is denoted by $\mathcal{O}(D)$ and explicitely given over every open set $U_i$ by free line bundle $\frac{1}{f_i} O_{U_i}$: they are glued together in the obvious way. Similarly we will denote by $\mathcal{O}(-D)$ the line bundle $\mathcal{O}(D)^{\otimes -1}$. If $L$ is a line bundle on $X$, we will denote by $L(D)$ the line bundle $L \otimes \mathcal{O}(D)$: similarly for $L(-D)$.*

Conversely, given a line bundle $L$ over $X$; we can describe it as a covering $\{U_i\}$ and cocycle functions $\{g_{ij}\}$. A Cartier divisor $D$ is such that $\mathcal{O}(D) = L$ if $D$ can be described by functions $f_i$ on $U_i$ with $f_i = g_{ij} f_j$.

**3.18 Definition.** *Let $X$ be a smooth projective curve and $L$ be a line bundle on it. A meromorphic section of $L$ is a Cartier divisor $D$ such that $\mathcal{O}(D) \simeq L$.*

More in general we can prove the following proposition:

**3.19 Proposition.** *Let $X$ be a scheme and $L$ be a line bundle over it. We can associate to every morphism of sheaves $\varphi : \mathcal{O}_X \to L$ a Cartier divisor $D_\varphi$ on $X$ which is effective and $\mathcal{O}(D) = L$. Conversely, every effective Cartier divisor $D$ on $X$ such that $\mathcal{O}(D) \simeq L$ give rise to an injective morphism $\psi_D : \mathcal{O}_X \to L$. Given $\varphi : \mathcal{O}_X \to L$, we can find a unit $u \in \Gamma(X, \mathcal{O}_X)$ such that $\psi_{D_\varphi} = u \cdot \varphi$.*

*Sketch of Proof:* Suppose that we have a map $\varphi : \mathcal{O}_X \to L$. We can choose a covering $\{U_i\}_{i\in I}$ of $X$ by affine open sets such that $L|_{U_i}$ is free. Consequently $\varphi|_{U_i}$ is a morphism $\mathcal{O}_{U_i} \to \mathcal{O}_{U_i}$. Define $f_i = \varphi(1)$ on $U_i$. By definition $f_i$ is in $\mathcal{O}_{U_i}$ and $f_i/f_j$ is a unit of $U_i \cap U_j$. So $(U_i, f_i)$ is an effective Cartier divisor on $X$. We leave as exercice to prove the fact that $\mathcal{O}(D) = L$ and the other part of the proposition.

One easily checks (by local computations again) that this construction is compatible with the construction given on curves.

**3.20 Definition.** *By the proposition above, given an $D$ effective line bundle on a Scheme $X$, it corresponds to a map $\psi_D : \mathcal{O}_X \to \mathcal{O}(D)$. The* support *of $D$ is the closed set of $X$ where the map $\psi_D$ is* not *an isomorphism.*

**3.21 Example.** Let $X$ be a smooth projective curve. Suppose that $D$ is the divisor $\sum n_p[p]$. Then the support of $D$ is the set of points $p$ such that $n_p \neq 0$

Let' s consider again a smooth projective curve $X$.

**3.22 Definition.** *Let $L$ be a line bundle over $X$. The* degree $\deg(L)$ *of $L$ is the degree of any meromorphic section of $L$. This definition is well posed (it is independent on the choice of the meromorphic section).*

Let $\eta : Spec(\mathbb{C}(X)) \to X$ be the generic point of $X$. Given a line bundle $L$ on $X$, we define $L_\eta$ the pull back $\eta^*(L)$ of it to $Spec(\mathbb{C}(X))$. It is a vector space of dimension one on $\mathbb{C}(X)$. It is easy to see that to give a meromorphic section of $L$ is equivalent to give a non zero element of $L_\eta$. For each closed point $p \in X$, Let $\mathcal{O}_{X,p}$ be the local ring of $X$ at $p$. We have a sequence of inclusions of schemes

$$Spec(\mathbb{C}(X)) \hookrightarrow Spec(\mathcal{O}_{X,p}) \hookrightarrow X.$$

We will denote by $L_p$ the free $\mathcal{O}_{X,P}$–module of rank one obtained taking the restriction of $L$ to $Spec(\mathcal{O}_{X,p})$. Observe that $L_p \otimes_{\mathcal{O}_{X,p}} \mathbb{C}(X) = L_\eta$.

This defines, for each closed point $p \in X$, naturally a norm on $L_\eta$ in the following way: fix $p \in X$; take $m \in L_\eta$ non zero and define

$$\|m\|_p := \sup\{\|\lambda\|_p \ / \ \lambda \in \mathbb{C}(X) \ \text{ and } \ \lambda^{-1}m \in L_p\}.$$

One easily check that, for the trivial line bundle this is the norm we defined before. never the less observe that the definition of the norm on $L_\eta$ *depends* on the choice of the norm on $\mathbb{C}(X)$. Moreover, one prove (exercise) that

**3.23 Theorem.** *Let $L$ be a line bundle over a smooth projective curve as before and $m$ a meromorphic section over it then:*

$$\deg(L) = -\sum_{p\in X} \log(\|m\|_p).$$

This formula tells us what we need to do to obtain a similar definition in the arithmetic case: Let $O_K$ be the ring of integers of a number field. Let $L$ be a locally free $O_K$ module of rank one (it is a line bundle over $Spec(O_K)$). For every embedding $\sigma : K \to \mathbb{C}$ we can consider the one dimensional $\mathbb{C}$ vector space $L \otimes \mathbb{C}$ (where $\mathbb{C}$ is a $O_K$ algebra via $\sigma$). We will denote it $L_\sigma$.

**3.24 Definition.** *An hermitian line bundle over $Spec(O_K)$ is a locally free $O_K$ module with, for every embedding $\sigma$ of $K$ in $\mathbb{C}$ an hermitian norm $\|\cdot\|_\sigma$ on $L_\sigma$ with the following restriction: If $\sigma = \overline{\tau}$ (complex conjugate embedding) then the norm at $L_\sigma$ is the conjugate of the norm at $L_\tau$.*

**3.25 Remark.** It is important to observe that, exactly as in the geometric case, for every maximal ideal $\mathfrak{p}$ of $O_K$, the module $L_\mathfrak{p}$ give rise to a non archimedean norm $\|\cdot\|_\mathfrak{p}$ on the $K$ vector space $L_K$: let $m \in L_K$ and $\mathfrak{p} \in M_{fin}$ then

$$\|m\|_\mathfrak{p} := \sup\{\|\lambda\|_\mathfrak{p} \ \ / \ \ \lambda \in K \ \text{ and } \ \lambda^{-1}m \in L_\mathfrak{p}\}.$$

We can now state t the following theorem

**3.26 Theorem.** *The compactified Picard group $\widehat{Pic}(O_K)$ is canonically isomorphic to the group of hermitian line bundles of $O_K$ modulo isomorphism. Thus we can speak about the degree of an hermitian line bundle of $O_K$.*

*If $L$ is an hermitian line bundle on $O_K$ and $m \in L_K$ is a non zero element. Then we can compute the degree of $L$ by the formula*

$$\widehat{deg}(L) = -\sum_\sigma \log\|m\|_\sigma - \sum_{\mathfrak{p}\in Specmax(O_K)} \log\|m\|_\mathfrak{p}.$$

The reader can prove it by exercise following the strategy used in the geometric situation. It is worth remarking how the two theories, the geometrical and the arithmetic are similar and formally identical.

**3.27 Example.** It is possible to prove that $Pic(\mathbb{P}^1) \simeq \mathbb{Z}$ and it is generated by the line bundle $O_\mathbb{P}(1)$ associated to a point: $\mathcal{O}_\mathbb{P}(1) = \mathcal{O}_\mathbb{P}(p)$ where $p \in \mathbb{P}^1(\mathbb{C})$ is a closed point. Its degree is 1.

**3.28 Example.** $\widehat{Pic}(\mathbb{Z})$ is isomorphic to $\mathbb{R}_{>0}$ with the multiplicative structure: if $\lambda \in \mathbb{R}_{>0}$ the corresponding hermitian line bundle on $\mathbb{Z}$ is given by $(\mathbb{Z}, \|1\| = \lambda)$; its degree is $-\log\lambda$.

**3.29 Definition.** *Let $X$ be a smooth projective curve and $L$ be a line bundle on it. Let $m \in L_\eta$ a meromorphic section of $L$. We associate to it the divisor*

$$div(m) = \sum_{p \in X} - \log(\|m\|_p)[p].$$

*One checks that $\mathcal{O}(div(m)) = L$. Similarly, if $L$ is an hermitian line bundle over the ring of integers of a number field $O_K$, and $m \in L_K$, we associate to it the compactified divisor*

$$\widehat{div}(m) = - \sum_{\sigma \in M_\infty} \log \|m\|_\sigma [\sigma] - \sum_{\mathfrak{p} \in M_fin} \frac{\log \|m\|_p}{d_\mathfrak{p} \log(p)} \cdot [\mathfrak{p}].$$

*where $p$ is the positive prime number such that $(p) = \mathfrak{p} \cap \mathbb{Z}$ and $d_\mathfrak{p}$ is the residual degree of $\mathfrak{p}$. Of course the hermitian line bundle $\mathcal{O}(\widehat{div(m)})$ is isomorphic, as hermitian line bundle to $L$.*

The definition above shows the correspondence between line bundles and divisors. One remark again that, *mutatis mutandis*, the arithmetic and the geometric situations are formally similar.

One can check that, essentially by definition the following holds

**3.30 Proposition.** *Let $X$ be a smooth projective curve and $L$ be a line bundle over it. Let $U$ be a open set of $X$. Then*

$$\Gamma(U, L) = \{m \in L_\eta \ / \ \|m\|_p \leq 1 \ \ \forall \, p \in U\}.$$

*In particular $H^0(X, L) = \{m \in L_\eta \ / \ \|m\|_p \leq 1 \ \ \forall \, p \in X\}$.*

Indeed a section $m \in L_\eta$ is in $\Gamma(U, L)$ if it belongs to $L_p$ for every $p \in U$, which is equivalent to say that $m$ has norm at $p$ less or equal then one. Conversely a section in $\Gamma(U, L)$ can be extended to a section in $L_\eta$ because $\eta$ is contained in every open set.

In particular we obtain that

**3.31 Proposition.** *Let $L$ be a line bundle on $X$, then if $H^0(X, L) \neq \{0\}$ we have that*

$$\deg(L) \geq 0$$

.

**3.32 Exercise.** Prove that if $\deg(L) = 0$ and $H^0(X, L) \neq \{0\}$, then $L \simeq \mathcal{O}_X$.

Remark that one can prove that $H^0(X, L)$ is a *finite dimensional* vector space over $\mathbb{C}$.

By analogy with the geometric case we can give the following definition

**3.33 Definition.** *Let $O_K$ be the ring of integers of a number field and $L$ be an hermitian line bundle over it. We define*

$$H^0_{Ar}(L) := \{m \in L_K \ / \ \|m\|_{\mathfrak{p}} \leq 1 \ \ \forall \mathfrak{p} \in M_K\}.$$

The set $H^0_{Ar}(L)$ is *only a set*: there is no structure of module on something; for instance the sum of two elements in $H^0_{Ar}(L)$ is not always an element of $H^0_{Ar}(L)$ (one can check that it is a module over the group of roots of unities of $K$).

Observe that the set of elements $\{m \in L_K \ /\|m\|_{\mathfrak{p}} \leq 1 \ \ \forall \mathfrak{p} \in M_{fin}\}$ is the underlying $O_K$ module $L$; consequently, the points $\sigma \in M_\infty$ may be considered as points at infinity of a scheme which is obtained by "compactifying" the scheme$Spec(O_K)$.

We can prove that

**3.34 Proposition.** *The set $H^0_{Ar}(L)$ is finite*

*Sketch of proof*: Let $M'_\infty$ be a subset of $M_\infty$ obtained taking all the real embedding and only one representative within two complex conjugate embedding. The real vector space $L_{\mathbb{R}} := \prod_{\sigma \in M'_\infty} L_\sigma$ is a finite dimensional vector space of dimension $[K : \mathbb{Q}]$ and equipped with a norm $\|(m_\sigma)_{\sigma \in M'_\infty}\| = \sup \|m_\sigma\|_\sigma$.

One can check that the image of the natural map

$$\begin{aligned} L &\longrightarrow \prod_{\sigma \in M'_\infty} L_\sigma \\ m &\longrightarrow (m \otimes_\sigma 1)_{\sigma \in M'_\infty} \end{aligned}$$

Is a lattice in $L_{\mathbb{R}}$. Moreover it is easy to see that $L$ is the set of elements $m$ in $L_K$ for which $\|m\|_{\mathfrak{p}} \leq 1$ for every $\mathfrak{p} \in M_{fin}$. The conclusion follows because $H^0_{Ar}(L)$ is the intersection of the unit ball of $L_{\mathbb{R}}$, which is compact, with $L$ which is a lattice.

In order to check that $L$ is a lattice in $L_{\mathbb{R}}$ it suffices to prove that $L \otimes_{\mathbb{Z}} \mathbb{R} = L_{\mathbb{R}}$ and this is an exercise on flat base change.

Given an hermitian line bundle $L$ over $O_K$ and a section $m \in L$. We can compute the degree of $L$ by the following formula

**3.35 Proposition.** *If $m \in L$ then*

$$\widehat{\deg}(L) = \log(Card(L/mO_K)) - \sum_{\sigma \in M_\infty} \log \|m\|_\sigma. \tag{3.35.1}$$

*Proof:* We need to prove that $-\sum_{\mathfrak{p} \in M_{fin}} \log \|m\|_{\mathfrak{p}} = \log(Card(L/mO_K))$. We start by remarking that if $R$ is a Dedekind domain, $L$ is a trivial $R$ module and $m \in L$, then the Chinese remainders theorem tells us that the map $L/mR \to \prod_{\mathfrak{p} \in Specmax(R)} L_{\mathfrak{p}}/mR_{\mathfrak{p}}$ is an isomorphism. Thus, by localization we see that

$$L/mO_K \simeq \prod_{\mathfrak{p} \in Specmax(O_K)} L_p/mO_{K,\mathfrak{p}}.$$

Consequently we need to show that $-\log \|m\|_p = \log(Card(L_{\mathfrak{p}}/mO_{K,\mathfrak{p}}))$ and this is true by definition.

It is important to read the similarities between the formula of the degree in the geometric and the arithmetic contest:

Suppose that $X$ is an affine algebraic curve and $\overline{X}$ is its projective compactification. Then $\overline{X} \setminus X = D$ where $D = \sum_i^n p_i$ is a divisor which we can call *the divisor at infinity of* $\overline{X}$. Suppose that $L$ is a line bundle on $\overline{X}$ and $s$ is a meromorphic function over it. Then $div(s) = \sum_{q\in X} v_q(s)q + \sum_{p\in D} v_p(s)p$ and we obtain the formula

$$\deg(L) = \sum_{q\in X} v_q(s) - \sum_{p\in D} \log \|s\|_p. \tag{3.37.1}$$

Similarly, let $K$ and $O_K$ be its ring of integers. We can consider the embeddings $\sigma : K \hookrightarrow \mathbb{C}$ as points at infinity of a sort of compactification of $S = Spec(O_K)$. denote by $M_\infty$ the set of such embeddongs, we call it t*he set of points at infinity of* $S$. Let $L$ be an hermitian line bundle over $S$ and $s$ be a non zero element of $L_K$. We associate to $s$ the compactified divisor $\widehat{div}(s) = \sum_{\mathfrak{p}\in Specmax(O_K)} v_{\mathfrak{p}}(s)\mathfrak{p} - \sum_{\sigma\in M_\infty} \log \|s\|_\sigma[\sigma]$ and we have the formula

$$\widehat{\deg}(L) = \sum_{\mathfrak{p}\in Specmax(O_K)} v_{\mathfrak{p}}(s) \log(Card(O_K/\mathfrak{p})) - \sum_{\sigma\in M_\infty} \log \|s\|_\sigma. \tag{3.38.1}$$

So we see that the degree of a line bundle may be computed by using a meromorphic section of it in both cases. Moreover this degree is sum of two terms: the first one is a sum over the zeroes or the poles of the sections in the fixed affine part of the scheme. The second part is obtained computing norms at places corresponding to points at infinity.

# 4 Lecture III.

**4.1** *Construction of the pull back of line bundles.* First of all we recall the notion of pull back of coherent sheaf. Let $f : X \to Y$ be a morphism between schemes and $F$ a coherent sheaf on $Y$. We define the sheaf $f^{-1}(F)$ on $X$ in the following way: for every open set $U \subseteq X$ we consider the presheaf $U \to \lim_{f(U)\subseteq V} \Gamma(V, F)$ where the limit is on all the open sets containing $f(V)$. Then $f^{-1}(F)$ is the sheaf associated to this presheaf. In particular, if $F = \mathcal{O}_Y$, then $f^{-1}(\mathcal{O}_Y)$ is a sheaf in algebras with a natural map $f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$. If $B$ is a coherent sheaf over $Y$, then $f^{-1}(F)$ is a sheaf in $f^{-1}(\mathcal{O}_Y)$ modules. We eventually define

$$f^*(F) := \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(F).$$

It will be a coherent sheaf on $X$.

**4.1 Example.** Suppose that $X$ and $Y$ are affine: $X = Spec(B)$ and $Y = Spec(A)$. Then the morphism $f : X \to Y$ corresponds to a morphism of rings $A \to B$ and $F$ corresponds to a $A$–module $\tilde{F}$. In this case the sheaf $f^*(F)$ is the sheaf associated to the $B$–module $B \otimes \tilde{F}$.

Observe that if $F$ is a line bundle on $Y$ then $f^*(F)$ is a line bundle on $X$. One easily verify that the natural map $f^* : Pic(Y) \to Pic(X)$ is a group morphism.

**4.2** *Construction of the pull back of a divisor.* Suppose that $D$ is an effective divisor on $Y$ and suppose that the image of $f : X \to Y$ is not contained in the support of $D$. Choose a covering of $Y$ by affine open sets $U_i = Spec(A_i)$ (which "trivializes" $D$) and a consequent covering of $X$ by affine open sets $V_{ij} = Spec(B_{ij}$ such that $f(V_{ij}) \subseteq U_i$. So the restriction of the map $f$ to $V_{ij}$ corresponds to a morphism of rings $A_i \to B_{ij}$. If $D$ is given by the data $(U_i, f_i)$ with $f_i \in A_i$ then we define an effective Cartier divisor $f^*(D)$ on $X$ by $(V_{ij}; f_i \otimes_{A_i} 1)$. The reader can check that the condition that $f(X)$ is not contained in the support of $D$ exactly means that $f_i \otimes_{A_i} 1$ is not a zero divisor on $B_{ij}$. Moreover

$$\mathcal{O}_X(f^*(D)) \simeq f^*(\mathcal{O}_Y(D)).$$

One can check that, set theoretically, $f^*(D)$ is the preimage of $D$ via $f$. The definition is more complicated because in this way we give a structure of closed subscheme to it (and not just of closed subset!)

To resume:

**4.2 Proposition.** *Given a morphism between two schemes $f : X \to Y$ then we have a natural morphism of groups*

$$f^* : Pic(Y) \to Pic(X).$$

*Moreover, each time we have an effective divisor $D$ on $Y$ such that $f(X)$ is not contained in the support of it, then we can define a cartier divisor $f^*(D)$ on $X$. Set theoretically it is the preimage of $D$. The two constructions above are compatible.*

**4.3 Remark.** One should remark that the pull back of a divisor is, when defined, quite easy to define and clear to understand geometrically: it is essentially the preimage of it. The schematic structure can be computed locally. But the pull back of a divisor is not always defined. The pull back of a line bundle is less intuitive but always defined. The fact that they are compatible will be the crucial point of the theory.

**4.4 Example.** If $X$ is the projective space $\mathbb{P}^N$. Every hyperplane is a Cartier divisor of it. It is not difficult to say that all the hyperplanes are linearly equivalent and the associated line bundle is called the tautological line bundle of $\mathbb{P}^N$ and denoted by $\mathcal{O}_{\mathbb{P}}(1)$. The line bundle $\mathcal{O}_{\mathbb{P}}(1)^{\otimes n}$ is usually denoted with $\mathcal{O}_{\mathbb{P}}(n)$. If $n \geq 0$, then the global sections of $\mathcal{O}_{\mathbb{P}}(n)$ are the hypersurfaces of degree $n$. Remark that through each point there we can find a hypersurface of fixed degree and given two points, we can find an hypersurface which pass through one point and not through the other.

**4.5 Definition.** *Let $X$ be an algebraic variety. A line bundle $L$ on $X$ is said to be* ample *if, for some positive integer $n > 0$ and a closed embedding $\iota : X \hookrightarrow \mathbb{P}^N$ such that $\iota^*(\mathcal{O}_{\mathbb{P}}(1)) \simeq L^{\otimes n}$. The line bundle $L$ is said to be* very ample *if $n = 1$.*

**4.6 Remark.** Observe that if a line bundle is ample then for every $n \gg 0$ the vector space $H^0(X, L^n)$ is not empty and, for every couple of points $p$ and $q$ of $X$ there is a global section of $L^n$ whose support contains $p$ and not $q$.

**4.7** *Points from a geometric point of view.* Before we state the properties of the heights we need to generalize the notion of point of a scheme.

Given a scheme $C$, one can introduce the notion of $C$–point of a variety $X$. This notion generalize the notion of point of a variety

**4.7 Definition.** *If $X$ is a variety (or a scheme) and $C$ is a scheme , in the sequel we will denote by $X(C)$ set $\{p : C \to X\}$ of morphisms of $C$ in $X$. An element of $X(C)$ is called a $C$–point of $X$.*

Observe that:

(1) The set of points $X(C)$ is in bijection with the set of section of the projection $X \times C \to C$. Indeed the graph of a point is a section of the projection.

(2) If $f : X \to Y$ is a morphism of varieties (or schemes), then there is a natural map $f : X(C) \to Y(X)$: indeed $f(p) = f \circ p : C \to Y$.

(3) If $C$ is a point (the spectrum of a field), then we see that $X(C)$ is the set of closed points of $X$.

(4) We can generalize the notion of point to a relative situation: Suppose that $C$ is a scheme and $f : X \to C$ a morphism of schemes, then we define $X(C)$ as the set of morphisms $p : C \to X$ such that $f \circ p = Id_C$.

The main example which explain the definition is the following:

**4.8 Example.** Let $K$ be any field and $L/K$ be an extension of it. Consider the scheme $\mathbb{A}^N_K = Spec(K[z_1 \ldots, z_N])$. Suppose that we have a $L$–rational point of it $p : Spec(L) \to \mathbb{A}^N_K$. It corresponds to a map $K[z_1, \ldots, z_N] \to L$. Denote by $a_i$ he image of $z_i$; then the point $p$ is the point of coordinates $(a_1, \ldots, a_N)$ in $\mathbb{A}^N_K$. Thus we see that a $L$–rational point is a point with coordinates in $L$. The geometric definition coincides with the intuitive definition but it is intrinsic and do not depend on the choice of the coordinates.

**4.9** *Height theory in the geometric contest.* We will now explain how the theory of degrees of line bundles on curves can be used to measure the first degree of complexity of a curve in a variety. Of course the basic measure of complexity is the fact that the curve is a subvariety of dimension one.

**4.9 Definition.** *Suppose that $X$ is a smooth projective variety and $L$ a line bundle over it. Let $C$ a smooth projective curve and suppose we have a morphism $p : C \to X$. The integer* $\deg(p^*(L))$ *is called the* height *of $p$ with respect to $L$ and denoted by $h_L(p)$.*

**4.10 Example.** Suppose that $C = \mathbb{P}^1$ and $X = \mathbb{P}^N$. let $p : C \to X$; to simplify, suppose that it is an embedding. The intersection of an hyperplane with the image of $C$ is a divisor of degree $d$ for a suitable positive integer $d$ (which is usually called the degree of $C$). Consequently, $p^*(\mathcal{O}(1)) = \mathcal{O}(d)$. From this we deduce that $h_{\mathcal{O}(1)}(p) = d$.

The height is the second degree of complexity of a curve in a variety; as we said before, the first degree of complexity is the fact that the curve is a subvariety of dimension one.

The following properties of heights are essentially evident:

**4.11 Proposition.** Properties of geometric heights *: Let $C$ be a smooth projective curves and $X$ a projective variety. Then:*

*(1) If $L_1$ and $L_2$ are line bundles of $X$ and $p \in X(C)$ then*

$$h_{L_1 \otimes L_2}(p) = h_{L_1}(p) + h_{L_2}(p).$$

*(2) (Functoriality of heighs)If $f : X \to Y$ is a morphisms of varieties, $p \in X(C)$ and $L$ a line bundle on $Y$, then*

$$h_L(f(p)) = h_{f^*(L)}(p).$$

*(3) If $D$ is an effective Cartier divisor on $X$ and $p \notin D$ (this means that $p$ do not factorizes through the support of $D$) then*

$$h_{\mathcal{O}(D)}(p) \geq 0.$$

*Proof:* Each of the properties are easy consequences of the definition. The last one is proved as follows: Since $p$ is not contained in $D$, the pull back, via $p$ of $D$ is an effective divisor on $C$ whose associated line bundle is $p^*(\mathcal{O}(D)$; consequently the degree of $p^*(\mathcal{O}(D)$ must be positive.

**4.12 Example.** : The following example shows that, if $p$ is contained in the support of $D$, then it may happen that the height of $p$ with respect to $\mathcal{O}(D)$ is negative: Let $X$ be a surface and $q$ be a closed point on it. Let $\tilde{X} \to X$ be the blow up of $X$ in $q$. Let $E$ be the exceptional divisor of $\tilde{X}$ and $p : \mathbb{P}^1 \to \tilde{X}$ the inclusion of $E$ in $\tilde{X}$. Then $p^*(\mathcal{O}(E)) = \mathcal{O}(-1)$ thus the height of $p$ with respect to $\mathcal{O}(E)$ is negative.

One easily see that the property (3) of proposition 4.11 implies the following corollary

**4.13 Corollary.** *Let $X$ be a projective variety and $L$ be an ample line bundle on $X$. Then for every $p \in X(C)$ we have $h_L(p) \geq 0$*

*Proof:* By property (2) and the definition of ample line bundle we may suppose that $X$ is $\mathbb{P}^N$ and $L = \mathcal{O}(1)$. In this case, either $p$ is constant, and consequently $h_L(p) = 0$ or $p$ is not constant, so $h_L(p)$ is a multiple of the degree of the image of the curve $C$, so a positive number.

**4.14** *Bounded families.* The most important property of height is that, if we can bound the height of a set of $C$–points on a projective variety, then this set is "controllable". In the best situation, this set will be finite.

**4.14 Example.** Let $C = \mathbb{P}^1_k$, and $X = \mathbb{P}^1_F$. We take $L = \mathcal{O}(1)$. Rational points $p \in X(C)$ are algebraic maps $\varphi_p : \mathbb{P}^1 \to \mathbb{P}^1$. Let $p \in X(C)$ then, as we saw before, $h_L(p) = \deg(\varphi_p(\mathcal{O}(1))) = \deg(\varphi_p)$. Suppose that $S \subset X(C)$ is a set of rational points and suppose that it is a set of bounded height. This means that there exists a constant $A$ such that, for every $p \in S$ we have that $h_L(P) \leq A$. Consequently, the corresponding set of maps $\varphi_p$, for $p \in S$ is a set of rational maps of bounded degree from $\mathbb{P}^1_k$ to $\mathbb{P}^1_k$.

Fix a positive integer $n$. The set of degree $n$ maps from $\mathbb{P}^1_k$ to $\mathbb{P}^1_k$ is in bijection with the set of lines in $H^0(\mathbb{P}^1; \mathcal{O}(n))$ which is isomorphic to set of homogeneous polynomials in two variable ; thus it is in bijection with the $\mathbb{C}$ rational points of $\mathbb{P}^{n-1}$.

From the observation above, we deduce that there exists a variety $Y$ and a natural injection $S \subset Y(k)$.

The example above can be generalized. If $C$ is a smooth projective curve, we will see that the set of $C$ points of bounded height of a variety can be parametrized by the $\mathbb{C}$–rational points of a variety defined over $\mathbb{C}$.

We concluded the example above by saying that there is a "natural inclusion" $S \subset Y(k)$. We would now clarify what we mean by "natural inclusion".

**4.15 Definition.** *Let $C$ and $Y$ be two varieties over $\mathbb{C}$. Suppose that $f_1 : C \to Y$ and $f_2 : C \to Y$ are two morphisms. We will say that $f_1$ is equivalent to $f_2$ are equivalent if there exist an isomorphism $\varphi : C \to C$ such that the following diagram is commutative:*

$$\begin{array}{ccc} C & \xrightarrow{f_1} & Y \\ {\scriptstyle\varphi}\downarrow & \nearrow_{f_2} & \\ C & & \end{array} \quad .$$

One can remark that in this case, the images of the morphisms $f_1$ and $f_2$ are the same.

Suppose that $C$ and $Y$ are projective varieties. Let $U$ a quasi projective variety. A morphism $F : C \times U \to Y$ may be seen as a family of morphisms from $C$ to $Y$ parametrized by $U(\mathbb{C})$. For every point $u \in U$ the map

$$\begin{aligned} F_u : & X \longrightarrow Y \\ & p \longmapsto F(p, u) \end{aligned}$$

is a morphism from $X$ to $Y$ so a $C-$ point of $Y$.

**4.16 Definition.** *Let $C$ and $Y$ be projective varieties over $\mathbb{C}$. Let $S$ be a set of $C-$ points of $Y$. We will say that $S$ is a* bounded family of morphisms *if there exists a quasi projective variety $U$ a morphism $F : C \times U \to Y$ a subset $V \subseteq U(\mathbb{C})$ such that every $s \in S$ is equivalent, as a $C$–point, to a morphism $F_u$ for some $u \in V(\mathbb{C})$.*

Essentially, a bounded family of $C$–pointsis a set of $C$–points which appear as a subset of a parametrized family of $C$–points.

One of the cornerstones of the theory of heights in the geometric contest is the following: bounded height means bounded family:

**4.17 Theorem.** *Suppose that $C$ is a smooth projective curve and $X$ is a projective variety. Let $L$ be an ample line bundle on $X$ and $S \subset X(C)$ be a set of $C$–points. Then $S$ is a bounded family of $C$–points of $X$ if and only if there exists a constant $B$ such that, for every $p \in S$ we have*

$$h_L(p) \leq B.$$

The proof of this theorem is complicate and we will not give it here. We just remark that one can suppose that $X$ is $\mathbb{P}^N$ and $L = \mathcal{O}_{\mathbb{P}}(1)$. So we need to prove that the set of curves $\mathbb{P}^N$ isomorphic to a fixed curve and with bounded degree is a bounded family. The reader may prove this as an exercise in the case when $C = \mathbb{P}^1$: In this case each morphism of $\mathbb{P}^1$ in $\mathbb{P}^N$ such that the image is a curve of degree $d$, corresponds to a subspace of dimension $N + 1$ in the space of the homogeneous polynomials of degree $d$ in two variables. So each map corresponds to a point in a suitable Grassmannian...

# 5 Lecture IV.

**5.1** *Heights in the arithmetic contest.* Once the theory of heights has been developed in the geometric contest, the arithmetic theory become clear. One recall that in order to define the degree of line bundles, we need to introduce an hermitian structure over line bundles on "arithmetic curves". Consequently, in order to introduce height theory in the geometric contest, one needs to introduce hermitian structure on line bundles on varieties.

**5.1** *Arithmetic varieties and models of varieties.* First of all we fix a number field $K$ and its ring of integers $O_K$.

Let $X_K$ be a variety (or more generally a scheme) defined over $K$; we recall that this means that we have a morphism of schemes $X_K \to Spec(K)$ which is locally of finite presentation: $X$ is covered by a finite set of open affine sets $U_i = Spec(A_i)$ and for every $i$ there exists a surjection of $K$ algebras $K[x_1, \dots, x_{n_i}] \to A_i$. We also fix a line bundle $L_K$ over $X_K$.

For every infinite place $\sigma : K \hookrightarrow \mathbb{C}$ we will denote by $X_\sigma$ the variety $X_K$ seen as a $\mathbb{C}$ variety via $\sigma$: to be more precise the variety $X_\sigma$ is the variety for which the following diagram is cartesian

$$\begin{array}{ccc} X_\sigma & \xrightarrow{\sigma_X} & X_K \\ \downarrow & & \downarrow \\ Spec(\mathbb{C}) & \xrightarrow{\sigma} & Spec(K). \end{array}$$

We will denote by $L_\sigma$ the line bundle $\sigma_X^*(L)$.

When we study varieties over number fields, one is often interested to study their properties when we restrict our attention to finite fields. For this reason we introduce the notion of model, which generalize the concept introduced in (4) of definition 4.7.

**5.1 Definition.** *(a) An arithmetic variety is a scheme $\mathcal{X}$ with a faithfully flat morphism of finite presentation $f : \mathcal{X} \to Spec(O_K)$. The generic fibre of $\mathcal{X}$ is the $K$ variety $X_K$ for which the following diagram is cartesian*

$$\begin{array}{ccc} X_K & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ Spec(K) & \longrightarrow & Spec(O_K). \end{array}$$

*(b) Let $X_K$ be a variety over $K$. A model of $X_K$ over $O_K$ is an arithmetic scheme $\mathcal{X}$ whose generic fibre is isomorphic to $X_K$.*

**5.2 Example.** The scheme $\mathbb{P}^N_{O_K} = Proj[z_0, \dots z_N] \to Spec(O_K)$ is an arithmetic variety and it is a model of $\mathbb{P}^N$; it is called the arithmetic projective space over $O_K$. Similarly $\mathbb{A}^N_{O_K} = Spec(O_K[z_1 \dots . z_N]) \to Spec(O_K)$ is an affine arithmetic variety, which is a model of $\mathbb{A}^N_K$ and it is called the arithmetic affine space.

**5.3** *Models always exist.* If $X_K$ is a projective variety, it is not difficult to find a model for it: since $X_K$ is projective and defined over $K$, it is defined as the zeroes of an homogeneous ideal $I_{X_K} = (F_1, \dots, F_s)$ where $F_i \in K[z_0, \dots, z_N]$ are homogeneous polynomials. Since we may multiply the $F_j$ by constants, we may suppose that $F_i \in O_K[z_0, \dots, z_N]$. Suppose that $n$ is an integer such that $O_K[\frac{1}{n}]$ is a principal ideals domain. Then we may suppose that the coefficients of $F_i$ are coprime in $O_P[\frac{1}{n}]$. So $X_n := Proj(O_K[\frac{1}{n}][z_0 \dots, z_N]/(F_1, \dots, F_s))$ is a closed set of $\mathbb{P}^N_{O_K[\frac{1}{n}]}$ which is an open set in $\mathbb{P}^N_{O_K}$. We take as a model of $X_K$ the closure $\mathcal{X}$ in the Zariski topology of $X_n$ in $\mathbb{P}^N_{O_K}$. Remark that, $\mathcal{X}$ may be very singular as a scheme, even if $X_K$ is a smooth variety.

It is important to remark that, given a variety $X_K$, the models of it over $O_K$ are not unique. One can show that they are all birational between them.

In the case of line bundles over $Spec(O_K)$ we introduced the notion of hermitian line bundles. We now need to introduce a similar notion in the higher dimensional situation.

First of all we recall the notion of hermitian line bundles over complex varieties. Let $X$ be a complex variety and $L$ be a line bundle over it. An hermitian structure over $L$ is essentially an hermitian metric on the fibres of $L$ which varies smoothly (or just continuously in some cases). More precisely: Suppose that $L$ is given by a covering $\{U_i\}$ and a corresponding cocycle $\{g_{ij}\}$ on $U_{ij} := U_i \cap U_j$; recall that $g_{ij}$ are non vanishing holomorphic functions on $U_{ij}$ verifying the cocycle condition $g_{ij}g_{jk}g_{ki} = 1$.

**5.3 Definition.** *A smooth (continuous) hermitian metric* $(\|\cdot\|)$ *on $L$ is a collection of smooth (continuous) functions*

$$\rho_i : U_i \longrightarrow \mathbb{R}_{>0}$$

*such that on $U_{ij}$ we have $\rho_i = |g_{ij}|^2\rho_j$. A line bundle equipped with an hermitian metric is called* hermitian line bundle.

If $V$ is an open set of $X$ and $f \in \Gamma(V; L)$ we define the norm of $f$ in the following way: $f$ is a collection of holomorphic functions $f_i : V \cap U_i \to \mathbb{C}$ such that $f_i = g_{ij}f_j$; thus we define, for $z \in V$

$$\|f\|^2(z) := \frac{|f_i|^2}{\rho_i};$$

It is easy to check that the definition is well posed.

Observe that if $L$ is equipped with a metric, then, for every integer $n$ it is easy to equip $L^{\otimes n}$ with a induced metric.

If $X$ is a point then a line bundle on it is just a vector space of dimension one and a metric is an hermitian metric on it.

**5.4 Example.** Let $X = \mathbb{P}^N$ and $L = \mathcal{O}(1)$. Fix homogeneous coordinates $[x_0 : \ldots : x_N]$ on $\mathbb{P}^N$. Every global $s$ section of $\mathcal{O}(1)$ may be represented by a homogeneous linear form $s = a_0x_0 + \ldots + a_Nx_N$ with $a_i$ not all zero. We may define metrics on $\mathcal{O}_{\mathbb{P}}(1)$ in the following way

$$\|s\|^2_{L_2}([x_0 : \ldots : x_N]) := \frac{|a_0x_0 + \ldots + a_Nx_N|^2}{|x_0|^2 + \ldots |x_N|^2}$$
$$\|s\|^2_{\sup}([x_0 : \ldots : x_N]) := \frac{|a_0x_0 + \ldots + a_Nx_N|^2}{\sup\{|x_0|^2; \ldots |x_N|^2\}}.$$

One can easily check that this formulæ define a metrics on $\mathcal{O}(1)$. Observe that the first metric is smooth while the second is just continuous. These two metrics are called respectively the $L_2$ and the sup Fubini–Study metrics.

**5.5 Exercise.** Let $K$ be a local ring and $\mathfrak{p}$ be a maximal ideal of $O_K$. Denote by $O_{\mathfrak{p}}$ the localized of $O_K$ at $\mathfrak{p}$. Consider the tautological line bundle $\mathcal{O}_{\mathbb{P}}(1)$ of $\mathbb{P}^N_{O_{\mathfrak{p}}}$. For every point $p \in \mathbb{P}^N(\mathcal{O}_{\mathfrak{p}})$ the line bundle $p^*(\mathcal{O}_{\mathbb{P}}(1))$ over $Spec(O_{\mathfrak{p}})$ is naturally equipped with a norm. Let $p := [z_0 : \ldots : z_N]$ such a point and $s := a_0z_0 + \ldots a_Nz_N$ be a global section of $\mathcal{O}_{\mathbb{P}}(1)$. Prove that

$$\|s\|(p) = \frac{\|a_0z_0 + \ldots + a_Nz_N\|_{\mathfrak{p}}}{\sup\{\|z_0\|_{\mathfrak{p}}; \ldots : \|z_N\|_{\mathfrak{p}}\}}.$$

The exercise above shows that the Fubini Study metric seems to hide an integral structure at infinite places of the projective space.

It is easy to see that, if $f : X \to Y$ is an analytic map and $(L, \|\cdot\|)$ is an hermitian line bundle, then the pull back line bundle $f^*(L)$ is naturally equipped with the structure of an hermitian line bundle: we take as functions defining the metric the functions $\rho_i \circ f$. In particular, if $p \in X$ is a point, then the $\mathbb{C}$ vector space $L|_p$ is naturally equipped with the structure of hermitian vector space of dimension one.

**5.6 Definition.** *Let $\mathcal{X}$ be an arithmetic variety and $L$ be a line bundle over it. An hermitian structure on $L$ is, for every $\sigma \in M_\infty$, an hermitian metric on the complex line bundle $L_\sigma$: with the additional condition that if $\sigma = \overline{\tau}$ then the metric on $L_\sigma$ is the complex conjugate of the metric on $L_\tau$ (which essentially means that the involved $\rho_i$ are the same). A line bundle over $\mathcal{X}$ equipped with an hermitian structure will be called an hermitian line bundle.*

Of course we can define the Compactified Picard group $\widehat{Pic}(\mathcal{X})$ of $\mathcal{X}$: it will be the abelian group of hermitian line bundles up to isometry (isomorphisms of line bundles over $\mathcal{X}$ which preserve the metrics). we have a surjective morphism of groups:

$$\widehat{Pic}(\mathcal{X}) \longrightarrow Pic(\mathcal{X}).$$

The surjectivity follows from the fact that, by using partitions of unities, one can always equip a line bundle with an hermitian metric.

Before we can develop the arithmetic heights theory, we need to explain the relation between models and rational points.

Let $X_K$ be a variety defined over $K$. Suppose that we have a model $\mathcal{X} \to Spec(O_K)$ of $X_K$. One easily see that we have an inclusion

$$\mathcal{X}(O_K) \subseteq X_K(K).$$

Indeed, let $p : \mathrm{Spec}(O_K) \to \mathcal{X}$. The map $Spec(K) \to Spec(O_K) \xrightarrow{p} \mathcal{X}$ and the identity map on $Spec(K)$ give rise, since $X_K$ is the generic fibre of $\mathcal{X}$ to a map $Spec(K) \to X_K$. The map $\mathcal{X}(O_K) \to X_K(K)$ is an inclusion because $Spec(K)$ is dense in $Spec(O_K)$ and $X_K$ is dense in $\mathcal{X}$.

For general $X_K$ the inclusion above is not a surjection: for instance the point $1/2 \in \mathbb{A}^1_{\mathbb{Q}}(\mathbb{Q})$ cannot be extended to a morphism from $Spec(\mathbb{Z})$ to $\mathbb{A}^1_{\mathbb{Z}}$. Nevertheless if $X_K$ is projective the inclusion is a bijection:

**5.7 Theorem.** *Let $R$ be a Dedekind domain and $K = Frac(R)$. Let $\mathcal{X} \to Spec(R)$ be a projective $R$–scheme and $X_K$ the generic fibre of it. Suppose that $p \in X_K(K)$ is a $K$-rational point. Then $p : Spec(K) \to X_K$ extends, in a unique way, to a $R$–morphism $P : Spec(R) \to \mathcal{X}$.*

*Proof:* We need only to show that the morphism extends. Since $\mathcal{X}$ is a closed subset of $\mathbb{P}^N_R$ for a suitable $N$, we may suppose that $\mathcal{X} = \mathbb{P}^N_R$. Indeed $X_K$ will be defined as the vanishing set of a suitable ideal of the ring of polynomials; if a homogeneous polynomial vanishes on $p$, then it will vanish on the curve image of $P$ because $Spec(K)$ is dense in $Spec(R)$.

In order to prove the theorem we may also suppose that $R$ is a discrete valuation ring because if the morphism extends to every local ring of an affine scheme it extends on the scheme itself.

Let $R$ be a discrete valuation ring with uniformizer $t$. A $K$ rational point has homogeneous coordinates $[t^{n_0}u_0 : \ldots : t^{n_N}u_N]$, where $n_j$ are integers and $u_j$ are units in $R$. We may multiply each coordinate by $t^n$ with $n$ sufficiently big in such a way that each $n_i$ is positive and at least one of the $n_i$ is zero. Thus the point has homogeneous coordinates in $R$ and extends to a morphism $Spec(R) \to \mathbb{P}^N$.

In the proof above we used the fact that, if $R$ is a discrete valuation ring, the points of $\mathbb{P}^N_R$ can be described with homogeneous coordinates $[a_0 : \ldots : a_N]$. Remark that this is not true for an arbitrary ring. It will be true for instance if the ring is principal.

**5.8** *models and arithmetic heights.*

**5.8 Definition.** *Let $\mathcal{X} \to Spec(O_K)$ be an arithmetic scheme and $L$ be an* hermitian *line bundle on it. Let $p : Spec(O_K) \to \mathcal{X}$ be a $O_K$ rational point. The line bundle $p^*(L)$ is an hermitian line bundle on $O_K$ and the real number*

$$h_L(p) := \widehat{\deg}(p^*(L))$$

*is called the* height *of $p$ with respect to $L$.*

Of course we have arithmetic analogue of the properties 4.11:

**5.9 Proposition.** Properties of heights *: Let $\mathcal{X} \to Spec(O_K)$ be be an arithmetic variety. Then:*

*(1) If $L_1$ and $L_2$ are hermitian line bundles of $\mathcal{X}$ and $p \in \mathcal{X}(O_K)$ then*

$$h_{L_1 \otimes L_2}(p) = h_{L_1}(p) + h_{L_2}(p).$$

*(2) (Functoriality of heights) If $f : \mathcal{X} \to \mathcal{Y}$ is a morphisms of arithmetic varieties, $p \in \mathcal{X}(O_K)$ and $L$ an hermitian line bundle on $\mathcal{Y}$, then*

$$h_L(f(p)) = h_{f^*(L)}(p).$$

Observe that we have also an analogue of property (3) of proposition 4.11:

**5.10 Proposition.** *Let $\mathcal{X}$ be a projective arithmetic scheme and $L$ be an hermitian line bundle over it. Let $D$ be an effective Cartier divisor on $X$ such that $\mathcal{O}(D) = L$. Then there is a constant $C$ (depending on $D$, and $L$) with the following property: for every rational point $p \in \mathcal{X}(O_K)$ not contained in the support of $D$, we have*

$$h_L(p) \geq -C.$$

*Proof:* There is a global section $s \in H^0(\mathcal{X}, L)$ such that $div(s) = D$. For every infinite place $\sigma$ let $C_\sigma$ be the real number $\log \sup_{x \in \mathcal{X}_\sigma(\mathbb{C})}\{\|s\|_\sigma(x)\}$. Take as $C$ a constant which is bigger then $[K : \mathbb{Q}] \sup_\sigma\{C_\sigma\}$. Take $p \in \mathcal{X}(O_K)$ which is not in the support of $D$. The Cartier divisor $p^*(D)$ is effective and it corresponds to the the element $p^*(s)$ in the locally free hermitian $O_K$ module $p^*(L)$. The conclusion follows from 3.35 because $\log(Card(p^*(L)/p^*(s)O_K))$ is a positive number.

**5.11 Corollary.** *Suppose that in the hypotheses above $L$ is an ample hermitian line bundle on $\mathcal{X}$:*

*(a) There exists a constant $C$ such that, for every $p \in \mathcal{X}(O_K)$ we have*

$$h_L(p) \geq -C.$$

*(b) For every hermitian line bundle $M$ on $X_K$, there exists a positive constants $A$ and $B$ such that*

$$h_M(p) \leq A h_L(p) + B$$

*for every rational point $p \in \mathcal{X}(O_K)$.*

*Proof:* (a) It suffices to remark that there exists a positive integer $n$ such that $L^{\otimes n}$ is very ample. The conclusion follows from the functorial property of the heights and remark 4.6.

(b) There exists a positive integer $A$ such that $L^{\otimes A} \otimes M^{\otimes -1}$ is ample. Thus the property follows from (a) and (1) of 5.9.

The height theory for varieties over number fields depends on the theory of hermitian line bundles over arithmetic varieties. Essentially the height associated to a line bundle is a function on the rational points of a variety which is defined up to bounded functions. The ambiguity depends on the fact that given a projective variety and a line bundle over it, we have many choices of models of them and many structures of hermitian line bundle on the same line bundle.

**5.12 Definition.** *Let $X_K$ be a projective variety defined over a number field $K$ and $L_K$ be a line bundle over it. Suppose that $f : \mathcal{X} \to Spec(O_K)$ is a model of $X_K$ and $L_K$ is an hermitian line bundle over it which is a model of $L_K$. For every $p \in X_K(K)$ let $P \in \mathcal{X}_{O_K}$ be the corresponding $O_K$ point (cf. Theorem 5.7). The function*

$$\begin{aligned} h_L : X_K(K) &\longrightarrow \mathbb{R} \\ p &\longrightarrow h_L(P) \end{aligned}$$

*is called* **a** *height on $X_K$ associated to $L_K$.*

Since every line bundle can be written as the difference of two ample line bundles. Given a line bundle $L$ we can always find a positive integer $n$ such that $L^N$ has a model over $O_K$. Thus, given a line bundle over a projective variety, we can always find a height function associated to it.

It is very important to observe that, given a line bundle $L_K$ we may have many heights associated. Nevertheless we have the following theorem (which we will not prove here)

**5.13 Theorem.** *Let $X_K$ be a projective variety defined over $K$ and $L_K$ be a line bundle over it. Let $h^1_{L_K}$ and $h^2_{L_K}$ be two heights on $X_K$ associated to $L_K$. Then there exists a constant $C$, depending on $L_K$ and the two heights functions, such that, for every $p \in X_K(K)$ we have*

$$|h^1_{L_K}(p) - h^2_{L_K}(p)| \leq C.$$

For a proof cf. for instance [BGS].

Thus we may consider the height associated to a line bundle as a function on rational points defined up to a bounded function.

The main example we have to keep in mind is the following:

**5.14 Example.** (The height on projective space): We equip the tautological line bundle $\mathcal{O}_{\mathbb{P}}(1)$ of $\mathbb{P}^N$ with one of the sup Fubini–Study metrics. Let $p \in \mathbb{P}^N(K)$ be the $K$–rational point with homogeneous coordinates $[z_0 : \ldots : z_N]$ then

$$h_{\mathcal{O}(1)}(p) = \sum_{v \in M_K} \log(\sup\{\|z_0\|_v, \ldots, \|z_N\|_v\}.$$

*Proof:* Suppose that $z_0 \neq 0$, this means that $p$ is not contained in the support of the hyperplane $z_0 = 0$. We may then apply the definitions of Fubini Study metric 5.4 and exercise 5.5.

One of the main theorems of the heights theory is the Northcott theorem. It tells us something similar to the bounded families properties in geometry: every set of bounded height is finite:

**5.14 Theorem.** *(Northcott) Let $X_K$ be a projective variety defined over a number field and $L_K$ be an ample line bundle over it. Let $h_L$ be an height on $X_K$ associated* *to $L_K$ and $A$ be a positive constant. Then the set of rational points $p \in X_K(K)$ such that $h_L(p) \leq A$ is finite.*

The proof requires some lemmas:

**5.15 Definition.** *Let $K$ be a number field. If $a \in K$ we define the height $h(a)$ of $a$ to be the* sup *Fubini–Study height of the point $[a:1] \in \mathbb{P}^1_K$*

**5.16 Example.** Suppose that $K = \mathbb{Q}$ and $a = b/c$ with $b$ and $c$ coprime integer numbers. Then $h(a) = \log \sup\{|b|;|c|\}$

We will use properties of heights to prove the following standard facts. It is a useful exercise to prove as consequences of the definitions.

**5.17 Proposition.** *Suppose that $a$ and $b$ are two elements of $K$ then:*
*(1) if $a^\sigma$ is a conjugate of $a$ then $h(a) = h(a^\sigma)$;*
*(2) There exist constants $A_1$ and $A_2$ such that $h(a+b) \leq h(a) + h(b) + A_1$ and $h(ab) \leq h(a) + h(b) + A_2$*

*Proof:* We leave (1) as exercise to the reader.
To prove (2) consider the morphisms

$$+ : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$$
$$(x,y) \longrightarrow x+y$$

and

$$x : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$$
$$(x,y) \longrightarrow xy$$

They can be compactified to morphisms

$$\tilde{+} : \mathbb{P}^1 \tilde{\times} \mathbb{P}^1 \to \mathbb{P}^1 \quad \text{and} \quad \tilde{x} : \overline{\mathbb{P}^1 \times \mathbb{P}^1} \to \mathbb{P}^1$$

where $q_1 : \mathbb{P}^1 \tilde{\times} \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ and $q_2 : \overline{\mathbb{P}^1 \times \mathbb{P}^1} \to \mathbb{P}^1 \times \mathbb{P}^1$ are suitable blow ups with exceptional divisors $\tilde{E}$ and $\overline{E}$ respectively.

On $\mathbb{P}^1 \times \mathbb{P}^1$ we have the line bundle $\mathcal{O}(1,1) := p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{O}(1))$ where $p_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ are the two projections. The variety $\mathbb{P}^1 \times \mathbb{P}^1$ admits as a model the arithmetic scheme $\mathbb{P}^1_{O_K} \times_{Spec(O_K)} \mathbb{P}^1$ and the line bundle $\mathcal{O}(,1,1)$ admits as hermitian model over $O_K$ the corresponding line bundle over $\mathbb{P}^1_{O_K} \times_{Spec(O_K)} \mathbb{P}^1$. Consequently, if $(a,b) \in \mathbb{P}^1 \times \mathbb{P}^1(K)$ then $h_{\mathcal{O}(1,1)}(a,b) = h_{\mathcal{O}(1)}(a) + h_{\mathcal{O}(1)}(b)$.

By construction we have that $\tilde{+}^*(\mathcal{O}(1)) = q_1^*(\mathcal{O}(1,1))(-\tilde{E})$ and similarly $\tilde{x}^*(\mathcal{O}(1)) = q_2^*(\mathcal{O}(1,1))(-\overline{E})$. Property (b) follows from the property 5.10

The proof of the proposition above is perhaps not the easier one, one can easily deduce these properties from the definitions. Nevertheless we find useful the proposed proof because it shows the geometric way to prove it.

From 5.17 we find this important theorem:

**5.18 Theorem.** *Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$. Let $A$ and $R$ be positive constants. The set of numbers $a \in \overline{\mathbb{Q}}$ such that*

$$h(a) \leq A \quad \text{and} \quad [\mathbb{Q}(a) : \mathbb{Q}] \leq R$$

*is finite.*

*Proof:* First we remark that if $R = 1$ the proposition is evident. A point of $\mathbb{P}^1(\mathbb{Q})$ may be written in a unique way as $[a : b]$ with $a$ and $b$ coprime integers and its height is $\log \sup\{|a|, |b|\}$. Thus there are only finitely many points $p \in \mathbb{P}^1(\mathbb{Q})$ with bounded height. Let $a$ be a number which satisfy the hypotheses of the theorem. Let $F(z)$ be his minimal polynomial over $\mathbb{Q}$. Since the coefficients of $F(z)$ are linear combinations of products of algebraic numbers conjugate to $a$, prop 5.17 implies that each of these coefficients has height at most $R!A$ (we did not write the best constant). Thus, for the first part of the proof, the polynomial $F(z)$ lies in a finite list. The conclusion follows.

Now we come to the proof of 5.14:

*Proof:* (*of 5.14*) By the properties of the heights we may suppose that $X_K$ is the projective space $\mathbb{P}^N$. As a consequence of 5.18 we find that the theorem holds for $X_K = \mathbb{P}^1 \times \ldots \mathbb{P}^1$ ($N$ times) with the line bundle $\mathcal{O}(1, \ldots, 1) := p_1^*(\mathcal{O}(1)) \otimes \cdots \otimes p_N^*(\mathcal{O}(1))$ where $p_i : \mathbb{P}^1 \times \cdots \mathbb{P}^1 \to \mathbb{P}^1$ is the $i$-th projection. The Symmetric group $\Sigma_R$ acts on $X_K := \mathbb{P}^1 \times \ldots \mathbb{P}^1$ and the quotient $X_K / \Sigma_N$ is isomorphic to $\mathbb{P}^N$. Moreover if $q : X_K \to \mathbb{P}^N$ is the projection, we have that $q^*(\mathcal{O}(1)) = \mathcal{O}(1, \ldots, 1)$. If $p \in \mathbb{P}^N(K)$ then the points of fiber $q^{-1}(p)$ are at most $N!$ and defined over an extension of $K$ of degree at most $N!$. The conclusion follows from the functoriality of the heights and the first part of the proof.

## 6 lecture V.

In the previous lectures we described the height theory in the geometric and arithmetic contests as a theory of degree of line bundles. To do this we had to introduce the norms and the product formula. It is important to notice that, from a geometric point of view, the archimedean norms did not, until now, play any specific role. In the arithmetic contest, instead, it was indispensable to use archimedean metrics at infinite places.

To push forward the analogy, we would like to understand the relation between metrics on line bundles on complex varieties and degree of line bundles on compact Riemann surfaces. In order to undestand this we will introduce the first Chern class and the first Chern form of an hermitian line bundle over a complex variety. Its relation with the degree which is obtained by the Poincaré–Lelong formula.

One also remark that, in the arithmetic contest, the infinite places seems to play a special role. They are similar to the points at infinity of a curve. So one may consider the arithmetic theory as a theory where naturally one has a border. We will see that we may introduce the theory of analytic maps from affine curves to projective varieties and obtain a theory which is analogue to the geometric and the arithmetic theories just described and where again, metrics play special roles and the border too.

Thus one may have the impression that, geometry, arithmetic and analysis have interesting similar descriptions. In the sequel we will see how they can interact together.

In order to relate the metrics with the degree, we need to do an excursus in the complex differential geometry:

**6.1** *The Laplace operator on a Riemann surface.* Let $\Omega$ be a domain on a Riemann surface with regular border. We denote the border of $\Omega$ by $\Gamma$. We suppose that $\Omega$ is relatively compact and denote by $\overline{\Omega}$ the closure of it. If $\alpha$ is a smooth 1 form on a open set containing $\overline{\Omega}$ we have the *Stokes formula*:

$$\int_\Gamma \alpha = \int_\Omega d(\alpha).$$

On a domain as above we introduce the two operators $\partial$ and $\overline{\partial}$ in the following way:

Locally on $\Omega$ we have a holomorphic coordinate $z$ and an anti–holomorphic coordinate$\overline{z}$. The operators are defined as follows

$$\begin{aligned} \partial : A^0(\Omega) &\longrightarrow A^{(1,0)}(\Omega) \\ f(z,\overline{z}) &\longrightarrow \frac{df}{dz}dz \end{aligned} \qquad \text{and} \qquad \begin{aligned} \overline{\partial} : A^0(\Omega) &\longrightarrow A^{(0,1)}(\Omega) \\ f(z,\overline{z}) &\longrightarrow \frac{df}{d\overline{z}}d\overline{z} \end{aligned}$$

One may check that these definitions do not depend on the holomorphic coordinates and that the operator $d : A^0(\Omega) \to A^1(\Omega)$ is given by $\partial + \overline{\partial}$. Observe that $f(z)$ is holomorphic if and only if $\overline{\partial}(f) = 0$.

We introduce the new operator $d^c : A^0(\Omega) \to A^1(\Omega)$ as follows: $d^c := \frac{1}{4\pi i}(\partial - \overline{\partial})$.

Observe that $dd^c$ is $\frac{1}{2\pi i}\partial\overline{\partial}$ which, up to a constant is the Laplacian.

Let $g$ be a smooth function. By abuse of notation, we will call the $(1,1)$ form $dd^c(g)$ the Laplacian of $g$.

Without entering in the theory of the distributions, we will explain a operative way to extend the laplacian operator to locally integrable functions.

**6.1 Definition.** *A function $g(z)$ on $\Omega$ is said to be locally integrable if it is defined almost everywhere, measurable and every point has a neighborhood $U$ such that $\left|\int_U g dz \wedge d\overline{z}\right| < \infty$.*

The main example of locally integrable function is the following: Let $f(z)$ be a meromorphic function in $\Omega$, then $\log|f(z)|^2$ is locally integrable. Observe that, the function $\log|f(z)|^2$ is not defined near the points where $f(z)$ has zeroes or poles.

We may extend the operator $dd^c$ to the locally integrable functions in the following way:

**6.2 Definition.** *If $g(z)$ is a locally integrable function, then we define the distribution $dd^c(g)$ as follows: Let $h$ be a smooth function with support contained in $\Omega$ then*

$$dd^c(g)(h) := \int_\Omega g \cdot dd^c(h).$$

Let's show that this definition extends the Laplacian on smooth functions: let $g$ be a smooth function on $\Omega$ and $h$ be a smooth function with compact support. We want to show that

$$\int_\Omega h \cdot dd^c(g) = \int_\Omega g \cdot dd^c(h). \tag{6.3.1}$$

**6.4 Proposition.** *Let $h$ and $g$ be two smooth functions on a neighborhood of $\Omega$, then*

$$\int_\Gamma d^c(h) \cdot g - h \cdot d^c(g) = \int_\Omega g \cdot dd^c(h) - \int_\Omega h \cdot dd^c(g). \tag{6.4.1}$$

*Proof:* For every couple of smooth functions $g$ and $h$ one has $d(g) \wedge d^c(h) - d(h) \wedge d^c(g) = 0$. Thus we have that $d(gd^c(h) - hd^c(g)) = gdd^c(h) - hdd^c(g)$ consequently, by Stokes theorem, the conclusion follows.

If the support of $h$ is contained in $\Omega$, then the first integral of 6.4.1 is zero. Thus 6.3.1 holds.

Let $f(z)$ be a *non vanishing* holomorphic function on $\Omega$. The function $\log|f(z)|^2$ is a well defined smooth function on $\Omega$ and since $\partial(f(z)) = \overline{\partial}(\overline{f(z)}) = 0$ we have that $dd^c(\log|f(z)|^2) = 0$. This is not the case when $f(z)$ has zeroes. Remark that in the zeroes of $f(z)$ the function $\log|f(z)|^2$ is not defined but it is integrable in a neighborhood of them.

We want now to compute the distribution $dd^c \log(|f(z)|^2$ when $f(z)$ is an arbitrary meromorphic function on $\Omega$.

**6.6 Definition.** *Let $P := \{p_n\}_{n \in \mathbb{N}}$ be a sequence of points in $\Omega$ with no accumulation point. Let $M := \{m_n\}_{n \in \mathbb{N}}$ be a sequence of integers. We associate to the senquences a formal sum $D := \sum_n m_n p_n$ and a distribution (a current) $\delta_D$ on $\Omega$ defined in the following way, let $h$ be a smooth function with compact support on $\Omega$ then:*

$$\delta_D(h) = \sum_n m_n \cdot h(p_n).$$

*The current $\delta_D$ is called the Dirac distribution associated to $D$.*

Observe that, even if $D$ is a finite sum, every compact set of $\Omega$ contains only finitely many points of the sequence $P$; thus, for every $h$ smooth with compact support the sum in $\delta_D(h)$ is a finite sum.

The typical example of such a $D$ is the set of zeroes and poles of a meromorphic function defined over $\Omega$.

**6.7 Theorem.** *Let $f(z)$ be a meromorphic function over $\Omega$ and for every point in $\Omega$ denote by $v_p(f)$ the order of zero or pole of $f$ at $p$. Let $div(f)$ the formal sum $div(f) := \sum_{p\in\Omega} v_p(f)p$. Then*

$$dd^c(\log|f(z)|^2) = \delta_{div(f)}.$$

*Proof:* First of all we remark that we already proved the theorem in the case when $f(z)$ is holomorphic and non vanishing. We need to prove that. given a smooth function with compact support $h$ on $\Omega$ then

$$\int_\Omega \log|f(z)|^2 dd^c(h) = \sum_{p\in\Omega} v_p(f)h(p).$$

For each $p \in \Omega$ with $v_p(f) \neq 0$ let $B(p, \epsilon_p)$ be a small open open neighborhood of $p$ contained in $\Omega$, holomorphically equivalent to a disk centered in $p$ and such that for every $q$ in the closure of $B(p, \epsilon_p)$ we have that $v_q(f) = 0$. Similarly let $B(p, \epsilon_p/2)$ be a open set, holomorphically equivalent to a disk centered at $p$ and the closure of which is contained in $B(p, \epsilon_p)$. Let $\Omega' = \Omega \setminus \cup_{v_p(f)\neq 0}\overline{B(p, \epsilon_p/2)}$ where $\overline{B(p, \epsilon_p/2)}$ is the closure of $B(p, \epsilon_p/2)$ and $\{\rho_{\Omega'}, \rho_p \;\; v_p(f) \neq 0\}$ be a partition of unity subordinated to the covering $\{\Omega', B(p, \epsilon_p) \;\; v_p(f) \neq 0\}$. We need to compute

$$\int_\Omega \log|f(z)|^2 dd^c(\rho_{\Omega'}h + \sum_{v_p(f)\neq 0} \rho_p h).$$

By what we noticed before, $\int_{\Omega'} \log|f(z)|^2 dd^c(\rho_{\Omega'}h) = 0$ because $f(z)$ is a holomorphic non vanishing function on $\Omega'$ and $\rho_{\Omega'}h$ is a smooth function with compact support in $\Omega'$. Consequently

$$\int_\Omega \log|f(z)|^2 dd^c(h) = \sum_{v_p(f)\neq 0} \int_{B(p,\epsilon_p)} \log|f(z)|^2 dd^c(\rho_p h).$$

Thus, since $\rho_p h$ is a smooth function with compact support contained in $B(p, \epsilon_p)$, we may suppose that $\Omega$ is a disk and it suffices to prove the following:

**6.8 Lemma.** *Let $g(z)$ be a non vanishing holomorphic function on the unit disk $B(0, 1)$ of the plane and $f(z) = z^n g(z)$ with $n \in \mathbb{Z}$. Let $h(z)$ be a smooth with compact support* *in* $B(0,1)$

$$\int_{B(0,1)} \log|f(z)|^2 dd^c(h) = nh(0).$$

*Proof:* Since $\log|f(z)|^2 = n\log|z|^2 + \log|g(z)|^2$ and $g(z)$ is holomorphic non vanishing, we need to prove that

$$\int_{B(0,1)} \log|z|^2 dd^c(h) = h(0).$$

For every $\epsilon \in ]0,1[$ denote by $C(\epsilon)$ the annulus $\{z \in B(0,1) \ / \ \epsilon < |z| < 1\}$ and by $S(\epsilon)$ the circle $\{|z| = \epsilon\}$. Notice that $dd^c \log|z|^2 = 0$ on $C(\epsilon)$ and $h\log|z|^2 = 0$ on the circle $\{|z| = 1\}$. Thus we may apply formula 6.4.1 to $h$ and $\log|z|^2$ to the domain $C(\epsilon)$ and obtain that

$$\begin{aligned}
&\int_{B(0,1)} \log|z|^2 dd^c(h) \\
&= \lim_{\epsilon\to 0} \int_{C(\epsilon)} \log|z|^2 dd^c(h) - hdd^c\log|z|^2 \\
&= \lim_{\epsilon\to 0} \left( -\int_{S(\epsilon)} \log|z|^2 d^c(h) + \int_{S(\epsilon)} hd^c\log|z|^2 \right).
\end{aligned}$$

(remark that we changed the sign because of the inverse orientering of the circle) Write $z = re^{i\theta}$. Since $h$ is a smooth function, we can find a constant $B$ such that if $d^c(h) = ad(r) + bd(\theta)$ then $|b| \leq B$. Consequently there exists a constant $B_1$ independent on $\epsilon$ such that

$$\left| \int_{S(\epsilon)} \log|z|^2 d^c(h) \right| \leq \epsilon\log\epsilon^2 B_1$$

Thus $\lim_{\epsilon\to 0} \int_{S(\epsilon)} \log|z|^2 d^c(h) = 0$. The conclusion will be a consequence of the following lemma:

**6.9 Lemma.** *With the notations as above, for every* $\epsilon \in ]0;1[$ *we have*

$$\int_{S(\epsilon)} d^c \log|z|^2 = 1.$$

*Proof:* We have that

$$\begin{aligned}
&d^c\log|z|^2 \\
&= \frac{1}{4\pi i}(\partial - \overline{\partial})(\log z\overline{z}) = \frac{1}{4\pi i}\left(\frac{dz}{z} - \frac{d\overline{z}}{\overline{z}}\right)
\end{aligned}$$

Write $z = re^{i\theta}$, then $\frac{dz}{z} = \frac{dr}{r} + id\theta$ and $\frac{d\overline{z}}{\overline{z}} = \frac{dr}{r} - id\theta$. From this we deduce $d^c\log|z|^2 = \frac{1}{4\pi i}\cdot 2id\theta = \frac{1}{2\pi}d\theta$. The conclusion follows.

To show the relation with the degree and what we described in the previous lectures, we deduce from what we did the two following basic facts:

**6.10 Corollary.** *(a) Let $X$ be a* compact *Riemann surface.*

*(a) Let $f$ be a holomorphic function on $X$; then $f$ is constant.*

*(b) let $g$ be a meromorphic function on $X$: then the degree of $div(g) := \sum v_p(g)p$ is zero.*

*Proof:* (a) Suppose that $f$ is not constant, changing $f$ by $f - f(p)$ where $p \in X$, we may suppose that $f$ has at least a zero. Consequently $div(f) \neq 0$ and since $f$ is holomorphic, we should have $deg(div(f)) > 0$. The function $1(p) = 1$ on $X$ is a smooth function with compact support on $X$, because $X$ is compact; thus formula 6.4.1 becomes, since $dd^c(1) = 0$,

$$0 = \int_X dd^c \log |f(z)|^2$$

But this is in contradiction with 6.7 which should give $\int_X dd^c \log |f(z)|^2 = deg(div(f))$.

(b) The proof of (b) is similar. Let $d = deg(div(g))$; then the same argument as above gives $0 = \int_X dd^c \log |g(z)|^2 = d$.

Remark that the same proof gives the following proposition which will be useful in the sequel:

**6.11 Proposition.** *Let $X$ be a compact Riemann surface and $g$ be a locally integrable function on it. Then*

$$\int_X dd^c(g) = 0.$$

The proposition above may be seen as a version of Stokes theorem on $X$: the differential of a function on a surface with no border is zero.

**6.12** *Chern classes and the Poincaré–Lelong formula.* In the previous subsection we defined the Laplace operator on a RIemann surface. An interesting consequence of its property is the new proof of the Product formula 6.10. We will see that it can be interpreted as the case of the trivial line bundle with trivial metric of the Chern forms theory.

Let $X$ be a Riemann surface and $L$ be a line bundle on it. We suppose that $L$ is equipped with a smooth hermitian metric $\| \cdot \|$.

**6.12 Definition.** *Let $s$ be a meromorphic section on $L$ and $U$ the open set $X \setminus \{Supp(div(s))\}$ where $Supp(div(s))$ is the support of the divisor $div(s)$. The $(1,1)$ form on $U$ defined by*

$$c_1(L, \| \cdot \|) := -dd^c \log \|s\|^2$$

*is called the Chern form of the hermitian line bundle $(L; \| \cdot \|)$.*

We show now that the form $c_1(L, \| \cdot \|)$ extends to a smooth form on $X$.

**6.13 Proposition.** *Let $s_1$ and $s_2$ two meromorphic sections of $L$ and $U$ be the open set of $X$ where $s_i$ have no zeros and poles. Then, over $U$ we have*

$$dd^c \log \|s_1\|^2 = dd^c \log \|s_2\|^2.$$

*Proof:* The statement is local on $U$. Consequently we may suppose that the line bundle $L$ is trivial, the $s_i$ are non vanishing holomorphic functions and that there exists a smooth positive function $\rho$ such that $\|s_i\|^2 = \frac{|s_i|^2}{\rho}$. Thus, since $dd^c \log |s_i|^2 = 0$,

$$dd^c \log \|s_1\|^2 = dd^c \log |s_1|^2 - dd^c \log(\rho) = dd^c \log |s_2|^2 - dd^c \log(\rho) = dd^c \log \|s_2\|^2.$$

The proposition above shows that, given an hermitian line bundle $(L, \|\cdot\|)$ on $X$, we can associate to it a global smooth $(1,1)$ form $c_1(L, \|\cdot\|)$ which we will call the *Chern form of* $(L, \|\cdot\|)$: To compute the Chern form of $(L, \|\cdot\|)$ in the neighborhood of a point $p$ of x, take a meromorphic section $s$ which do not have zeroes or poles in $p$, the Chern form $c_1(L : \|\cdot\|)$ near $p$ is given by the $(1,1)$ form $-dd^c \log \|s\|^2$.

The following proposition is easy and left as exercise:

**6.14 Proposition.** *Let $L_1$ and $L_2$ be two hermitian line bundles on $M$, then*

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).$$

If $s$ is a meromorphic section of an hermitian line bundle $L$ then, a local computation shows that the function $\log \|s\|^2$ is locally integrable even near the poles and the zeroes of $s$. Thus we may compute the laplacian $dd^c \log \|s\|^2$ of it as a distribution.

The fomula which compute it is called the *Poincaré–Lelong* formula:

**6.15 Theorem.** *(Poincaré–Lelong formula) Let $X$ be a Riemann surface and $(L, \|\cdot\|)$ be an hermitian line bundle on it. Let $s$ be a meromorphic section of $L$ and $div(s)$ the formal sum $\sum v_p(s)p$ of points of $X$ (for arbitrary $X$ and $s$ the sum $div(s)$ may have infinitely many terms so it is not a divisor). Then the following equality of distributions holds:*

$$dd^c \log \|s\|^2 = \delta_{div(s)} - c_1(L, \|\cdot\|)$$

*Where $\delta_{div(s)}$ is the Dirac distribution defined in 6.6.*

The proof is a direct application of 6.7: again the statement is local on $X$ so we may suppose that $L$ is trivial, $s$ is a meromorphic function and there exists a non negative smooth function $\rho$ such that $\|s\|^2 = \frac{|s|^2}{\rho}$. From 6.7 we get

$$dd^c \log \|s\|^2 = dd^c \log |s|^2 - dd^c \log \rho = \delta_{div(s)} - c_1(L\|\cdot\|).$$

The Poincaré–Lelong formula above is very important because it shows the strict relation between the *hermitian* metrics on line bundles over a compact Riemann surface and the degree of the line bundle itself:

We recall that every compact Riemann surface is algebraic: it can be realized as a Smooth projective curve in a suitable projective space. Thus, given a line $L$ bundle on it, we can speak about the degree $deg(L)$. Here we show that the Poincaré–Lelong formula allows to compute this degree using a metric on $L$.

**6.16 Theorem.** *Let $X$ be a compact Riemann surface and $(L, \|\cdot\|)$ be an hermitian vector bundle on it. Then*

$$\int_X c_1(L, \|\cdot\|) = deg(L).$$

*Proof:* Let $s$ be a meromorphic section of $L$ and $div(s)$ the associated divisor. Let $1(z)$ be the (smooth with compact support) function on $X$ such that $1(z) = 1$. Then $\delta_{div(s)}(1) = \deg(div(s)) = \deg(L)$.

By proposition 6.11, we have that $\int_X dd^c \log\|s\|^2 = 0$ thus, by Poincaré–Lelong formula

$$0 = \int_X dd^c \log\|s\|^2 = \delta_{div(s)}(1) - \int_X c_1(L, \|\cdot\|).$$

The conclusion follows.

**6.17 Remark.** The theory of the first Chern form holds on any smooth complex variety. Once one defines the operator $dd^c$ on a variety (and this can be done with no difficulties), we define the first Chern form of an hermitian line bundle $L$ on a complex variety $X$ exactly in the same way we defined it over a Riemann surface. It is not difficult to prove the following functorial property: if $f : Y \to X$ is an analytic morphism, then $c_1(f^*(L)) = f^*(c_1(L))$. Moreover one can prove an analogue of the Poincaré–Lelong formula on arbitrary variety. We refer to the literature for more details on the subject

# 7 Lecture VI.

**7.1** *Enters Nevanlinna theory.* The height theory of points over arithmetic varieties is strictly related to the introduction of metrics over the restriction of line bundles to infinite places. One should remark that in order to define heights, we need to (a) introduce a sort of compactification of the spectra of ring of integers of number fields. (b) introduce metrics on the compactified objects.

The first point is achieved considering the infinite places as if they were some "new points" of the curves. This is similar to the compactification of affine curves. Nevertheless what is natural to consider is the "open" affine object $Spec(O_K)$. Thus when we consider the geometric analogue of it, we should deal with morphisms from "open" affine curves to projective varieties. The difference is not without interest because, when we deal with morphisms from affine curves to varieties, we have a much wider class of morphisms: the class of analytic morphisms.

**7.1 Example.** The map $f : \mathbb{C} \to \mathbb{P}^2$ defined as $f(c) = [1, z, e^z]$ is an analytic map which do not extends to an algebraic map from the projective line to $\mathbb{P}^2$.

Thus one would like to find analogy between the height theory and the theory of analytic maps from an affine curve to a projective variety. We will show now that, if we properly interpret the Nevanlinna theory, once again the analogy is striking.

Before we start, let's make a remark: the theorem 6.16 tells us that the degree of a line bundle is obtained by integrating a suitable $(1,1)$ form on a Riemann surface $Y$. A form $\omega$ is said to be positive if locally we write it as $\omega = iF(z)dz \wedge d\overline{z}$ with $F(z)$ a smooth positive real function. When the for is positive, we may see it as a measure on $X$. Consequently we may conclude that the degree is the area of $Y$ with respect to the metric of the line bundle.

Let $X$ be a smooth projective variety equipped with an hermitian ample line bundle $L$. We denote by $c_1(L)$ the first Chern form of $L$. We want to study an *analytic* map $f : \mathbb{C} \to X$.

Instead of computing the area of $\mathbb{C}$ with respect to $f^*(c_1(L))$, which may very well be infinite, we compute a sort of averaged area of the disks of $\mathbb{C}$.

**7.2 Definition.** *We define the analytic characteristic function of $f$ or* the analytic heigh of $r$ with respect to $f$ *to be the number*

$$T_f(r, L) = \int_0^r \frac{dt}{t} \int_{|z|<t} f^*(c_1(L)).$$

Some explanations to the definition above are necessary:

(a) If we consider the standard measure $idz \wedge d\overline{z}$ on $\mathbb{C}$ we have that

$$\int_0^r \frac{dt}{t} \int_{|z|<t} idz \wedge d\overline{z} = \pi r^2 :$$

Thus if we compute the analytic height of $r$ with respect to the standard Lebesgue measure on $\mathbb{C}$ we obtain the area of the disk of radius $r$.

(b) Let $\omega = F(z)idz \wedge d\overline{z}$ be a $(1,1)$ form on $\mathbb{C}$. Near the origin, we can find a constant $A$ such that $|F(z)| \leq A$. Thus $\left|\int_{|z|<\epsilon} \omega\right| \leq 2\pi A\epsilon^2$. Consequently the integral in the definition of the analytic height do not diverges near the origine and it is well defined.

(c) By the Poincaré–Lelong formula, if $f : \mathbb{C} \to \mathbb{P}^1$ is the natural inclusion and $L = \mathcal{O}(1)$ with the Fubini–Study metric (or any other metric) then, if we fix $\epsilon > 0$, for every $\epsilon_1$ and $r$ sufficiently big, for every integer $n$ we have

$$\left| \int_\epsilon^r \frac{dt}{t} \int_{|z|<t} f^*(c_1(L^{\otimes n})) - n \log \frac{r}{\epsilon} \right| \leq \epsilon_1 \log \frac{r}{\epsilon}.$$

(The reader is invited to prove this by exercise) Thus, in this situation, the analytic degree is similar to the standard degree (geometric height) multiplied by a factor $\log(r)$.

(d) Formula in (c) shows that the analytic height has something in common with the geometric heigh. Observe that, in order to introduce it, we need to work with line bundles equipped with metrics. This feature is similar to the introduction of arithmetic heights.

(e) A curve, with map in a projective variety is the geometric analogue of a point in the variety. In the analytic contest, *we start with a given analytic map* $f : \mathbb{C} \to X$ and each positive real number $r$ is the analytic analogue of a rational point in the arithmetic contest.

(f) In order to explain the analogue of Northcott theorem 5.14 we need to introduce the notion of positive metric on a line bundle. Let $\Delta$ be the unit disk and $\omega := F(z)idz \wedge d\overline{z}$ be a smooth $(1,1)$–form on it. We recall that $\omega$ is said to be *positive* if $F(z)$ is a smooth function with values in $\mathbb{R}_{>0}$. A smooth positive form defines a positive measure on the disk: the area of every open set with compact closure is strictly positive. We generalize this to a variety:

**7.3 Definition.** *Let $X$ be a projective variety and $L$ be an hermitian line bundle on it. The metric on $L$ is said to be* positive *if, for every analytic map $f : \Delta \to X$ the restriction $f^*(c_1(L))$ of the first Chern form of $L$ to $\Delta$ is a smooth positive $(1,1)$ form on it.*

**7.4 Example.** Let $X = \mathbb{P}^N$ and $L = \mathcal{O}(1)$. One can check that the Fubini–Study metric on $L$ is a smooth positive metric.

One of the basic theorems of the theory of hermitian line bundles, is a theorem due to Kodaira:

**7.5 Theorem.** *(Kodaira) Let $X$ be a smooth projective variety and $L$ a line bundle on it. Then $L$ admits a smooth positive metric if and only if $L$ is ample.*

The analogue of Northcott theorem in this contest is the following "trivial observation":

**7.6 Proposition.** *Let $X$ be a smooth projective variety and $L$ be an hermitian ample line bundle equipped with a positive metric. Let $f : \mathbb{C} \to X$ be an analytic map and $A$ be a positive constant. Then the set $A(r) := \{r \in \mathbb{R} \ / \ T_f(r, L) \leq A\}$ is a compact set.*

*Proof:* It suffices to remark that , since the metric is positive, for every positive $t$ the integral $\int_{|z|\leq t} f^*(c_1(L))$ is positive. Consequently $T_f(r, L)$ is a increasing function of $r$: its derivative is $\frac{1}{r}\int_{|z|\leq r} f^*(c_1 L))$ which is positive. Thus $A(r)$ is compact.

(g) The most striking analogy between the geometric, the arithmetic and the analytic theory is the analogue of properties 4.11 (c) and 5.10: the fact that if a point $p$ do not lie on a effective Cartier divisor $D$, then the height if $p$ with respect to $D$ is uniformly lower bounded. This is the main issue of the so called "Nevanlinna First Main Theorem":

In order to explain and prove this property we need to introduce some new concepts. We will see that the formulas we find will have also a much wider interpretation and application.

**7.7 Definition.** *Let $r > 0$, we define the Green function of the origin $0$ in the disk of radius $r$ to be the function*

$$g_0^r(z) := \begin{cases} \log(\dfrac{r^2}{|z|^2}) \quad \text{if} \quad |z| \leq r \\ 0 \quad \text{if} \quad |z| \geq r. \end{cases}$$

The Green function is a continuous, thus locally integrable, function on $\mathbb{C}$. Consequently we can compute its laplacian. We introduce some notation:

For every positive real number $r$ we denote

$$B(0, r) := \{z \in \mathbb{C} \ / \ |z| < r\} \quad \text{and} \quad S(r) := \{z \in \mathbb{C} \ / \ |z| = r\}.$$

**7.8 Theorem.** *With the notations above, there exists a positive measure $\mu_r$ on $\mathbb{C}$ of total mass $1$ and supported on $S(r)$ such that*

$$dd^c g_0^r = \mu_r - \delta_0.$$

*Proof:* Let $\epsilon > 0$ be a real number strictly less then $r$. The restriction of $dd^c(g_0^r)$ to the ball $B(0\epsilon)$ is $-\delta_0$ because of theorem 6.7. We claim that the restriction of $dd^c(g_0^r)$ to $\mathbb{C} \setminus \{0\}$ is a positive measure $\mu_r$ supported on $S(r)$. This suffices to prove the theorem: indeed choose $r > \epsilon' > \epsilon > 0$ and consider the covering of $\mathbb{C}$ made with the two open sets $U_1 = \{|z| > \epsilon\}$ and $U_2 := \{|z| < \epsilon'\}$. Let $\{\rho_1 \ , \ \rho_2\}$ be a partition of unity subordinated to the covering. Let $h$ be a smooth function with compact support on $\mathbb{C}$. To compute $\int_{\mathbb{C}} h dd^c(g_0^r)$ it suffices to compute $I_1 = \int_{U_1} \rho_1 h dd^c(g_0^r)$ and $I_2 = \int_{U_2} \rho_1 h dd^c(g_0^r)$; but the claim implies that $I_1 = \int_{S(r)} h\mu_r$ and $I_2 = h(0)$.

Let's prove the claim. Let $h$ be a smooth function with compact support contained in $\mathbb{C} \setminus \{0\}$. We may suppose that there exists $\epsilon > 0$ such that the support of $h$ is contained in $\{|z| > \epsilon\}$. Let $\tilde{g}_0^r(z) := \log \frac{r^2}{|z|^2}$. it is a smooth function on $\mathbb{C} \setminus \{0\}$. We apply formula

6.4.1 to the domain $B(\epsilon,r):=\{\epsilon<|z|<r\}$ and obtain

$$\int_{B(\epsilon,r)}\tilde{g}_0^r dd^c h - h dd^c\tilde{g}_r^0=\int_{\partial B(\epsilon,r)}\tilde{g}_0^r d^c h - h d^c\tilde{g}_0^r.$$

Observe that:

– Since $g_0^r$ vanishes outside the ball $B(0,r)$, we have that

$$\int_{\mathbb{C}\setminus B(0,\epsilon)} g_0^r dd^c h=\int_{B(r,\epsilon)}\tilde{g}_r^0 dd^c h:$$

– the distribution $dd^c(\tilde{g}_0^r)$ vanishes on $B(r,\epsilon)$;

– $\partial B(\epsilon,r)=S(r)\cup S(\epsilon)$;

– Since $h$ has compact support on $\{|z|>\epsilon\}$, the restriction of both $h$ and $d^c(h)$ to $S(\epsilon)$ vanish.

– The function $\tilde{g}_0^r$ vanishes on $S(r)$. Thus we obtain

$$\int_{\mathbb{C}} h dd^c g_0^r=\int_{B(r,\epsilon)}\tilde{g}_0^r dd^c h=-\int_{S(r)} h d^c\tilde{g}_0^r.$$

This prove that the distribution $dd^c(g_0^r)$ is supported on $S(r)$. An explicit computation gives that $-d^c\tilde{g}_0^r = d^c\log|z|^2$ and if we parametrize the cercle $S(r)$ by the coordinate $z=re^{i\theta}$, then the restriction of $d^c\log|z|^2$ to $S(r)$ is the measure $\frac{1}{2\pi r}d\theta$. The claim follows.

We can now state and prove the First Main Theorem.

**7.9** *The First Main Theorem of Nevanlinna theory.* Let $X$be a smooth projective variety and $L$ be an hermitian line bundle over it. Let $f:\mathbb{C}\to X$ be an analytic map.

**7.9 Theorem.** *Let $s\in H^0(X,L)$ a non zero holomorphic global section. Suppose that the restriction of $s$ to $f(0)$ is not zero: $s|_{f(0)}\neq 0$. Then*

$$T_f(r,L)=\sum_{|z|<r} v_z(f^*(s))\log\frac{r}{|z|}-\int_{|z|=r}\log\|f^*(s)\|\mu_r+\log\|s\|(f(0)). \tag{7.9.1}$$

Let's make some comments to the formula 7.9.1 above:

(a) The term $-\int_{|z|=r}\log\|f^*(s)\|\mu_r$ is usually called *proximity function*, it is denoted by $m_f(s,r)$ and it compute, in some way, the average of the logarithm of the norm of the section on the border of the disk $B(0,r):=\{|z|\leq r\}$.

(b) The term $\sum_{|z|<r} v_z(f^*(s))\log\frac{r}{|z|}$ is the sum of the zeroes in $B(0,r)$ when we give to a number $z$ the weight $g_0^r(z)$. It is usually called the *counting function* and it is denoted by $N_f(s,r)$.

(c) The First Main theorem is often written as $T_f(r,L)=m_f(s,r)+N_f(s,r)+O(1)$. where the symbol $O(1)$ means that the left hand side and the right hand side are the same up to a bounded function (actually a constant).

(d) One should compare the formula 7.9.1 with formulas 3.37.1 and 3.38.1. As explained before, in the analytic contest, the analogue of a point is a positive real number $r$, and the corresponding disc $\{|z| < r\}$. In this situation the degree $T_f(r, L)$ is computed as sum of two terms: the first term is the weighted sum, on the points inside the disk , of zeroes and poles of the global section. The weight of each point is not 1 as in the geometric case, but it is $\log \frac{r}{|z|}$; thus the logarithm of something, as in the arithmetic case!. The second term is minus the sum (an integral) over the border of the point of the norm of the section.

(e) Suppose that $\sup_{x\in X}\{\|s\|(x)\} \leq 1$; since $X$ is compact, this condition can always be achieved multiplying $s$ by a constant. Then formula 7.9 implies that there exists a constant $C$ independent on $r$ such that $T_f(r, L) \geq C$. This is a feature similar to 5.10.

We now come to the proof of the Nevanlinna First Main theorem: It suffices to prove the following:

**7.11 Theorem.** *Let $L$ be an hermitian line bundle on $\mathbb{C}$ and $s \in H^0(\mathbb{C}, L)$ be a holomorphic section of it. We suppose that $s(0) \neq 0$. Then, for every positive real number $r$ we have*

$$\int_0^r \frac{dt}{t} \int_{|z|<t} c_1(L) + \int_{|z|=t} \log \|s\|\mu_r = \sum_{|z|<r} \log \frac{r}{|z|} + \log \|s\|(0).$$

*Proof:* Consider the functions $\log \|s\|(0)$ and $g_0^r(z)$. Their laplacian is not just a distribution but it is a measure (it can be computed on continuos functions). By Poincaré–Lelong formula and theorem 7.8 we have

$$dd^c \log \|s\|^2 = \delta_{div(s)} - c_1(L) \quad \text{and} \quad dd^c g_0^r = \mu_r - \delta_0.$$

Observe that, the fact that $s(0) \neq 0$ implies that we can apply $dd^c g_0^r$ to $\log \|s\|$ and $dd^c \log \|s\|$ to $g_0^r$.

An argument similar to the one used in the proof of theorem 6.7, authorizes to say that the following equality holds:

$$\int_{\mathbb{C}} \log \|s\|^2 dd^c g_0^r = \int_{\mathbb{C}} g_0^r dd^c \log \|s\|^2;$$

Indeed, even if neither $\log \|s\|^2$ or $g_0^r$ are smooth functions, nevertheless:

– they are both locally integrable functions;

– the function $g_0^r$ is smooth on $\mathbb{C} \setminus \{0\}$ and the function $\log \|s\|^2$ is smooth everywhere but a discrete set of points.

– The function $g_0^r$ vanishes on $\{|z| \geq r\}$ thus, at least formally, the formula above is an application of 6.4 on a disk of radius bigger then $r$.

– To rigorously justify the formula above we need to take a covering of the disk of radius $r$ made by a domain obtained removing from it small disks centered on 0 and the zeroes of $s$ and disks slightly bigger then the removed ones and centered on the same set. Then apply a partition of unity etc. The details are left to the reader, in any case they are exactly the same then the method used in proof of theorem 6.7

From the formula above we obtain:

$$\int_{|z|=r} \log \|s\|^2 \mu_r - \log \|s\|^2(0) =$$

$$\delta_{div(s)}(g_r^0) - \int_{|z|<r} g_r^0 c_1(L)$$

We remark that $\delta_{div(s)}(g_r^0) = \sum_{|z|<r} \log \frac{r^2}{|z|^2}$ and the conclusion follows from the following lemma:

**7.12 Lemma.** *Let $\alpha$ be a smooth $(1,1)$ form on $\mathbb{C}$. Then*

$$2\int_0^r \frac{dt}{t} \int_{|z|<t} \alpha = \int_{\mathbb{C}} g_0^r \alpha.$$

*Proof:* We consider the variety $\mathbb{C}_t(r) \subset \mathbb{C} \times \mathbb{R}_{\geq 0}$ defined in the following way: $\mathbb{C}_t(r) := \{(z,t) \quad / \quad |z| < t \quad t < r\}$. Let $p_1 : \mathbb{C}_t(r) \to \mathbb{C}$ and $p_2 : \mathbb{C}_t(r) \to \mathbb{R}$ the natural projections. Then we can compute the integral $\int_{\mathbb{C}_t(r)} p_1^*(\alpha) \wedge p_2^*(\frac{dt}{t})$ in two ways, using Fubini Theorem and we obtain:

$$\begin{aligned}\int_{\mathbb{C}_t(r)} p_1^*(\alpha) \wedge p_2^*(\frac{dt}{t}) &= \int_0^r \frac{dt}{t} \int_{|z|<t} \alpha \\ &= \int_{\mathbb{C}} \alpha \int_{|z|}^r \frac{dt}{t}.\end{aligned}$$

The conclusion follows from the fact that $\int_{|z|}^r \frac{dt}{t} = \frac{1}{2} g_0^r$.

Observe that the First Main theorem implies this, often useful, corollary.

**7.13 Corollary.** *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, then*

$$T_f(r, \mathcal{O}(1)) = \int_{|z|=r} \log \sqrt{1 + |f(re^{i\theta})|^2} \frac{d\theta}{2\pi} + \log |f(0)|.$$

*Proof:* It suffices to remark that the image of $f$ do not meet the divisor $[\infty]$ of $\mathbb{P}^1$ and the norm of the section $s \in H^0(\mathbb{P}^1, \mathcal{O}(1)$ such that $div(s) = [\infty]$ is, over the chart $\mathbb{C}$, given by $\|s\|(x) = \frac{1}{\sqrt{1+|x|^2}}$.

# 8 Lecture VII.

In the last lecture we developed some tools from Nevanlinna theory of analytic maps from $\mathbb{C}$ to a projective variety. We saw that, when duly interpreted, this theory has features very similar to the theory of heights and the theory of degree of line bundles on projective curves.

Of course one would like a more general Nevanlinna theory of analytic maps from Riemann surfaces to projective variety. In principle this can be developed but, if the Riemann surface do not have nice properties, it will not be very satisfactory.

We briefly describe here how the theory may be generalized to analytic map from *affine* Riemann surfaces to projective varieties. The wider class where we can develop a nice Nevanlinna theory is the so called class of Parabolic Riemann surfaces in the sense of Ahlfors Myrberg: every affine Riemann surface and every finite covering of the affine line is in this class.

Let $M$ be a smooth affine curve and $\overline{M}$ the smooh projective compactification of it. We denote by $D = \sum_{i=1}^{r}$ the divisor $\overline{M} \setminus M$. We also fix a point $p \in M$. Fix a smooth projective variety equipped with an hermitian line bundle $L$.

Let $f : M \to X$ be an analytic map.

Lemma 7.12 tells us that, in order to define the counting function when $M = \mathbb{C}$ and $p = 0$, the main tool is the introduction of the Green functions $g_0^r$. And these are obtained, essentially from the global function $\log |z|^2$ on $\mathbb{C}$. Indeed, if we denote by $g_0(z)$ the real function $\log |z|^2$ then we remark that the only properties we used to develop the Nevanlinna theory on $\mathbb{C}$ are:

– we have that $dd^c(g_0(z) = \delta_0$ and $g_0(z)$ is a smooth function outside 0;

– we have that $g_0(z) \to +\infty$ when $|z| \to +\infty$;

– If, for every real number $a$ we denote by $(a)^+$ the number $\sup\{a, 0\}$, then $g_0^r = (\log r^2 - g_0(z))^+$.

The reader may check by exercise that the only properties which are used to prove the first main theorem are the properties above.

Consequently, if on an arbitrary Riemann surface we have a function which has properties similar to the properties above, we can develop a Nevanlinna theory over it.

We now quote, without proof, some theorems which allow to develop Nevanlinna theory for analytic maps from $M$ to $X$. We will see that, at least formally, the definitions are very similar to the constructions we made in the previous section. The proofs require a little bit of more involved complex analysis and Hodge theory. Since we will not need the proofs, we omit them.

We recall that an harmonic function is a function $v$ on a Riemann surface, such that $dd^c(v) = 0$.

**8.1 Theorem.** *Up to an additive scalar, there exists a unique function* $g_p : \overline{M} \to [-\infty; +\infty]$ *with the following properties:*

*a) it satisfies the differential equation*

$$dd^c g_p = \delta_p - \frac{1}{d} \cdot \delta_D;$$

$\delta_p$ *(resp.* $\delta_D$*) being the dirac operator on* $p$ *(resp. on* $D$*).*

*b) It is a $C^\infty$ function on $\overline{M} \setminus \{p\} \cup \{|D|\}$.*
*c) There is a open neighborhood $U$ of $p$ and an harmonic function $v_p$ on $U$ such that*

$$g_p|_U = \log|z-p|^2 + v_p.$$

The function $g_p$ plays, over an arbitrary Riemann surface $\overline{M}$, the role played by $\log|z|^2$ on $\mathbb{C}$. Let's quote some properties of it.

**8.2 Example.** If $M = \mathbb{C}$, so $\overline{M} = \mathbb{P}_1$ then $g_p = \log|z-p|^2$ up to a constant.

The reader will notice the similarities with the function $g_0^r$ used in the previous section.
The following property tells us that $g_p$ is an exhaustion function over $M$:

**8.3 Proposition.** *For every constant $C$, we have that $g_p^{-1}((C, +\infty])$ is a non empty neighborhood of $D$ in $\overline{M}$.*

If $p$ and $q$ are two different points of $M$ then we can compare $g_p$ and $g_q$:

**8.4 Proposition.** *Suppose that $p$ and $q$ are two different points of $M$. For every relatively compact open neighborhood $U$ of $p$ and $q$ there exists a constant $C_U$ such that*

$$|g_p(z) - g_q(z)| \leq C_U$$

*for every $z \notin U$.*

We introduce then the *Green functions* over an arbitrary Riemann Surface.

**8.5 Definition.** *Let $U$ be a regular region on a Riemann surface $M$ and $p \in U$. A* Green function *for $U$ and $p$ is a function $g_{U;p}(z)$ on $U$ such that:*
*a) $g_{U;p}(z)|_{\partial U} \equiv 0$ continuously;*
*b) $dd^c g_{U;p} = 0$ on $U \setminus \{p\}$;*
*c) near $p$, we have $g_{U;p} = -\log|z-p|^2 + \varphi$, with $\varphi$ continuous, thus harmonic, around $p$.*

One extend $g_{U,p}$ to all of $M$ by defining $g_{U,p} \equiv 0$ outside the closure of $U$. We easily deduce from the definitions that $dd^c g_{U;p} + \delta_P = \mu_{\partial U;p}$ where $\mu_{\partial U;p}$ is a positive measure of total mass one and supported on $\partial U$.
Moreover the following is true:

**8.6 Proposition.** *The Green function, if it exists, it is unique.*

For every positive real number $r$, we consider the following two closed sets of $X$

$$B(r) := \left\{ z \in M \text{ s.t } \; g_p(z) \leq \log(r^2) \right\} \quad \text{and} \quad S(r) := \left\{ z \in M \text{ s.t } \; g_p(z) = \log(r^2) \right\}.$$

The function $g_p$ is strictly related with the Green function on $B(r)$:

**8.7 Proposition.** *Let $r$ be a positive real number. The function*

$$g_p^r := \log(r^2) - g_p|_{B(r)}$$

*is the Green function of $B(r)$ and $p$. Consequently, for every $p$ and $q$ in $X$ there is a constant $C$, depending on $p$ and $q$, such that, for every $r$ sufficiently big,*

$$\left|g_p^r(q) - \log(r)\right| \leq C.$$

For a proof of all these statements cf. [Ga] and the references there.

We may remark that proposition above is a formal consequence of the definitions and the properties of the $g_p$ and the Green functions. Observe that in also in this case, the measure $\mu\partial B(r), p$ is the restriction to $S(r)$ of the form $d^c g_p$.

**8.8** *The function $g_p$ and metrics.* Over the affine curve $\mathbb{C}$ the function $g_p(z)$ may be used to define a metric on the line bundle $\mathcal{O}(p)$: Indeed let $\mathbb{I}_p$ be the section of $\mathcal{O}(p)$ which defines the divisor $p$ then we define $\|\mathbb{I}_p\|(z) = e^{\frac{1}{2}g_p(z)}$; this is nothing else then $\|\mathbb{I}_p\|(z) = |z-p|$. Evidently this metric extends over $p$.

This procedure can be generalized to any affine curve. Let $M$ be an affine curve and $p \in M$ then we define a metric on $\mathcal{O}(p)$ in the following way: let $\mathbb{I}_p$ be the section of $\mathcal{O}(p)$ which defines the divisor $p$ then we define $\|\mathbb{I}_p\|(z) = e^{\frac{1}{2}g_p(z)}$.

Notice that $g_p$ is defined up to an additive constant and $\mathbb{I}_p$ is defined up to a multiplicative constant. The reader will check that all the statements will remain true if we make other choices.

It is important to observe that this also defines a metric on the stalk of the cotangent bundle at $p$:

Let $\Omega^1_M$ be the cotangent bundle of $M$. We have a *canonical isomorphism*

$$\Omega^1_M(p)_p \xrightarrow{\simeq} \mathbb{C}.$$
$$f(z-p)\frac{dz}{z-p} \longrightarrow f(p)$$

The isomorphism above, which is called *the residue isomorphism*, do not depend on the choice of the local coordinate around $p$ (the reader can check it by exercise). Thus, if we denote by $T_pM$ the tangent space of $M$ at $p$ it defines a *canonical* isomorphism

$$T_pM \xrightarrow{\simeq} \mathcal{O}(p)|_p.$$

Consequently the metric on $\mathcal{O}(p)$ defined above defines also a metric on the tangent space $T_pM$. This is defined as following: the space $T_pM$ is generated by the derivation $\frac{d}{dz}$ and

$$\left\|\frac{d}{dz}\right\| = e^{\frac{1}{2}v_p(p)}.$$

**8.9** *Nevanlinna theory over arbitrary affine curve.* We are now ready to generalize the notion of analytic height theory to an arbitrary analytic map $f : M \to X$. First we define the analytic degree of a hermitian line bundle $L$ on $M$ (it is also called the characteristic function of $L$ ad then the height as an analytic degree:

**8.9 Definition.** *(a) Let $N$ be an hermitian line bundle over $M$ and $r$ be a positive real number. We define*

$$T(r, L) := \int_0^r \frac{dt}{t} \int_{B(r)} c_1(N)$$

*to be the analytic degree of r with respect of $N$.*

*(b) Suppose that L is an hermitian line bundle on $X$ and r s a positive real number. Then we define*

$$T_f(r, L) := \int_0^r \frac{dt}{t} \int_{B(r)} f^*(c_1(L))$$

*to be the* the analytic height *of r with respect to $f$ and $L$.*

The reader will notice that the definition is formally identical to its analogue when $M = \mathbb{C}$.

**8.10 Example.** Suppose that we put on $\mathcal{O}(p)$ the metric such that, if $\mathbb{I}_p$ is the section of it vanishing at $p$ then $\|\mathbb{I}_p\|(z) = e^{\frac{1}{2} g_p(z)}$. Then outside $p$ we have $c_1(\mathcal{O}(p)) = -dd^c(g_p) = 0$. Thus $T(r, \mathcal{O}(p)) = 0$.

The Poincaré–Lelong formula and the same proof given before allows to obtain the First Main Theorem (and its consequences):

**8.11 Theorem.** *(a) Suppose that $N$ is an hermitian line bundle on $M$ and $s \in H^0(M, N)$ is such that $s(p) \neq 0$, then*

$$T(r, N) = -\int_{S(r)} \log \|s\| d^c(g_p) + \frac{1}{2} \sum_{q \in B(r)} v_q(s) g_p^r(q) + \log \|s(p)\|.$$

*(b) Suppose that $f : M \to X$ is as above. Let $s \in H^0(X, L)$ be a global section such that $s(f(p)) \neq 0$. Then*

$$T_f(r, L) = -\int_{S(r)} \log \|s\| d^c(g_p) + \frac{1}{2} \sum_{q \in B(r)} v_q(f^*(s)) g_p^r(q) + \log \|s(f(p))\|.$$

We may then resume the properties of the analytic heighs. The reader will compare these properties with the properties of geometric heights 4.11 and the properties of arithmetic heights 5.9:

**8.12 Theorem.** *:* (Properties of analytic heights) *Let $X$ be a projective variety and $f : M \to X$ be an analytic map.*

*(a) Let $L_1$ and $L_2$ be two hermitian line bundles on $X$ then*

$$T_f(r, L_1 + L_2) = T_f(r, L_1) + T_f(r, L_2);$$

*(b)* (functoriality of analytic heights) *Suppose that $Y$ is another projective variety, $L$ is an hermitian line bundle on $Y$ and $h : X \to Y$ is an analytic map then*

$$T_{h\circ f}(r, L) = T_f(r, h^*(L));$$

*(c) Suppose that $L$ is an hermitian line bundle on $X$ and $s \in H^0(X, L)$ is such that $s((f(p)) \neq 0$. Then we may find a constant $C$ independent on $r$ such that*

$$T_f(r, L) \geq C.$$

*(d) Suppose that $L$ is an ample line bundle equipped with a positive metric on $X$. Let $A$ be a posive constant, then the set $\{r \in \mathbb{R} \;\; / \;\; T_f(r, L) \leq A\}$ is compact in $\mathbb{R}$. Moreover there exists a constant $C$, independent on $r$ such that $T_f(r, L) \geq C$.*

**8.13** *Order of growth.* Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an entire function. We say that the function $f$ has *finite order of growth $\rho$* if we may find an constant $C > 0$ such that , for every positive real number $R$ we have

$$\log \sup\{|f(z)| \;\; / \;\; |z| \leq R\} \leq C \cdot R^\rho.$$

Most of the known functions are of finite order: for instance $f(z) = e^z$ has finite order of growth 1 and every polynomial $h(z)$ has even smaller order of growth: $\log \sup\{|h(z)| \;\; / \;\; |z| \leq R\} \leq C \log |z|$.

With a little bit of work one can also prove the following:

**8.13 Proposition.** *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. For every real number $r > 0$ denote by $M_f(r)$ the number $\sup_{|z|\leq r}\{\frac{1}{2}\log(1 + |f(z)|^2)\}$. For every couple of real numbers $0 < r < R$ the following inequalities hold*

$$T_f(r, \mathcal{O}(1)) \leq M_f(r) \leq \frac{R + r}{R - r} T_f(R, \mathcal{O}(1))$$

*where $\mathcal{O}(1)$ is the tautological line bundle on $\mathbb{P}^1$ with the Fubini–Study metric.*

Thus the order of growth of a function is essentially controlled by the analytic height function. For this reason we give the following definition.

Suppose that $M$ is an affine Riemann surface with a marked point $p \in M$. Let $X$ be a smooth projective variety equipped with an ample line bundle $L$. We suppose that the metric on $L$ is positive.

**8.14 Definition.** *We will say that the order of growth of $f$ is $\rho > 0$ if*

$$\limsup_{r\to\infty} \frac{\log(T(r, L))}{\log(r)} = \rho.$$

It is important to observe that:

(a) The order of growth do not depend on:

– the choice of the base marked point $p$,

– the choice of the ample line bundle $L$,

– the choice of the metric on $L$.

(b) If $X = \mathbb{P}^1$, and $f : \mathbb{C} \to \mathbb{P}^1$ is an entire function, then the two definition of order of growth coincide.

(c) One can prove that, $f : M \to X$ extends to an algebraic map $f : \overline{M} \to X$ if and only if

$$\limsup_{r\to\infty} \frac{T_f(r, L)}{\log(r)} < +\infty.$$

# 9 Lecture VIII.

In the previous lectures we developed three parallel theories, the arithmetic, the geometric and the analytic height theory and we showed interesting common features. One would hope that the list of these common features will increase in the future. Many interesting conjectures are made about this.

On the other direction one would like to see if these three theories may be part of a unified theory where arithmetic, geometry and analysis where may interact together and give informations which may be interesting for all of them.

The theory where they can interact is the *geometric transcendence theory*.

The principal objects of the analytic theory are analytic maps from affine curves to projective varieties. Similarly, the principal objects of the arithmetic theory are rational points on projective varieties again. Thus one would like to study the interactions between analytic maps of affine varieties and rational points. This is exactly the subject of the geometric transcendence theory!.

Of course there should be some relationship between the analytic map we are looking to and the arithmetic of the variety otherwise it is easy to show that everything may happen.

Suppose that $f : M \to X$ is an analytic map. And, in order to deal with arithmetic, we suppose that $X$ is a projective variety defined over a number field $K$. The first condition we may suppose verified is that locally around the points $p$ of $f(M) \cap X(K)$, the analytic variety $f(M)$ is in some way defined over $K$. By this we mean the following: let $p \in M$ a point such that $f(p) = q \in X(K)$. Let $z_1, \dots, z_n$ algebraic coordinates around $q$, then we can choose a coordinate $t$ around $p$ in such a way that the function $f$ is given by power series $z_i = z_i(t) = \sum_{j=1}^{\infty} a_{ij} t^j$ with $a_{ij} \in K$. In this case we say that the germ of $f$ at $p$ is defined over $K$. This condition, even if quite natural, is difficult to be verified for an arbitrary analytic map $f$. But it is verified when $f$ is for instance a solution of an algebraic differential equation defined over $K$.

**9.1 Example.** Consider the differential equation $y' = y$. A solution of it defines an analytic map $f : \mathbb{C} \to \mathbb{C}^2$ given by $f(z) = (z, e^z)$. A point in $f(\mathbb{C}) \cap \mathbb{A}^2(\mathbb{Q})$ would be a rational number $z$ such that $e^z \in \mathbb{Q}$. Near the point $z$ the map $f$ is given by $f(z) = (z, \sum_{i=0}^{\infty} \frac{z^n}{n!})$. We will prove in the sequel that the only possible $z$ is $z = 0$.

We saw that a good method to study rational points on projective varieties over number fields has been to choose models of these and extend the rational points to morphisms from $Spec(O_K)$ to the model. Of course it would be difficult to extend analytic maps to models. But the germs of analytic maps are easier to handle and easier to extend.

**9.2** *Overview of formal schemes.* In order to work with germs of analytic maps, the natural language is the language of *formal schemes.* Formal geometry is a way to do local analysis on arbitrary schemes. Since it is not possible to give here an account of the theory of formal schemes, we will simply define and recall here the properties we need:

– Let $X$ be a scheme and $Y$ be a closed subscheme of it. We define the formal neigborhood of $Y$ in $X$ in the following way:

Let $I_Y$ be the sheaf of ideals defining $Y$ in $X$. The completion $\hat{X}_Y$ of $X$ around $Y$ is the following ringed topological space: As topological space it is $Y$ and as a ring of sheafs one has the limit $\lim_{\leftarrow} \mathcal{O}_X/I_Y^n$.

– We will call the subscheme of $X$ given by the ideal $I_Y^n$ the *i-th infinitesimal neighborhood of* $Y$ *in* $X$ and denote it by $(\hat{X}_Y)_n$. Observe that we have natural closed inclusions of schemes: $(\hat{X}_Y)_{n-1} \hookrightarrow (\hat{X}_Y)_n$.

**9.2 Example.** Let $A$ be a ring and $X = \widehat{\mathbb{A}}^1_A$. Let 0 be the origin of $\widehat{\mathbb{A}}^1_A$. The completion $\mathbb{A}^1_0$ is the formal spectrum of the ring $A[\![z]\!]$. By this we mean the projective limit of the spectra of the rings $A[t](t^n)$ with the natural maps $A[t]/(t^n) \to A[t]/(t^{n-1})$. We have to imagine the $i$–th infinitesimal neighborhood of 0 as the ring of formal power series truncated at the order $i$.

– In particular, if $A$ is a ring and $I$ is an ideal, we denote by $Spf(A_I)$ the projective limit of the schemes $A_n := A/I^n$. Observe that in this situation we have an exact sequence

$$0 \to I^{n-1}/I^n \longrightarrow A_n \longrightarrow A_{n-1} \to 0.$$

Observe that $I^{n-1}/I^n$ is a sheaf supported on the closed subscheme $A/I$.

– Denote by $X = Spec(A)$ and $Y = Spec(A/I)$. One can prove that, if $I$ is defined by a regular sequence (for instance if $X$ is the ring of regular function of a smooth variety and the closed subvariety $Y$ is also smooth), then the sheaf $I/I^2$ is locally free on $Y$; it is usually called the *conormal sheaf of* $Y$ *in* $X$ and moreover $I^{n-1}/I^n = Sym^{n-1}(I/I^2)$.

– The construction above generalize to arbitrary closed schemes of a scheme. A subscheme locally defined by an ideal which is generated by a regular sequence is called a *locally complete intersection subscheme.* If $Y$ is a closed subscheme of a scheme $X$

with ideal sheaf $I_Y$, then $N_X(Y) := I_Y/I_Y^2$ is a sheaf with support on $Y$, thus a sheaf on $Y$. It is called the *conormal sheaf of* $Y$ *in* $X$. If $Y$ is a locally complete intersection of a scheme $X$ then $N_X(Y)$ is locally free. In this situation we have a generalization of the exact sequence above

$$0 \to Sym^{n-1}(I_Y/I_Y^2) \longrightarrow \mathcal{O}_{(\hat{X}_Y)_n} \longrightarrow \mathcal{O}_{(\hat{X}_Y)_{n-1}} \to 0.$$

– A typical example of locally complete intersection is the following: $\mathcal{X} \to Spec(O_K)$ an arithmetic scheme and $P : Spec(O_K) \to \mathcal{X}$ a section of it. If the local sheaf $\mathcal{O}_{X,P}$ is regular, then $P$ is locally complete intersection in $\mathcal{X}$.

– If $\mathcal{X}$ is as above and there exists a neighborhood of $P$ which is smooth of relative dimension $N$ over $Spec(O_K)$ then, *non canonically*

$$\widehat{\mathcal{X}}_P \simeq Spf(O_K[\![z_1, \ldots, z_N]\!]).$$

– Moreover in the situation above, the restriction $\Omega^1_{\mathcal{X}/O_K}$ of $\Omega^1_{\mathcal{X}/O_K}$ to $P$ is locally free and $Sym^n(\Omega^1_{\mathcal{X}/O_K})$ is free and generated by the "monomials" $(dz_1)^{\otimes i_1} \cdots (dz_N)^{\otimes i_N}$ with $i_1 + \ldots + i_N = n$.

– Similarly, if $M$ is a Riemann surface and $p \in M$ is a point, then, *non canonically*

$$\widehat{M}_P \simeq Spf(\mathbb{C}[\![t]\!]).$$

– More generally, if $X$ is a smooth algebraic variety of dimension $N$, and $p \in X$, then *non canonically*,

$$\widehat{X}_p \simeq Spf(\mathbb{C}[\![z_1, \ldots, z_N]\!].$$

If $f : M \to X$ is an analytic map. For every $z \in M$, the map $f$ induces a map of formal schemes

$$Spf(\mathbb{C}[\![t]\!]) \simeq \widehat{M}_z \xrightarrow{\hat{f}_z} \widehat{X}_{f(p)} \simeq Spf(\mathbb{C}[\![z_1, \ldots, z_N]\!].$$

Conversely suppose that $q \in X$ and

$$\hat{h} : Spf(\mathbb{C}[\![t]\!]) \longrightarrow \widehat{X}_q$$

is a map of formal schemes, we will say that $\hat{h}$ *has positive radius of convergency* if there exists a Riemann surface $M$, a point $p \in M$ and an analytic map $f : M \to X$ such that $h(p) = q$ and $\hat{h} = \hat{f}_p$.

**9.3 Example.** (a) The morphism of formal germs $\hat{h}(t) = \sum_{i=1}^{\infty} t^i$ defines a map with positive radius of convergency $\hat{h} : Spf(\mathbb{C}[\![t]\!]) \to \widehat{\mathbb{C}}_0$.

(b) The morphism of formal germs $\hat{h}(t) = \sum_{i=1}^{\infty} i! t^i$ is a map of formal schemes $\hat{h} : Spf(\mathbb{C}[\![t]\!]) \to \widehat{\mathbb{C}}_0$ which *do not* have positive radius of convergency.

– Suppose that $\mathcal{X} \to Spec(O_K)$ is an arithmetic scheme and $P : Spec(O_K) \to \mathcal{X}$ be a point of it. Denote by $X_K$ the generic fibre of $\mathcal{X}$ and $p \in X_K(K)$ the induced point.

We have a commutative diagram

$$\begin{array}{ccc} \widehat{(X_K)}_p & \longrightarrow & \widehat{\mathcal{X}}_P \\ \big\downarrow & & \big\downarrow \\ Spec(K) & \longrightarrow & Spec(O_K). \end{array}$$

Remark that this diagram *is not* cartesian.

**9.4** *Germs of type E.* Let $K$ be a number field, $O_K$ be its ring of integers, $M_K$ the set of places of $K$. Denote with $M_\infty \subset M_K$ the set of infinite places of $K$ and with $M_{fin} \subset M_K$ the set of finite places of $K$. In the sequel we will denote by $\widehat{\mathbb{A}}_0^N$ the $O_K$ formal scheme $Spf(O_K[\![Z_1,\dots,Z_N]\!])$ and by $(\widehat{\mathbb{A}}_0^N)_K$ the $K$–formal scheme $Spf(K[\![Z_1,\dots,Z_N]\!])$. We fix a $\sigma \in M_\infty$. For every geometric object $X$ defined over $O_K$, we will denote $X_K$ its restriction to the generic fibre $K$ and $X_\sigma$ its extension to $\mathbb{C}$ via the embedding $\sigma : K \hookrightarrow \mathbb{C}$.

**9.4 Definition.** *Let $\underline{C} := (C_v)_{v\in M_{fin}}$ be a sequence of numbers indexed by the finite places of $K$. We will say that $\underline{C}$ is an* admissible sequence *if:*
– $C_v \geq 1$ *for every* $v \in M_{fin}$*;*
– $\prod_{v\in M_{fin}} C_v = C < \infty$.
*We will say that $C$ is the radius of $\underline{C}$.*

We denote by $\underline{1}$ the admissible sequence with $C_v = 1$ for every $v \in M_{fin}$. Once we fix an admissible sequence we may define series which are suitable for the study of the geometric transcendence theory:

**9.5 Definition.** *Let $\underline{C}$ an admissible sequence and $\alpha$ be a non negative real number. We will say that a formal power series $f(t) = \sum_{i=0}^{\infty} a_i t^i \in K[\![t]\!]$ is a $E$* germ of type $(\underline{C},\alpha)$ at finite places *if, for every finite place $v \in M_{fin}$ we have*

$$\|a_i\|_v \leq \frac{C_v^i}{\|i!\|_v^\alpha}.$$

*The set of $E$–germs of type $(\underline{C},\alpha)$ at finite places is a ring containing $O_K[\![t]\!]$, it will denoted by $R_{\underline{C},\alpha}$ and called* the ring of $E$–germs of type $(\underline{C},\alpha)$ at finite places.

It is easy to see that $O_K[\![t]\!] = R_{(\underline{1},0)}$. We also have a natural inclusion

$$i_K : R_{\underline{C},\alpha} \otimes_{O_K} K \hookrightarrow K[\![t]\!] \tag{9.6.1}$$

In the sequence, we will denote by $\mathbb{D}_{\underline{C},\alpha}$ the $O_K$–formal scheme $Spf(R_{\underline{C},\alpha})$.

A way to think about $\mathbb{D}_{\underline{C},\alpha}$ is to imagine it as a sequence of disks, one for each place of $K$. The radius of each disk being not too small. For instance $\mathbb{D}_{\underline{1},0}$ is a sequence of disks of radius one.

We remark that for every $\underline{C}$ we have a canonical morphism $\iota_{\underline{C},\alpha} : \mathbb{D}_{\underline{C},\alpha} \to \mathbb{D}_{\underline{1},0} = Spf(O_K[\![t]\!])$. The natural inclusion $Spf(K[\![t]\!]) \hookrightarrow (\mathbb{D}_{\underline{1},0})_K$ factorizes:

$$Spf(K[\![t]\!]) \hookrightarrow (\mathbb{D}_{\underline{C},\alpha})_K \hookrightarrow (\mathbb{D}_{\underline{1},0})_K.$$

Similarly if $\sigma$ is an infinite place of $K$, the natural inclusion $\sigma : K \hookrightarrow \mathbb{C}$ induces a series of inclusions: $Spf(\mathbb{C}[\![t]\!]) \hookrightarrow Spf(K[\![t]\!]) \hookrightarrow (\mathbb{D}_{\underline{C},\alpha})_K \hookrightarrow (\mathbb{D}_{\underline{1},0})_K$.

As we remarked before, the sheaf of relative differentials $\Omega^1_{\mathbb{D}_{\underline{1}}/O_K}$ is free or rank one and canonically generated by $dt$. We will denote by $T_{\underline{1}/O_K}$ the free $O_K$ module of rank one $P^*_C(\Omega^1_{\mathbb{D}_{\underline{1}}/O_K})^\vee$.

For every non negative integer $i$, we denote by $(R_{\underline{C},\alpha})_i$ the ring $R_{\underline{C},\alpha}/t^{i+1}$ and by $(\mathbb{D}_{\underline{C},\alpha})_i$ the scheme $\mathrm{Spec}((R_{\underline{C},\alpha})_i))$. The scheme $(\mathbb{D}_{\underline{C},\alpha})_i$ is the $i$–th infinitesimal neighborhood of the center of $\mathbb{D}_{\underline{C},\alpha}$.

For every positive integer $i$ the canonical closed inclusions $(\mathbb{D}_{\underline{C},\alpha})_{i-1} \hookrightarrow (\mathbb{D}_{\underline{C},\alpha})_i \hookrightarrow \mathbb{D}_{\underline{C},\alpha}$ induces a canonical exact sequence

$$0 \longrightarrow (t)^{i-1}/(t)^i \longrightarrow (R_{\underline{C},\alpha})_i \longrightarrow (R_{\underline{C},\alpha})_{i-1} \longrightarrow 0. \tag{9.6.1}$$

Moreover the inclusion $\iota_{\underline{C},\alpha}$ give rise to a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & (t)^{i-1}/(t)^i & \longrightarrow & (R_{\underline{C},\alpha})_i & \longrightarrow & (R_\alpha\underline{C})_{i-1} & \longrightarrow 0. \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \longrightarrow & T^{\otimes-(i-1)}_{\underline{1}/O_K} & \longrightarrow & (O_K[\![t]\!])_i & \longrightarrow & (O_K[\![t]\!])_{i-1} & \longrightarrow 0. \end{array}$$

where all the vertical arrows are inclusions. Moreover observe that when we tensorize by $K$ we obtain an isomorphism

$$\iota_K : (T^{\otimes-i}_{\underline{1}/O_K})_K \to ((t)^i/(t)^{i+1})_K. \tag{9.7.1}$$

This shows in particular that the rings $R_{\underline{C}}$ are not regular when $\underline{C} \neq \underline{1}$.

Let $\mathcal{X} \to Spec(O_K)$ be an arithmetic scheme and $P : Spec(O_K) \to \mathcal{X}$ be a point of it. Denote by $X_K$ the generic fibre of $\mathcal{X}$ and $p \in X_K(K)$ the induced point. Will denote again by $\widehat{\mathcal{X}}_P$ the completion of $\mathcal{X}$ around $P$ and by$\widehat{(X_K)}_p$ the completion of $X_K$ around $p$ respectively.

Let $\gamma_K : Spf(K[\![t]\!]) \to \widehat{(X_K)}_p$ be a morphism of formal schemes. For every $\sigma \in M_\infty$ the inclusion $K[\![t]\!] \hookrightarrow \mathbb{C}[\![t]\!]$ induces a morphism of formal schemes $\gamma_\sigma : \mathbb{C}[\![t]\!] \to \widehat{(X_\sigma)}_p$.

We give now the definition of $E$ germ; it is a particular map of formal schemes $\gamma_K : Spf(K[\![t]\!]) \to \widehat{(X_K)}_p$ with good property of convergency at every place:

**9.8 Definition.** *A $E$–germ of type $(\underline{C},\alpha)$ of $X_K$ is a $K$–morphism of formal schemes*

$$\gamma_K : Spf(K[\![t]\!]) \to \widehat{(X_K)}_p$$

*with the following properties:*

*(a) For every $\sigma \in M_\infty$ the induced map $\gamma_\sigma : \mathbb{C}[\![t]\!] \to \widehat{(X_\sigma)}_p$ has positive radius of convergency;*

*(b) There exists a morphism of $O_K$–formal schemes $\gamma : \mathbb{D}_{\underline{C},\alpha} \to \widehat{\mathcal{X}}_P$ for which the following diagram is commutative:*

$$\begin{array}{ccc} Spf(K[\![t]\!]) & \xrightarrow{\gamma_K} & \widehat{(X_K)}_p \\ \downarrow & & \downarrow \\ (\mathbb{D}_{\underline{C},\alpha})_K & \xrightarrow{(\gamma)_K} & (\widehat{\mathcal{X}}_P)_K. \end{array}$$

At this point we are ready to study arithmetic properties of analytic maps.

Let $K$ be a number field, $O_K$ be its ring of integers and $\sigma$ an infinite place of $K$. We fix an admissible sequence $\underline{C}$ and $\alpha \geq 0$

Suppose that $\mathcal{X} \to Spec(O_K)$ is a projective arithmetic variety. We denote by $X_K$ its generic fibre and by $X_\sigma$ its restriction to the place $\sigma$. We suppose that $X_K$ is a smooth variety of dimension $N > 1$. For every rational point $p \in X_K(K)$ denote by $P : Spec(O_K) \to \mathcal{X}$ the corresponding section.

Let $M$ be a Rieman surface and $f : M \to X_\sigma(\mathbb{C})$ be an analytic map. As explained before, for every point $z \in M$ the map $f$ induces a map

$$Spf(\mathbb{C}[\![t]\!]) \simeq \widehat{M}_z \xrightarrow{\hat{f}} \widehat{(X_\sigma)}_z.$$

Denote by $S_f(\alpha)$ the subset of points $p$ of $M$ such that $f(p) \in X_K(K)$ and for wich there exists a $(\underline{C}, \alpha)$ germ $f_p : Spf(K[\![t]\!]) \to \widehat{(X_K)}_p$ such that the following diagram is commutative

$$\begin{array}{ccc} Spf(\mathbb{C}[\![t]\!]) & \xrightarrow{\hat{f}} & \widehat{(X_\sigma)}_p \\ \downarrow & & \downarrow \\ Spf(K[\![t]\!]) & \xrightarrow{(\gamma)_K} & \widehat{(X_K)}_p. \end{array}$$

Observe that $\underline{C}$ may vary for different points.

The set $S_f(\alpha)$ it the set of rational points in the pre–image of $X_K(K)$ for which the germ of $f$ in their neighborhood is defined over $K$ and have good arithmetic properties.

**9.9 Example.** Suppose that we have a differental form $\omega := f(z,w)dz + g(z,w)dw$ defined over $K$ ($f$ and $g$ are algebraic functions defined over $K$) and q is a point where either $f(q) \neq 0$ or $g(q) \neq 0$ (or both). Suppose that $f^*(\omega) = 0$, this means that $f$ is the unique solution of the differential equation $\omega = 0$. Then one can show that every point in $f^{-1}(q)$ is in $S_f(\underline{C}, 1)$ for some suitable $\underline{C}$.

The main theorem of these lectures is:

**9.10 Theorem.** *Suppose , in the hypotheses above that the order of growth of $f$ is $\rho$ and the image of $f$ is Zariski dense. Then the cardinality of $S_f(\alpha)$ is at most $\frac{N+1}{N-1}\rho \cdot \alpha[K : \mathbb{Q}]$.*

This theorem implies at once the transcendence of $e$ and the transcendence of $\pi$:

– Suppose that $e$ is algebraic. Then since the function $f : \mathbb{C} \to \mathbb{C}^2$ defined by $f(z) = (z, e^z)$ defines $(\underline{1}, 1)$ germs in every integral point, the set $\mathbb{Z} \subset \mathbb{C}$ will be contained in $S_f(1)$ and this is a contradiction.

– Suppose that $\pi$ is algebraic, then $i\pi$ is algebraic and again the set $i\pi \cdot \mathbb{Z} \subset \mathbb{C}$ will be contained in $S_f(1)$ and this is a contradiction.

## 10 Lecture IX.

We will now introduce some other general tools from Arakelov geometry necessary to the proof of Theorem 9.10. The reader will observe that this proof requires an interaction of the three theories we described so we go over the simple analogy.

Unfortunately the proof requires some other tools which are a little bit more involved of what we proved until now. Once again we try to give details of the proofs but sometimes we will need to use some facts without proofs.

We suppose that $K$, $O_K$, $\sigma$, the arithmetic variety $\mathcal{X}$, the analytic map $f : M \to X_\sigma$ etc. are fixed as in the previous lecture. for every $\tau \in M_\infty$ we fix a smooth metric $\eta_\tau$ on $X_\tau$. We also fix an ample line bundle $L$ on $\mathcal{X}$ equipped with, for every $\tau \in M_\infty$, with a positive metric on $L_\tau$. For every positive integer $N$, the $\mathbb{C}$–vector space $H^0(X_\tau, L_\tau^N)$ is equipped with a natural norm $\| \cdot \|_\tau$ and a natural hermitian product:

– if $s \in H^0(X_\tau, L_\tau^N)$ then we define $\|s\|_\tau := \sup_{z \in X_\tau(\mathbb{C})}\{\|s\|_\tau(z)\}$. This is a norm on $H^0(X_\tau, L_\tau^N)$.

– If $s_1$ and $s_2$ are in $H^0(X_\tau, L_\tau^N)$ then we define $\langle s_1; s_2\rangle_\tau := \int_{X_\tau} \langle s_1; s_2\rangle_\tau \eta_\tau$. This defines an hermitian structure on $H^0(X_\tau, L_\tau^N)$. The corresponding norm will be denoted by $\|s\|_{\tau, L_2}$.

Even if a priory the sup and the $L_2$ norms may be very different, asymptotically the things are not too bad:

**10.1 Theorem.** *(Gromov) Let $X$ be a compact complex variety of dimension $N$ equipped with a measure $\eta$. Let $L$ be an hermitian line bundle over it. Then there exists positive constants $C_i$ such that, for every positive integer $D$ and for every global section $s \in H^0(X, L^{\otimes D})$ the following inequality holds:*

$$C_1\|s\|_{L_2} \leq \|s\|_{\sup} \leq C_2 D^N \|s\|_{L_2}.$$

We will not need the theorem above but nevertheless we provide a proof for sake of completeness.

*Proof:* The first inequality is evident: $\int_X \|s\|^2\eta \leq \|s\|^2 \int_X \eta = \|s\|^2 Vol(X, \eta)$.

We may cover $X$ with a finite set of open set which are biholomorphic to the disk of dimension $N$. We may also suppose that the disks of radius $\epsilon$ also cover $X$. Let $B$

such a disk and $d\lambda$ the standard Lebesgue measure on it. There is a constant, which depends only on $X$ such that $\eta|_B \geq Cd\lambda$. The restriction of $L$ to $B$ is trivial and there exist a uniformly on $X$ bounded function $p(z)$ such that if $f \in H^0(B, L^D|_B)$ then $\|f\|^2(z) = |f|^2 p(z)^D$. Since $X$ is compact and $p$ continuous, so uniformly continuous, we may find a constant $C$, which depends only on $X$ and $L$ such that, for every $z$ and $y$ in $B$ we have $p(y) \geq p(z) - C|z - y|$. As soon as $|y - z|$ is sufficiently small we have that $p(z) - C|z - y| > 0$ and again this uniformly on $X$.

Suppose that $s \in H^0(X, L^D)$ and $z_0$ is a point such that $\sup\{\|s\|(z)\} = \|s\|(z_0)$ We may suppose that $z_0$ is center of a disk $B_r$ of radius $r$ entirely contained in one of the disks of the covering and that $r$ is independent of $z_0$. We can find a positive smooth function $p(z)$ such that $B_r$ we have that $\|s\|^2(z) = |s|^2 p(z)^D$. Denote by $S_t$ the border of the disk of radius $t$ and centered in $z_0$ and by $d\mu_t$ the standard Lebesgue measure on it. Thus

$$\int_X \|s\|^2 \ \eta \geq c_1 \int_{B_r} |s|^2 p(z)^D d\lambda = c_1 \int_0^r dt \int_{S_t} |s|^2 p(z)^D d\mu_t.$$

Since the function $|f(z)|^2$ is plurisubharmonic, we can find a constant $c_2 > 0$, depending only on $N$ , such that $\int_{S_t} |f(z)|^2 d\mu_t \geq c_2 t^{N-1} |f(z_0)|^2$. Consequently we may find a positive constant $c_3$ such that

$$\int_X \|s\|^2 \ \eta \geq c_3 |f(z_0)|^2 \int_0^r t^{N-1} (p(z_0) - Ct)^D dt.$$

As soon as $D$ is sufficiently big, the last integral is lower bounded by

$$p(z_0)^N) \int_0^{\frac{p(z_0)}{CD}} t^{N-1} (1 - \frac{1}{D})^D dt$$

which is uniformly lower bounded by $p(z_0)^D c_4$ for a suitable constant $c_4 > 0$. The conclusion follows.

Similarly to the geometric case, the theory of bundles over spectra of rings of integers of number fields may be generalized to higher rank. We briefly explain here what we need:

**10.2 Definition.** *An Hermitian rm vector bundle of rank $r$ on $Spec(O_K)$ is a couple $(E, \langle \cdot ; \cdot \rangle_\sigma)_{\sigma \in M_\infty}$ where:*

*– $E$ is a locally free $O_K$ module;*

*– for every complex embedding $\sigma : K \to \mathbb{C}$ the $\mathbb{C}$–vector space $E_\sigma$ is equipped with an hermitian metric $\langle \cdot ; \cdot \rangle_\sigma$ with the condition that, if $\sigma$ is the conjugate of $\tau$ then the hermitian metric $\langle \cdot ; \cdot \rangle_\sigma$ is the conjugate of $\langle \cdot ; \cdot \rangle_\tau$.*

It is very important to observe that, as in the case of rank one, a hermitian $O_K$ module is equipped with norms *at every place* even at finite places; this is due to the following lemma:

**10.3 Lemma.** *Let $K_v$ be a non archimedean field with ring of integers $R$ and $V$ be a finite dimensional vector space on it. To give a norm on $V$ is equivalent to have a $R$–module $V_R$ such that $V_R \otimes_R K_v = V$*

*Proof:* Suppose that $V_R$ is such a module, then if $v \in V$ we define

$$\|v\| = \inf\{|\lambda|^{-1} \ / \ \lambda \in K_v \ \text{ s.t. } \ \lambda \cdot v \in V_R\}.$$

Conversely, suppose that $\| \cdot \|$ is a norm on $V$, then we define $V_R$ to be $V_R = \{v \in V \ \ \|v\| \leq 1\}$. The reader will check by exercise that the first one is a norm that $V_R$ is a $R$ module and the two constructions are on e the inverse of the other.

If $E_1$ and $E_2$ are two hermitian vector bundles, then $E_1 \otimes E_2$ and $E_1^\vee := Hom(E_1, O_K)$ are hermitian vector bundles. If $F \hookrightarrow E$ is a sub $O_K$ module of an hermitian vector bundle, then $F$ has also the structure of hermitian vector bundle (it has the restriction of the metric structure of $E$). Also the quotient of an hermitian vector bundle has a natural structure of hermitian vector bundle. So, being quotient of the tensor copies of a vector bundle, the symmetric and the exterior powers of a hermitian vector bundle have a natural structure of hermitian vector bundles. In particular if $E$ has rank $r$, then $\bigwedge^r E$ is an hermitian *line bunde* so we can give the

**10.4 Definition.** *Let $E$ be an hermitian vector bundle of rank $r$ then we define $\widehat{\deg}(E)$ to be $\deg(\bigwedge^r E)$. We will also define the slope of $E$ to be $\mu(E) := \frac{\widehat{deg}(E)}{r}$*

Here we list some properties of the degree of higher rank vector bundles:
– $\widehat{\deg}(E^\vee) = -\widehat{\deg}(E)$ and $\mu(E^\vee) = -\mu(E)$;
– if we have an exact sequence of locally free $O_K$ modules

$$0 \to F_1 \longrightarrow E \longrightarrow F_2 \to 0$$

and $E$ is equipped with the structure of hermitian vector bundle, then $F_1$ and $F_2$ also have the structure of hermitian vector bundles and we have

$$\widehat{\deg}(E) = \widehat{\deg}(F_1) + \widehat{\deg}(F_2). \tag{10.4.1}$$

The reader will check the two previous properties by exercise.

– Property 10.4.1 implies that if $E = E_0 \supseteq E_1 \supseteq E_2 \supseteq \ldots \supseteq \{0\}$ is a filtration of an hermitian vector bundle $E$. Each $E_i$ and $E_i/E_{i+1}$ will be endowed with the induced metric and we have

$$\widehat{\deg}(E) = \sum_{i=0}^{\infty} \widehat{\deg}(E_i/E_{i+1}). \tag{10.5.1}$$

– Suppose that $E$ is an hermitian vector bundle of rank $r$ and $E_1 \hookrightarrow E$ is a subbundle of the same rank (we put on $E_1$ the induced metric). Then $E/E_1$ is a torsion module and

$$\widehat{\deg}(E_1) \leq \widehat{\deg}(E); \tag{10.6.1}$$

*Proof:* Suppose for the moment that $r = 1$. Let $s \in E_1$, then $E_1/sO_K \hookrightarrow E/sO_K$ consequently $Card(E_1/sO_K) \leq Card(E/sO_K)$. So the statement is consequence of 3.35.1. The general case follow from the rank one case because if $E_1 \hookrightarrow E$ then we have an inclusion of line bundles $\bigwedge^r E_1 \hookrightarrow \bigwedge^r E$.

– Suppose that $E$ is an $O_K$ module with *two* hermitian structures $E_1$ and $E_2$. The identity morphism is a map $\iota : E_1 \to E_2$ between the hermitian vector bundles. For every infinite place $\sigma$ denote by $\|\iota\|_\sigma$ the norm of $\iota$ as map between normed vector spaces. Then

$$\mu(E_1) \leq \mu(E_2) + \sum_\sigma \log \|\iota\|_\sigma \tag{10.7.1}$$

*Proof:* . Suppose that $E$ is of rank one. In this case the statement is evident: let $e \in E$ be a non zero element, then $\|\iota(e)\|_{2;\sigma} = \|\iota\|_\sigma \cdot \|e\|_{1,\sigma}$. Thus

$$\begin{aligned}
\widehat{\deg}(E_2) &= \log(Card(E/sO_K)) - \sum_\sigma \log \|s\|_{2,\sigma} \\
&= \log(Card(E/sO_K)) - \sum_\sigma (\log \|s\|_{1,\sigma} - \log \|\iota\|_\sigma \\
&= \widehat{\deg}(E_1) - \sum_\sigma \log \|\iota\|_\sigma.
\end{aligned}$$

Suppose now that the rank of $E$ is arbitrary. By the spectral theorem, for every infinite place $\sigma$ we may find a basis $\{e_1, \ldots, e_r\}$ of $E_\sigma$ such that $\langle e_i; e_j\rangle_{1,\sigma} = \delta_{ij}$ (Kronecker symbol) and $\langle e_i; e_j\rangle_{1,\sigma} = \lambda_i \delta_{ij}$ for suitable real numbers $\lambda_i$. Be aware that, in general the $e_i$ are not elements of $E$. The norm of $\iota$ at the place $\sigma$ is $\|\iota\|_\sigma = \max(|\lambda_i|)$ and the norm of $\wedge^r(\iota) : \bigwedge^r(E_i) \to \bigwedge^r(e_2)$ is $\prod |\lambda_i|$ which is bounded above by $\|\iota\|_\sigma^r$. The conclusion follows then from the case of rank one vector bundles.

– Suppose that $L_1, \ldots, L_n$ are hermitian line bundles over $O_K$ and let $E \hookrightarrow \bigoplus_{i=1}^r L_i$ is a subbundle, then

$$\mu(E) \leq \max\{\widehat{\deg}(L_i)\}. \tag{10.8.1}$$

*Proof:* Suppose for the moment that $E$ is of rank one. Then the inclusion $E \hookrightarrow \bigoplus_{i=1}^r L_i$ implies that there exists $i$ and a non zero map $E \to L_i$; this map must be injective because $E$ is of rank one and $L_i$ is locally free. Thus $\widehat{\deg}(E) \leq \widehat{\deg}(L_i)$ because of formula 10.6.1. Suppose that the rank of $E$ is $r$. Then we have an inclusion

$$\bigwedge^r E \hookrightarrow \bigwedge^r (\bigoplus_{i=1}^n L_i) = \bigoplus_{j_1+j_2+\ldots j_n=r} L_1^{j_1} \otimes L_2^{j_2} \otimes \ldots \otimes L_n^{j_n}.$$

The conclusion follows from the fact that, because of the additivity of the degrees, $\max\{\widehat{\deg})(L_1^{j_1} \otimes L_2^{j_2} \otimes \ldots \otimes L_n^{j_n})\} = r \max\{\widehat{\deg}(L_i)\}$.

– From formulae 10.6.1, 10.7.1 and 10.8.1 we get that if $E$ is *any* hermitian vector bundle and $\iota : E \hookrightarrow \bigoplus_{i=1}^r L_i$ (we do not suppose that the metric on $E$ is the induced metric and that the quotient is locally free), then

$$\mu(E) \leq \max\{\widehat{\deg}(L_i)\} + \sum_{\sigma} \log \|\iota\|_{\sigma}. \tag{10.9.1}$$

Remark that the norm of a linear map is defined as soon as we have a linear map between normed spaces, in particular also on non archimedean spaces. In particular observe that if $h : V_1 \to V_2$ is an injective linear map between two finite dimensional normed vector spaces over a non archimedean field, then, $\|h\| \leq A$ implies that $\| \bigwedge^r(h)\| \leq A^r$. Consequently we get the following generalization:

– Suppose we have an injective map $\iota : E \longrightarrow \left(\bigoplus_{i=1}^r L_i\right)_K$, then

$$\mu(E) \leq \max\{\widehat{\deg}(L_i)\} + \sum_{v \in M_K} \log \|\iota\|_v. \tag{10.10.1}$$

– Suppose that $\mathcal{X} \to Spec(O_K)$ is an arithmetic projective variety of relative dimension $N$. We suppose that $X_K$ is smooth and that, for every $\sigma$ it is equipped with a metric $\eta_\sigma$. Let $L$ be an ample line bundle on $\mathcal{X}$ equipped with a smooth hermitian structure. For every integer $D$, the $O_K$ module $H^0(\mathcal{X}, L^D)$ is equipped with the structure of an hermitian $O_K$ module: at every place at infinity we put the $L_2$ structure defined before. The following estimate is crucial:

**10.11 Theorem.** *Suppose that $L$ is relatively ample, then we can find a constant $C$ independent on $D$, such that*

$$\mu(H^0(\mathcal{X}, L^D)) \geq CD. \tag{10.11.1}$$

*Proof:* (Sketch) The proof of this fact requires some algebraic geometry: Let $R_L(\mathcal{X})$ be the algebra $\oplus_{D=0}^{\infty} H^0(\mathcal{X}, L^D)$. As a consequence of Serre asymptotic vanishing theorem we have that $R_L(\mathcal{X})$ is a $O_K$ finitely generated algebra (the reader who is not acquainted with this, is suggested to admit this fact). If $s$ is a global section of $H^0(\mathcal{X}, L^D)$ we define $\|s\|_{\sup} := \sup_{\sigma \in M_\infty}\{\|s\|_{\sup}\}$. Let $s_1, \dots, s_r$ be a basis of it as $O_K$ algebra. Suppose that $\|s_i\|_{\sup} \leq A$ for a suitable constant $A \geq 1$. Then, since every element in $H^0(\mathcal{X}, L^D)$ may be written as a polynomial in the $s_i$, each $H^0(\mathcal{X}, L^D)$ is generated by elements $f_J$ such that $\|f_J\|_{\sup} \leq A^D$. Let $n_D$ be the dimension of $H^0(\mathcal{X}, L^D)_K$. We may find $n_D$ linearly independent elements between the $f_J$ which form a basis of $H^0(\mathcal{X}, L^D)_K$. Denote by $E_D$ the sub $O_K$–module of $H^0(\mathcal{X}, L^D)_K$ generated by them.

Let $M_D$ be the hermitian $O_K$ module $\oplus_{J=0}^{n_D} \mathcal{O}_K \cdot f_J$. We have that

$$\mu(M_D) = \frac{\sum_J \widehat{\deg}(O_J \cdot f_J)}{n_D} = \frac{-\sum_J \sum_\sigma \log \|f_J\|_\sigma}{n_D} \geq -[K : \mathbb{Q}] \log(A) D.$$

Call $-[K : \mathbb{Q}] \log(A) = C$. The norm of the identity map $\iota : M_D \to E_D$ is bounded above by 2 thus, by 10.7.1 we have that $\mu(E_D) \geq CD - \frac{2[K:\mathbb{Q}]}{n_D}$ and by 10.6.1 the conclusion follows.

**10.13 Remark.** A small improvement of the argument in the proof implies that we can suppose just that $L_K$ is ample on the generic fibre. The details are left as exercise.

– Recall that If $L$ is an ample line bundle on a projective variety $X_K$ of dimension $N$, then we can find positive constants $C_i$ such that $C_1 D^N \leq dim_K(H^0(X_K, L^D)) \leq C_2 D^N$ (cfr. for instance [Ha chap I]).

## 11 Lecture X.

**11.1** *The proof of the main theorem.* We can now start the proof of Theorem 9.10. We recall the situation: $\mathcal{X}$ is a projective arithmetic variety, $M$ is an affine curve and $f : M \to X_\sigma$ is an analytic map of finite order $\rho$; We fix a non negative $\alpha$ and we consider the set $S_f(\alpha)$. Let $p_1, \ldots, p_r$ be $r$ points in $S_f(\alpha)$. We fix an hermitian ample line bundle $L$ on $\mathcal{X}$ and a, for every $\sigma \in M_\infty$, a positive measure $\eta_\sigma$ on $\mathcal{X}_\sigma$. For every non negative integer $D$ we denote by $E_D$ the hermitian $O_K$–module $H^0(\mathcal{X}; L^D)$.

Let $p$ one of the $p_j$'s. By definition we have an admissible sequence $\underline{C}$ and a map

$$f_p : \mathbb{D}_{\underline{C}_p, \alpha} \longrightarrow \mathcal{X}$$

extending the germ of the map $f$. the map $f_p$ induces a map

$$f_p^* : E_D \longrightarrow H^0(\mathbb{D}_{\underline{C}, \alpha}; L^D).$$

This map is injective because the map $f$ is Zariski dense: indeed if it were not injective, we could find a divisor of $X_\sigma$ whose restriction to $M$ is locally defined by an analytic function with infinite order of vanishing at the point $p$, thus identically zero; which means that the image of $f$ is contained in a divisor of $X_\sigma$ and this contradicts the denseness of the image of $f$.

Consider the injection

$$f_1 := (f_{p_1}, \ldots, f_{p_n}) : E_D \longrightarrow \bigoplus_{j=1}^{n} H^0(\mathbb{D}_{\underline{C}_{p_j}, \alpha}; L^D).$$

For every positive integer $i$, the map above and the natural inclusions $(\mathbb{D}_{UC_{p_j}, \alpha})_i \hookrightarrow \mathbb{D}_{\underline{C}_{p_j}, \alpha}$ induces a map

$$f_1^i : E_D \longrightarrow \bigoplus_{j=1}^{n} H^0((\mathbb{D}_{\underline{C}_{p_j}, \alpha})_i, L^D)$$

(we omitted the symbol of restriction of line bundle to subschemes). denote by $E_D^i$ the kernel of this map. The snake lemma, canonical exact sequence 9.6.1 and the isomorphism 9.7.1 give rise to an inclusion

$$\gamma_D^i : E_D^i/E_D^{i+1} \longrightarrow \left(\bigoplus_{j=1}^{n} L^D|_{p_j} \otimes T^{\otimes -i}_{\underline{1}/O_K}\right)_K .$$

The idea of the proof is the following: Because of theorem 10.11 we have a lower bound for $\widehat{\deg}(E_D)$ in terms of $D$. Formula 10.10.1 imply that we can find a constant $C$ independent on $i$ and $D$ such that

$$\mu(E^i/E^{i+1}) \leq C(i+D) + \sum_{v \in M_K} \log \|\gamma_D^i\|_v.$$

We will see that formula 10.5.1 applied to the filtration $E_D \supset E_D^1 \supset \ldots \supset E_D^i \supset \ldots$ and a good estimate of $\log \|\gamma_D^i\|_v$ for every place $v \in M_K$ will give a contradiction as soon as $n$ is too big.

**11.1** *Estimation of the norm of $\gamma_D^i$ at infinite places.* First of all we deal with infinite places different from $\sigma$: Let $X$ be a smooth projective variety defined over $\mathbb{C}$. Let $L$ be an ample line bundle over it and by $E_D$ the vector space $H^0(X, L^D)$; this space is naturally equipped with the $L_2$ and the sup norms. Let $q \in X$ and $\gamma : Spf(\mathbb{C}[\![t]\!]) \to \widehat{X}_q$ be a map of formal schemes. Denote by $B_i$ the closed subscheme $Spec(\mathbb{C}[t]/(t)^i) \hookrightarrow Spf(\mathbb{C}[\![t]\!])$. We have an induced map $E_D \to H^0(B_i, L^D)$; denote by $E_D^i(q)$ the kernel of it. Thus, as before we get an injective map

$$\gamma_D^i(q) : E_D^i(q)/E_D^{i+1}(q) \hookrightarrow L^D|_q \otimes ((t)/(t^2))^{\otimes i}.$$

Suppose that $\gamma$ has positive radius of convergency. Thus there exists a disk $\mathbb{D}$ of radius 1 and a map $\gamma_{\mathbb{D}} : M \to X$ such that $\gamma_{\mathbb{D}}(0) = q$ and the germ of it at 0 coincides with $\gamma$. Consequently $(t)/(t^2)$ is the cotangent space of $M$ at 0 the class of $t$ corresponds to $dt$ and it is equipped with a metric. Thus $L^d|_q \otimes ((t)/(t^2))^{\otimes i}$ is naturally equipped with a metric.

**11.1 Proposition.** *With the notation as above, we can find a constant $C$ independent on $D$ and $i$ such that*

$$\log \|\gamma_D^i(q)\| \leq C(i+D).$$

*Proof:* we may suppose that on $E_D$ we put the sup norm because of theorem 10.1. Shrinking the radius of $\mathbb{D}$ if necessary, may suppose that the image of the disk $\mathbb{D}$ is contained in a open set $B$ which of $X$ which is biholomorphic to a disk of dimension $N$. The restriction of $L$ to $B$ is trivial and generated by an element $\mathbb{I}$. Moreover there are two positive constants $A_1$ and $A_2$ such that, denoting by $\|\mathbb{I}\|$ the norm of $\mathbb{I}$ we have $A_1 \leq \|\mathbb{I}\|\widehat{\mathbb{A}}_2$. Let $s \in E_D$ and denote by $s_B$ its restriction to $B$. We may find an analytic function $F(z)$ such that $s_B = F(z)\mathbb{I}^D$. Denote by $\|s_B\|_{\sup}$ the supremum of $\|s\|$ in $B$.

We have

$$\|s\|_{\sup} \geq \|s_B\| \geq \sup\{|F(z)|\} \cdot A_1^D.$$

Let's compute explicitely the map $\gamma_D^i$: We may suppose that the variable $t$ converges on the disk $\mathbb{D}$ thus $\gamma_{\mathbb{D}}^*(s) = F(z(t)) \cdot \mathbb{I}^D(z(t)) = (\sum_{j=1}^{\infty} a_j t^j) \cdot \mathbb{I}^D(z(t))$. Since $s \in E_D^i$ we have that $a_j = 0$ for $j = 1, 2, \ldots, i-1$ and $\gamma_D^i(s) = a_i \mathbb{I}^D(dt)^i$. Now we apply the classical Cauchy inequality in the version below and conclude.

**11.2 Proposition.** *Let $f(t) = \sum_{j=0}^{\infty} a_j t^j$ be an analytic function on the disk of radius $r$. Then*

$$|a_j| \leq \frac{\sup\{|f(z)| \ / \ |z| = r\}}{r^j}.$$

From 11.1 we get the estimate we want:

**11.3 Proposition.** *Let $\tau \in M_\infty$ then we can find a constant $C$ independent on $i$ and $D$ such that*

$$\log \|\gamma_D^i\|_\tau \leq C(i + D).$$

*Proof: If $h : V_1 \to \oplus_{j=0}^n W_j$ is a map of normed vector spaces, denote by $h_j : V \to W_j$ the map obtained composing it with the projection on the $j$-th factor. It is an easy exercise to prove that $\|h\| \leq r \cdot \sup\{\|h_j\|\}$. For each of the $p_j$ we have that $E_D^i$ is contained in $E_D^i(p_j)$ thus by prop 11.1 the restriction to it of $\gamma_D^i(p_j)$ is bounded by $C(i+D)$ for a suitable $C$. The conclusion follows from the fact that $\gamma_D^i = (\gamma_i^D(p_1), \ldots, \gamma_D^i(p_j))$.*

**11.3 Remark.** Since the Cauchy inequality holds also over $p$–adic fields, In principle we could bound the norm of the $\gamma_D^i$'s at every place (provided that the germ has positive radius of convergency at every place). Unfortunately the constant $C$ involved would depend on the place we are dealing with, thus the sum of all the $C$'s may be too big and we cannot conclude. The introduction of the $E$–germs is due to deal with this problem.

**11.4** *estimation of the norm of $\gamma_D^i$ at finite places.* We now deal with the finite places. Here is where the $E$–germs play a crucial role. Let $\mathfrak{p}$ be a maximal ideal of $Spec(O_K)$. We denote by $K_\mathfrak{p}$ the local field of $\mathfrak{p}$ and by $R_\mathfrak{p}$ its local ring. We will denote by $X_\mathfrak{p}$ the smooth projective variety $X_K \times_K Spec(K_\mathfrak{p})$ and by abuse of notation, we will denote again by $L$ the restriction of the line bundle $L$ to it. The $R_v$–module $H^0(\mathcal{X}, L^D) \otimes_{O_K} R_\mathfrak{p}$ give rise to a norm on $H^0(X_\mathfrak{p}, L^D)$. Again we denote by $\mathcal{X}_\mathfrak{p}$ the projective $R_\mathfrak{p}$–scheme $\mathcal{X} \times_{O_K} Spec(R_\mathfrak{p})$ and by abuse of notation, we denote by $E_D^i$ the restriction to $R_\mathfrak{p}$ of the $O_K$–module $E_D^i$ defined before. Observe that $X_\mathfrak{p} = \mathcal{X}_p \times_{R_\mathfrak{p}} Spec(K_\mathfrak{p})$.

We denote again by $p$ the section over $\mathcal{X}_\mathfrak{p}(R_\mathfrak{p})$ induced by $p \in X_K(K)$.

We fix an isomorphism *over* $R_\mathfrak{p}$

$$\widehat{(\mathcal{X}_\mathfrak{p})}_p \simeq Spf(R_V[\![z_1, \ldots, z_N]\!]).$$

We fix a trivializaton of $L$ near $p$, thus each section $s \in E_D$ give rise, by restriction, to an element $s_p$ of $R_V[\![z_1, \dots, z_N]\!]$.

The $E$–germ $f_p : \mathbb{D}_{\underline{C}_p,\alpha} \to \mathcal{X}$ induces a map

$$\gamma_{\mathfrak{p}} : R_V[\![z_1, \dots, z_N]\!] \longrightarrow R_{\underline{C},\alpha} \otimes_{O_K} R_V.$$

In particular,

$$\gamma_p(s_p) = s_p(t) = \sum_{n=1}^{\infty} a(s_p)_n t^n$$

Again, if $s \in E_D^i$ then $a_j(s_p) = 0$ for $n = 1, \dots, i-1$ and the term corresponding to $p$ of $\gamma_D^i(s)$ in $\left(\bigoplus_{j=1}^n L^D|_{p_j} \otimes T^{\otimes -i}_{\underline{1}/O_K}\right)_K$ is $a(s_p)_i$ (we omit the notation of the trivializations). Thus, since the norm is non archimedean and the map is an $E$–germ, we have $|i! \cdot a(s_p)_i| \leq C_v^i$ and we obtain:

**11.4 Proposition.** *With the notations as above we have that for every finite place* $\mathfrak{p}$

$$\log \|\gamma_D^i\|_{\mathfrak{p}} \leq i \log(C_v) - \alpha \log |i!|_{\mathfrak{p}}.$$

Thus, using the product formula, and the Stirling formula: $\log(i!) \leq i \log(i) + Ai$ we obtain

**11.5 Proposition.** *With the notation above we may find a constant $C$ such that*

$$\sum_{\mathfrak{p} \in M_{fin}} \log \|\gamma_D^i\| \leq [K : \mathbb{Q}]\alpha \cdot i \log(i) + C \cdot i.$$

**11.6** *estimation of the norm of $\gamma_D^i$ at the place $\sigma$.* Now we deal with the infinite place $\sigma$. Here we are going to use that the germs near the points are germs of an analytic map from an affine Riemann surface to $X$.

We recall the situation. We have an analytic map $f : M \to X_\sigma$ of finite order of growth $\rho$, where $M$ is an affine Riemann surface. We fixed the points $p_1, \dots, p_n \in M$.

For each of the $p_j$ we denote by $\mathbb{I}_j$ a section of $\mathcal{O}(p_j)$ vanishing exactly at $p_j$ with order of vanishing one and we will suppose that $\mathcal{O}(p_j)$ is equipped with the metric defined before $\|\mathbb{I}_j\|(z) = exp(\frac{1}{2} g_{p_j}(z))$.

Fix one of the $p_j$'s and denote it $p$. The statement below give the estimate we need:

**11.6 Theorem.** *Let*

$$\gamma_D^i(p) : E_D^i / E_D^{i+i} \longrightarrow L^D|_{f(p)} \otimes T_p M^{\otimes -i}$$

*be the map obtained composing $\gamma_D^i$ with the projection $\bigoplus_{j=1}^r L^D|_{f(p_j)} \otimes T_{p_j} M^{\otimes -i} \to L^D|_{f(p)} \otimes T_p M^{\otimes -i}$. Then we may find constants $C_1$ and $C_2$ independent on $i$ and $D$*

*such that*

$$\log \|\gamma_D^i(p)\| \leq -\frac{i \cdot n}{\rho} \cdot \log \frac{i}{D} + C_1(i+D).$$

*as soon as* $\frac{i}{D} \geq C_2$

**11.7 Remark.** Observe that if we have many points ($n$ is very big) then the norm of $\gamma_D^i$ is very small.

*Proof:* Let $s \in E_D^i$ and we suppose that the sup norm of it is one. The section $\tilde{s} := \frac{f^*(s)}{\prod_j \mathbb{I}_j^i}$ is a global section of $f^*(L^D)(-\sum p_j)$. The ratio between the norm of $\tilde{s}$ at $p$ and the norm of $\gamma_D^i(p)(s)$ is $\left(\prod_{p_j \neq p} exp(\frac{1}{2} g_{p_j}(p)\right)^i$. Thus it suffices to find an upper bound for $\|\tilde{s}\|(p)$.

We may suppose that $\tilde{s}(p) \neq 0$ and we apply the First Main Theorem to the section $\tilde{s} \in H^0(M, f^*(L^D)(-i\sum_j p_j))$ and the base point $p$ and we obtain:

$$T(r, f^*(L^D)(-i\sum_j p_j)) \geq -\int_{S(r)} \log \|\tilde{s}\| d^c g_p + \log \|\tilde{s}(p)\|.$$

Observe that:

– We can find a constant $\lambda$ such that $T(r, f^*(L^D)(-i\sum_j p_j)) = D \cdot T_f(r, L) \leq D \cdot \lambda r^\rho$. This is due to the choice of the metrics on $\mathcal{O}(p_j)$ (cf. remark 8.10) and the fact that $f$ has order of growth $\rho$.

– Since we may find a constant $C$ such that for every $j$ we have $\left|g_{p_j}(z) - g_p(z)\right| \leq C$ (cf. proposition 8.4) and the sup norm of $s$ is one, we may find a constant $B_1$ independent on $i$ and $D$ such that

$$\begin{aligned}
&\int_{S(r)} \log \|\tilde{s}\| d^c g_p \leq \\
&\int_{S(r)} \log \|s\| d^c g_p - \frac{i}{2} \sum_j \int_{S(r)} g_{p_j} d^c g_p \\
&\leq -i \cdot n \cdot \log(r) + B_1 \cdot i.
\end{aligned}$$

From these two inequalities we obtain

$$\log \|\tilde{s}(p)\| \leq D \cdot \lambda r^\rho - i \cdot n \log(r) + B_1 \cdot i.$$

We can choose $r$ in such a way the right hand side is minimal: this will be obtained for $r^\rho = \frac{n \cdot i}{\rho \cdot \lambda \cdot D}$ and for this value we obtain new constant $B_2$ and $B_3$ independent on $i$ and $D$ such that if $\frac{i}{D} \geq B_3$ then

$$\log \|\tilde{s}\| \leq -\frac{i \cdot n}{\rho} \log \frac{i}{D} + B_2 \cdot i.$$

The conclusion follows.

**11.8** *The conclusion of the proof.* We now have all the tools to conclude the proof of Theorem 9.10. We begin by resuming what we found on the estimates of the norms of the $\gamma_D^i$:

**11.8 Proposition.** *In the hypotheses of the previous subsection we may find constants $A_i$ with the following properties:*

*(a) If $i \leq A_1 D$ then*

$$\sum_{v \in MK} \log \|\gamma_D^i\|_v \leq \alpha[K:\mathbb{Q}] \cdot i \log(i) + A_2(i+D).$$

*(b) If $i \geq A_1 D$ then*

$$\sum_{v \in MK} \log \|\gamma_D^i\|_v \leq -\frac{i \cdot n}{\rho} \cdot \log \frac{i}{D} + \alpha[K:\mathbb{Q}] \cdot \log(i) + A_3(i+D).$$

**11.9 Remark.** Observe that the inequality in (a) holds for every $i$ and $D$, while the inequality in (b) holds only for $\frac{i}{D}$ sufficiently big.

*Proof:* (of Theorem 9.10). In the sequel a constant will be a non negative real number independent on $i$ and $D$ but depending on the other data. Call $B$ the number $\frac{n}{\rho}$.

Formula 10.10.1 and proposition 11.8 give that there are constants $A_i$ such that

– If $i \leq A_1 D$ then

$$\widehat{\deg}(E_D^i/E_D^{i+1}) \leq rk(E_D^i/E_D^{i+1}) \cdot (\alpha[K:\mathbb{Q}] \log(i) + A_2(i+D))\,;$$

– if $i \geq A_1 D$ then

$$\widehat{\deg}(E_D^i/E_D^{i+1}) \leq rk(E_D^i/E_D^{i+1}) \cdot \left(-B \log \frac{i}{D} + (\alpha[K:\mathbb{Q}] \log(i) + A_2(i+D)\right).$$

Thus we may find constants $A_i$ such that

$$\begin{aligned} A_1 \cdot D^{N+1} &\leq \widehat{\deg}(E_D) \leq \sum_{i=0}^{\infty} \widehat{\deg}(E_D^i/E_D^{i+1}) \\ &\leq \sum_{i=0}^{\infty} rk(E_D^i/E_D^{i+1}) \cdot \left( \sum_{v \in M_K} \log \|\gamma_D^i\|_v + A_2(i+D) \right). \end{aligned}$$

Choose $a$ such that $1 < a < \frac{N+1}{2}$. We may divide the last sum in the following way

$$\begin{aligned} A_1 \cdot D^{N+1} \leq &\sum_{i \leq D^a} rk(E_D^i/E_D^{i+1}) \cdot \left( \sum_{v \in M_K} \log \|\gamma_D^i\|_v + A_2(i+D) \right) \\ &+ \sum_{i > D^a} rk(E_D^i/E_D^{i+1}) \cdot \left( \sum_{v \in M_K} \log \|\gamma_D^i\|_v + A_2(i+D) \right). \end{aligned} \tag{11.10.1}$$

Let $\epsilon > 0$ very small. We suppose $D$ sufficently big to have that if $i > D^a$ then $i < \epsilon i \log(i)$. Moreover observe that $i > D^a$ implies that $\log(D) \leq \frac{1}{a} \cdot \log(i)$ and in particular $D \leq \epsilon i \log(i)$.

By 11.8 (a), the first term of the sum above may be bounded from above by

$$\begin{aligned} &\sum_{i=1}^{D^a} rk(E_D^i/E_D^{i+1})\,(A_2(i+D) + \alpha[K:\mathbb{Q}]i\log(i)) \\ &\leq \sum_{i=0}^{D^a} A_3(i+D) + A_3 i\log(i) \end{aligned}$$

because the rank of $E_D^i/E_D^{i+1}$ is bounded above by $n$. Since for every $\ell$ we have $\sum_{h=1}^{\ell} h\log(h) \leq \int_0^{\ell+1} tlog(t)dt \leq (\ell+1)^2\log(\ell+1)$, the sum above is bounded by

$$A_5(D^a+1)^2$$

for a suitable constant $A_5$. Since as soon as $D$ is sufficiently big, $A_5(D^a+1)^2\log(D) < \epsilon A_1 D^{N+1}$; thus the first sum on the right is bounded by something which is much smaller then the term on the right.

Now we deal with the second sum on the right. As soon as $D$ is very big and $i > D^a$ we have that

$$-Bi\log(i) + [K:\mathbb{Q}]\alpha \cdot i\log(i) + Bi\log(D) + A_2(i+D)$$

is bounded above by

$$-Bi\log(i) + [K:\mathbb{Q}]\alpha \cdot i\log(i) + \frac{B}{a}i \cdot \log(i) + \epsilon_1 i\log(i)$$

which is negative as soon as $\epsilon$ is sufficently small, $D$ sufficently big and

$$B - \frac{B}{a} - [K:\mathbb{Q}]\alpha > 0.$$

Consequently, if $B(1-\frac{1}{a}) \geq [K:\mathbb{Q}]\alpha$ the second sum on the right of 11.10.1 is negative and this is impossible. Thus

$$\frac{n}{\rho} = B \leq \frac{a}{a-1}\alpha[K:\mathbb{Q}] \leq \left(\frac{N+1}{N-1} + \epsilon_3\right)\alpha[K:\mathbb{Q}]$$

for an arbitrarily small $\epsilon_3 > 0$. The conclusion follows.

## 12 References.


[Bo] Bost, Jean-Benot, Algebraic leaves of algebraic foliations over number fields. Publ. Math. Inst. Hautes tudes Sci. No. 93 (2001).

[BGS] Bost, J.-B.; Gillet, H.; Soul, C. Heights of projective varieties and positive Green forms. J. Amer. Math. Soc. 7 (1994), no. 4, 9031027.

[De] Deligne, P. Le déterminant de la cohomologie. Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), 93–177, Contemp. Math., 67, Amer. Math. Soc., Providence, RI, 1987.

[Ga] Gasbarri, C. Analytic subvarieties with many rational points. Math. Ann. 346 (2010), no. 1, 199–243.

[GH] Griffiths, Phillip; Harris, Joseph, Principles of algebraic geometry. Reprint of the 1978 original. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. xiv+813 pp.

[Ha] Hartshorne, Robin, Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[La] Lang, Serge, Algebra. Revised third edition. Graduate Texts in Mathematics, 211. Springer-Verlag, New York, 2002. xvi+914

[La1] Lang, Serge, Introduction to transcendental numbers. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966 vi+105 pp.

[Sz] Szpiro, Lucien, Degrés, intersections, hauteurs. Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84). Astrisque No. 127 (1985), 1128.

[Vo] Vojta, Paul, Diophantine approximations and value distribution theory. Lecture Notes in Mathematics, 1239. Springer-Verlag, Berlin, 1987. x+132 pp.



Carlo Gasbarri
IRMA – Université de Strasbourg
7, Rue René Descartes
67000 Strasbourg – France

email: gasbarri@math.unistra.fr