ELEMENTARY NUMBER THEORY

1. Notations

$[x]$ denotes the integer part of x.
$\{x\}$ denotes the fractionnal part of x.
$d \mid n$ means d divides n.
$d \nmid n$ means d does not divide n.
The letter p denotes primes.
Given a prime power p^k, we note $p^k \parallel n$ if $p^k \mid n$ and $p^{k+1} \nmid n$.

2. Arithmetic Functions

Definition 1. *An arithmetic function is any real or complex valued function defined on some subset of the set* $\boldsymbol{N}$ *of positive integers.*

(1) The constant function c: Let c fixed and define $f(n) = c$ for all integer n. In particular the constant function equals 1, plays an important role.
(2) Unit function e (we shall explain later the word unit), defined by
$$e(n) = 1 \text{ if } n = 1 \text{ and } e(n) = 0, n \geq 2.$$
(3) Identity function
$$id(n) = n.$$
(4) The function number of divisors of n
$$\tau(n) = \sum_{d \mid n} 1.$$
(5) The function sum of divisors of n
$$\sigma(n) = \sum_{d \mid n} d.$$
(6) Logarithm function log
(7) Number of distinct prime factors on an integer n
$$w(n) = \sum_{p \mid n} 1.$$
(8) Number of prime powers divisors
$$\Omega(n) = \sum_{p^k \mid n} 1.$$
(9) Möbius function μ defined by $\mu(1) = 1, \mu(n) = 0$ if n has a square divisor, $\mu(n) = (-1)^j$ if $n = p_1.p_2...p_j, p_s \neq p_t$, when $s \neq t$.
(10) The characteristic function of square free integers μ^2.
(11) Euler function φ
$$\varphi(n) = card\left\{m \leq n, (m,n) = 1\right\}.$$

(12) Ramanujan function

$$c_q(n) = \sum_{a=1,(a,q)=1}^{q} e^{2\pi i \frac{an}{q}}.$$

(13) Von Mangold's function

$$\Lambda(n) = \begin{cases} \log p \text{ if } n = p^r, r \geq 1 \\ 0 \text{ otherwise} \end{cases}$$

2.1. **Additive, multiplicative functions.**

Definition 2. *An arithmetic function f is additive if*

$$f(m.n) = f(m) + f(n) \text{ when } (m,n) = 1.$$

Definition 3. *An arithmetic function f is multiplicative if*

$$\begin{aligned} & f \text{ is not the constant function equals zero and} \\ f(m.n) &= f(m).f(n) \text{ when } (m,n) = 1. \end{aligned}$$

Lemma 1. *f is multiplicative if and only if*

$$f(1) = 1 \text{ and } f(n) = \prod_{p^k \| n} f(p^k).$$

Proof. Let f be a multiplicative function, then there exists an n such that $f(n) \neq 0$. So $f(n) = f(n.1) = f(n)f(1)$, thus $f(1) = 1$. Now by induction on the number of prime divisors of n, we write $n = p^k.m$, $(p,m) = 1$, so $f(n) = f(p^k)f(m) = f(p^k)\prod_{p^l \| m} f(p^l)$.

Conversely if $(m,n) = 1$, then $m = \prod p_i^{\alpha_i}, n = \prod p_j^{\beta_j}$ and $p_i \neq p_j$ thus $m.n = \prod p_i^{\alpha_i} \prod p_j^{\beta_j}$ and

$$\begin{aligned} f(m.n) &= f(\prod p_i^{\alpha_i} \prod p_j^{\beta_j}) \\ &= \prod f(p_i^{\alpha_i}) \prod f(p_j^{\beta_j}) \\ &= f(m).f(n). \end{aligned}$$

□

Lemma 2. *f is additive if and only if*

$$f(1) = 0 \text{ and } f(n) = \sum_{p^k \| n} f(p^k).$$

2.2. **Dirichlet convolution.**

Definition 4. *Given two arithmetic functions f, g, we define the convolution arithmetic function $h = f * g$ as*

$$\begin{aligned} h(n) &= \sum_{d|n} f(d)g(n/d) \\ &= \sum_{ab=n} f(a)g(b). \end{aligned}$$

Lemma 3. *i)The function e defined earlier is the unit element of the convolution operation $*$.*
ii) $f * g = g * f$
iii) $(f * g) * h = f * (g * h)$.
iv) If $f(1) \neq 0$, then f has an unique inverse g,i.e. thre exists a unique function g such that $f * g = e$.

Proof. We neeed to show only the assertion iv). So suppose that $f(1) \neq 0$, define $g(1) = 1/f(1)$, then $(f * g)(1) = e(1) = 1$. By induction, we suppose $g(m)$ is defined for all $m's, 1 \leq m < n$ such that $(f * g)(m) = e(m)$. In order to get $(f * g)(n) = e(n) = 0$, we need to define $g(n)$ such that

$$f(1)g(n) + \sum_{\substack{d|n \\ d<n}} f(n/d)g(d) = 0,$$

or equivalently

$$g(n) = -\frac{1}{f(1)} \sum_{\substack{d|n \\ d<n}} f(n/d)g(d).$$

□

Lemma 4. *The convolution of two multiplicative functions is a multiplicative function.*

Proof. Let $(n_1, n_2) = 1$. If $d \mid n_1.n_2$ then $d = d_1.d_2$ where $d_i \mid n_i, i = 1, 2$. Converselly given $d_1 \mid n_1, d_2 \mid n_2$, the product $d_1.d_2 \mid n_1.n_2$. So

$$\begin{aligned} h(n_1.n_2) &= \sum_{d|n_{l1}.n_2} f(d)g(\frac{n_1.n_2}{d}) \\ &= \sum_{d_1|n_1} \sum_{d_2|n_2} f(d_1 d_2)g(\frac{n_1.n_2}{d_1.d_2}) \\ &= \sum_{d_1|n_1} \sum_{d_2|n_2} f(d_1)f(d_2)g(\frac{n_1}{d_1})g(\frac{n_2}{d_2}) \\ &= h(n_1).h(n_2). \end{aligned}$$

□

Lemma 5. *The inverse of a multiplicative function is a multiplicative function.*

Proof. As $f(1) = 1, f$ is invertible. Let g be the multiplicative function defined by $g(1) = 1, g(p^k) = f^{-1}(p^k)$. By the preceeding lemma, $f * g$ is multiplicative, $(f * g)(p^j) = 0 = e(p^j), j \geq 1$, so the two multiplicative functions e and $f * g$ agree on the set of prime powers, they are equal. thus g is the inverse of f. □

Remark 1. *It is easily seen that the functions*

$$1, e, id, \tau, \sigma, \mu \text{ and } \mu^2$$

are multiplicatives and the functions

$$\log, w \text{ and } \Omega$$

are additives.

Lemma 6.

$$\Lambda * 1 = \log$$

Proof. This is equivalent to the unique factorisation of an integer n as a product of prime powers. Let $n = \prod p_j^{k_j}$

$$\log n = \sum k_j \log p_j = \sum_{d|n} \Lambda(d).$$

□

2.3. **Möbius Formula.**

Lemma 7 (Möbius formula)**.**

$$\mu * 1 = e$$

In other words

$$\sum_{d|n} \mu(d) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n > 1 \end{cases} .$$

Proof. It is enough to show that the two multiplicative functions e and $\mu * 1$ are equal on prime powers $p^j, j \geq 1$. Now

$$(\mu * 1)(p^j) = \sum_{l=0}^{j} \mu(p^l) = 1 + \mu(p) = 0.$$

There is a more combinatory proof: Let $n > 1, n = \prod p_k^{\alpha_k}, \alpha_k \geq 1$. Then a divisor d of n satisfying $\mu(d) = 0$ is necessarily a divisor of $\prod p_k$, that is

$$d = 1 \text{ (and } \mu(1) = 1) \text{ or } d = p_{i_1}..p_{i_s},\ i_1 < .. < i_s, \text{ and } \mu(d) = (-1)^s.$$

So

$$\begin{aligned} \sum_{d|n} \mu(d) &= \sum_{s=0}^{k} (-1)^s card \{1 \leq i_1 < .. < i_s \leq k\} \\ &= \sum_{s=0}^{k} (-1)^s \binom{k}{s} = (1-1)^k = 0. \end{aligned}$$

□

Theorem 1 (Möbius inversion formula)**.** *Let f, g be two arithmetic functions, then*

$$g = f * 1 \Leftrightarrow f = g * \mu.$$

2.4. **Some applications.**

Corollary 1.

$$\Lambda = \log * \mu$$

Proof. We apply Möbius inversion formula to the relation $\Lambda * 1 = \log$,

$$\begin{aligned} \Lambda * 1 * \mu &= \log * \mu \\ \Lambda &= \log * \mu. \end{aligned}$$

□

Corollary 2. *We have*

$$\left|\sum_{n\le x}\frac{\mu(n)}{n}\right|\le 1, x\ge 1.$$

Proof. Let $x\ge 1$ and assume that x is an integer (as clearly the inequality depends only on $[x]$)

$$\begin{aligned}\sum_{n\le x} e(n) &= 1\\ &= \sum_{n\le x}\sum_{d|n}\mu(d)\\ &= \sum_{d\le x}\mu(d)\left[\frac{x}{d}\right]\\ &= x\sum_{d\le x}\frac{\mu(d)}{d}-\sum_{d\le x}\mu(d)\left\{\frac{x}{d}\right\},\end{aligned}$$

in the last sum we can ignore the term $d=1$ as $\{x/1\}=0$, so

$$\left|x\sum_{d\le x}\frac{\mu(d)}{d}\right|\le 1+\sum_{1<d\le x}|\mu(d)|\le x.$$

□

Corollary 3. *For any arithmetic function f,we have*

$$\sum_{n\le x,(m,n)=1} f(n)=\sum_{d|m}\mu(d)\sum_{n\le x/d} f(nd).$$

Proof.

$$\begin{aligned}\sum_{n\le x,(m,n)=1} f(n) &= \sum_{n\le x} f(n)\sum_{d|(m,n)}\mu(d)=\sum_{d|m}\mu(d)\sum_{n\le x,d|n} f(n)\\ &= \sum_{d|m}\mu(d)\sum_{n\le x/d} f(nd).\end{aligned}$$

□

Corollary 4.

$$\varphi(n)=n\sum_{d|n}\frac{\mu(d)}{d}$$

Proof.

$$\begin{aligned}\varphi(n) &= \sum_{m\le n,(m,n)=1} 1=\sum_{m\le n}\sum_{d|(m,n)}\mu(d)\\ &= \sum_{d|n}\mu(d)\sum_{m\le n,d|m} 1=n\sum_{d|n}\frac{\mu(d)}{d}.\end{aligned}$$

□

Lemma 8. *The Euler function φ is multiplicative and*

$$\varphi(n) = n \prod_{p|n} (1 - \frac{1}{p}).$$

Proof. Clearly $\varphi(n) = id(n).(\mu/id * 1)(n)$, as all those functions are multiplicative, φ is multiplicative. Now $\varphi(p^k) = p^k - p^{k-1} = p^k(1 - 1/p)$ for $k \geq 1$. □

Lemma 9. *We have*

$$n = \sum_{d|n} \varphi(d)$$

Proof. This can be deduced immediatly from the preceeding lemmas. We give an interesting direct proof. Let

$$A = \{m \leq n\}, \text{ for } d \mid n, \text{ define } A_d = \{m \leq n, (m,n) = d\}.$$

Clearly

$$A = \cup_{d|n} A_d, \; A_d \cap A_{d'} = \emptyset \text{ if } d \neq d'.$$

Let $m \in A_d$, then $m = dm', m' \leq n/d$ and $(m', n/d) = 1$, so

$$\begin{aligned} cardA_d &= \varphi(n/d) \\ cardA &= n = \sum_{d|n} cardA_d \\ &= \sum_{d|n} \varphi(n/d) \\ &= \sum_{k|n} \varphi(k). \end{aligned}$$

□

Corollary 5.

$$c_q(n) = \sum_{d|q, d|n} \mu(\frac{q}{d}) d.$$

In particular

$$c_q(1) = \mu(q).$$

Proof.

$$\begin{aligned} c_q(n) &= \sum_{a=1,(a,q)=1}^{q} e^{2\pi i \frac{an}{q}} = \sum_{a=1}^{q} e^{2\pi i \frac{an}{q}} \sum_{d|(a,q)} \mu(d) \\ &= \sum_{d|q} \mu(d) \sum_{a=1, d|a}^{q} e^{2\pi i \frac{an}{q}} = \sum_{d|q} \mu(d) \sum_{b=1}^{q/d} e^{2\pi i \frac{bn}{q/d}}) \end{aligned}$$

but

$$\sum_{j=1}^{l} e^{2\pi i \frac{jn}{l}} = \begin{cases} 0 \text{ if } l \nmid n \\ l \text{ if } l \mid n \end{cases},$$

so

$$c_q(n) = \sum_{\substack{d|q \\ (q/d)|n}} \mu(d)\frac{q}{d} = \sum_{\substack{d'|q \\ d'|n}} \mu(\frac{q}{d'})d',$$

(writing $q = dd'$). □

3. Estimates of some arithmetic sums

Lemma 10. *Let $f : [a,b] \to \mathbb{R}, f$ monotonic and suppose that a, b are integers. Then there exists a real number $\theta_{a,b} \in [-1,1]$ such that*

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)dt + (f(b) - f(a))\theta_{a,b}.$$

Proof. Suppose that f is decreasing, then

$$\begin{aligned} f(n) &\leq \int_{n-1}^{n} f(t)dt \leq f(n-1), \\ \sum_{n=a+1}^{b} f(n) &\leq \sum_{n=a+1}^{b} \int_{n-1}^{n} f(t)dt \leq \sum_{n=a+1}^{b} f(n-1), \\ \sum_{n=a+1}^{b} f(n) &\leq \int_a^b f(t)dt \leq \sum_{m=a}^{b-1} f(m), \end{aligned}$$

so

$$0 \leq \int_a^b f(t)dt - \sum_{n=a+1}^{b} f(n) \leq \sum_{n=a}^{b-1} f(n) - \sum_{n=a+1}^{b} f(n = f(a) - f(b).$$

□

Lemma 11 (Abel summation)**.** *Let g be an arithmetic function, f a complex-valued function defined on $[y,x]$ and having a continuous derivative there. Let $G(x) = \sum_{n\leq x} g(n)$, then*

$$\sum_{y<n\leq x} f(n)g(n) = -\int_y^x G(t)f'(t)dt + G(x)f(x) - G(y)f(y).$$

Proof. For any $(n,t)\in\mathbb{N}\times[y,x]$, let $\chi(n,t)=\begin{cases}1 \text{ if } n\le t\\ 0 \text{ if } n>t\end{cases}$. We have

$$\begin{aligned}\int_y^x G(t)f'(t)dt &= \int_y^x \sum_{n\le x} g(n)\chi(n,t)f'(t)dt\\ &= \sum_{n\le x} g(n)\int_y^x \chi(n,t)f'(t)dt\\ &= \sum_{n\le x} g(n)\int_{\max(n,y)}^x f'(t)dt\\ &= \sum_{n\le x} g(n)(f(x)-f(\max(n,y)))\\ &= f(x)\sum_{n\le x} g(n)-f(y)\sum_{n\le y} g(n)-\sum_{y<n\le x} g(n)f(n)\\ &= f(x)G(x)-f(y)G(y)-\sum_{y<n\le x} g(n)f(n).\end{aligned}$$

□

Corollary 6. *Let g be an arithmetic function, f a complex- valued function defined on $[1,x]$ and having a continuous derivative there. Let $G(x)=\sum_{n\le x} g(n)$, then*

$$\sum_{n\le x} f(n)g(n)=G(x)f(x)-\int_1^x G(t)f'(t)dt.$$

Proof. Let $y=1$ in the preceeding lemma and remark that $G(1)f(1)=g(1)f(1)$, so

$$\begin{aligned}f(1)G(1)+\sum_{y<n\le x} f(n)g(n) &= \sum_{n\le x} f(n)g(n)\\ &= f(1)g(1)+G(x)f(x)-G(1)f(1)-\int_1^x G(t)f'(t)dt\\ &= G(x)f(x)-\int_1^x G(t)f'(t)dt.\end{aligned}$$

□

Lemma 12 (Euler-Maclaurin 1)**.** *Let $0<y\le x$ and suppose that f is a complex-valued function defined on $[y,x]$ and having a continuous derivative there. Then*

$$\sum_{y<n\le x} f(n)=\int_y^x f(t)dt+\int_y^x \{t\}\, f'(t)dt-\{x\}\, f(x)+\{y\}\, f(y).$$

Proof. Put $g(n)=1,$ in the previous lemma and replace $G(x)=[x]$ by $x-\{x\}$. □

Lemma 13 (Euler-Maclaurin 2)**.** *With the same assumptions*

$$\sum_{n\le x} f(n)=\int_1^x f(t)dt+\int_1^x \{t\}\, f'(t)dt-\{x\}\, f(x)+f(1).$$

Proof. Apply the preceeding lemma with $y = 1$ and use the fact that $\sum_{n\le x} f(n) = f(1) + \sum_{1<n\le x} f(n)$. □

4. Applications

Lemma 14. *For* $x \ge 1, \gamma = 1 - \int_1^\infty \{t\} \frac{dt}{t^2}$

$$\sum_{n\le x} \frac{1}{n} = \log x + \gamma + O(\frac{1}{x}).$$

Proof. Applying Euler-Maclaurin's lemma

$$\begin{aligned}\sum_{n\le x} \frac{1}{n} &= \int_1^x \frac{dt}{t} - \int_1^x \{t\} \frac{dt}{t^2} + \frac{\{x\}}{x} + 1 \\ &= \log x - \int_1^\infty \{t\} \frac{dt}{t^2} + \int_x^\infty \{t\} \frac{dt}{t^2} + 1 + O(\frac{1}{x}) \\ &= \log x + \gamma + O(\frac{1}{x}).\end{aligned}$$

□

Lemma 15. *Let* $x \ge 2$, *then*

$$\sum_{n\le x} \log n = x \log x - x + O(\log x).$$

Proof. From Euler-Maclaurin's lemma

$$\begin{aligned}\sum_{n\le x} \log n &= [x] \log x - \int_1^x \frac{[t]}{t} dt \\ &= x \log x - (x-1) + \int_1^x \frac{\{t\}}{t} dt \\ &= x \log x - x + O(\log x).\end{aligned}$$

□

Another important application of Euler-Maclaurin is

Theorem 2 (Zeta function)**.** *Let for* $\Re s > 1$

$$\zeta(s) = \sum n^{-s}$$

then

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \{x\} x^{-s-1} dx.$$

This provides an analytic continuation of $\zeta(s)$ in the region $\Re(s) > 0$.

5. Dirichlet's hyperbola formula

Theorem 3. *Given two arithmetic functions* f, g, *we define*

$$h = f * g$$

and note

$$F(u) = \sum_{n\le u} f(n), G(u) = \sum_{n\le u} g(n),$$

then for any $y \leq x$

$$\sum_{n\leq x} h(n) = \sum_{n\leq y} f(n)G(x/n) + \sum_{n\leq x/y} g(n)F(x/n) - F(y)G(x/y).$$

Proof.

$$\begin{aligned}\sum_{n\leq x} h(n) &= \sum_{n\leq x}\left\{\sum_{ab=n, a\leq y} f(a)g(b) + \sum_{ab=n,a>y} f(a)g(b)\right\}\\ &= \sum_{a\leq y} f(a)\sum_{b\leq x/a} g(b) + \sum_{y<a\leq x} f(a)\sum_{b\leq x/d} g(b)\\ &= \sum_{a\leq y} f(a)G(x/a) + \sum_{b\leq x} g(b)\sum_{y<a\leq x/b} f(a)\end{aligned}$$

the condition $y < a \leq x/b$ implies that $b \leq x/y$, so

$$\begin{aligned}\sum_{n\leq x} h(n) &= \sum_{a\leq y} f(a)G(x/a) + \sum_{b\leq x/y} g(b)(F(x/b) - F(y))\\ &= \sum_{a\leq y} f(a)G(x/a) + \sum_{b\leq x/y} g(b)F(x/b) - F(y)G(x/y).\end{aligned}$$

□

5.1. **An important application.**

Lemma 16 (Dirichlet)**.** *Let $\tau(n) = \sum_{d|n} 1 = (1 * 1)(n)$, then*

$$\sum_{n\leq x} \tau(n) = x\log x + (2\gamma - 1)x + O(\sqrt{x}).$$

Proof. By the lemma

$$\begin{aligned}\sum_{n\leq x} \tau(n) &= \sum_{n\leq y}\left[\frac{x}{n}\right] + \sum_{n\leq x/y}\left[\frac{x}{n}\right] - [y]\,[x/y]\\ &= x\{\sum_{n\leq y}\frac{1}{n} + \sum_{n\leq x/y}\frac{1}{n}\} - x + \sum_{n\leq y}\left\{\frac{x}{n}\right\} + \sum_{n\leq x/y}\left\{\frac{x}{n}\right\}\\ &\quad -y\,\{x/y\} - \frac{x}{y}\,\{y\} - \{y\}\left\{\frac{x}{y}\right\}.\end{aligned}$$

We majorize the fractional parts by 1, getting an error term of order $y + x/y + 1$, and we choose $y = x/y = \sqrt{x}$; so

$$\sum_{n\leq x} \tau(n) = 2x\sum_{n\leq\sqrt{x}}\frac{1}{n} - x + O(\sqrt{x}),$$

we apply finally our estimate

$$\sum_{n\leq\sqrt{x}}\frac{1}{n} = \log\sqrt{x} + \gamma + O(\frac{1}{x}).$$

□

6. Landau's theorem

Let $M(x) = \sum_{n\le x}\mu(n), \pi(x) = \sum_{p\le x} 1$ and $\psi(x) = \sum_{n\le x}\Lambda(n)$. The Prime Number theorem asserts that

$$\frac{\pi(x)}{(x/\log x)} \to 1 \text{ when } x \to \infty.$$

We will prove later that this is equivalent to the assertion

$$\frac{\psi(x)}{x} \to 1 \text{ when } x \to \infty.$$

The following theorem gives another equivalent assertion:

Theorem 4 (Landau)**.**

$$\lim_{x\to\infty}\frac{M(x)}{x} = 0 \Leftrightarrow \lim_{x\to\infty}\frac{\psi(x)}{x} = 1.$$

Proof. We prove the $\Rightarrow$ part only.

We recall the following convolution identities:

$$\Lambda = \mu * \log; 1 = \mu * 1 * 1 = \mu * \tau, e = \mu * 1.$$

Let

$$f(n) = \log n - \tau(n) + 2\gamma,$$

then

$$\begin{aligned}\sum_{n\le x} f(n) &= \sum_{n\le x}\log n - \sum_{n\le x}\tau(n) + 2\gamma\sum_{n\le x} 1\\ &= O(\sqrt{x}).\end{aligned}$$

We remark that

$$\mu * f = \Lambda - 1 + 2\gamma e,$$

so

$$\sum_{n\le x}(\mu * f)(n) = \Psi(x) - [x] + 2\gamma.$$

We need to show that under the assumption $M(x) = o(x)$, we have $\Psi(x) - [x] + 2\gamma = o(x)$.

We apply Dirichlet hyperbola method

$$\begin{aligned}\sum_{n\le x}(\mu * f)(n) &= \Psi(x) - [x] + 2\gamma\\ &= \sum_{n\le y}\mu(n)F(x/n) + \sum_{n\le x/y} f(n)M(x/n) - M(y)F(x/y)\end{aligned}$$

where $F(u) = \sum_{n\le u} f(n)$. Let $0 < \varepsilon < 1$ fixed and $y = \varepsilon x$. If $n \le x/y$, then $x/n \ge y$, so $M(x/n) = o(x/n)$.

$$|\Psi(x) - [x] + 2\gamma| \le O(\sqrt{x}\sum_{n\le y}\frac{1}{\sqrt{n}}) + o(x\sum_{n\le x/y}\left|\frac{f(n)}{n}\right|) + o(\varepsilon x\sqrt{\frac{x}{\varepsilon x}});$$

Now

$$\begin{aligned}\sum_{n\le u}\left|\frac{f(n)}{n}\right| &\le \sum_{n\le u}\frac{\log n}{n}+\sum_{n\le u}\frac{\tau(n)}{n}+2\gamma\sum_{n\le u}\frac{1}{n}\\ &\le O\log^2 u)+\sum_{n\le u}\frac{1}{n}\sum_{m\le u/n}\frac{1}{m}+2\gamma\sum_{n\le u}\frac{1}{n}\\ &= O(\log^2 u).\end{aligned}$$

Also

$$\sum_{n\le u}\frac{1}{\sqrt{n}}=O(\sqrt{u}).$$

So

$$\begin{aligned}|\Psi(x)-[x]+2\gamma| &\le O(x\sqrt{\varepsilon})+o(x\log^2\frac{1}{\varepsilon})+o(x\varepsilon\sqrt{\frac{1}{\varepsilon}})\\ \limsup_{x\to\infty}\frac{|\Psi(x)-[x]+2\gamma|}{x} &= O(\sqrt{\varepsilon})\end{aligned}$$

and we get the result as $\varepsilon\to 0$. □

7. Chebyshev theorem

Theorem 5. *There exists 2 constants* A, B *satisfying* $0<A<1<B$ *such that*

$$Ax\le\Psi(x)\le Bx.$$

Chebyshev gave the values $A=0.92129, B=1.1055$.
We will give a complete proof of the weaker result

$$x\log 2+O(\log x)\le\Psi(x)\le 2x\log 2+O(\log^2 x).$$

Proof. As $\Lambda*1=\log$, we have $\Lambda*1*v=\log*v$. Let $w=1*v$, $W(u)=\sum_{n\le u}w(n)$, $Z(u)=\sum_{n\le u}(\log*v)(n)=\sum_{n\le u}\Lambda(n)W(x/n)$.
Suppose we can find a function v such that
$i)Z(x)\sim cx, x\to\infty, ii)W(u)\in[0,1], iii)W(u)=1$ when $1\le u<2$,
then we deduce from the relation

$$Z(x)=\sum_{n\le x}\Lambda(nW(x/n)$$

that

$$\psi(x)\ge Z(x)\ge(c-\varepsilon)x,$$

which is the lower bound and

$$\Psi(x)-\Psi(x/2)\le Z(x)\le(c+\varepsilon)x$$

which gives by iteration an upper bound for $\Psi(x)$.

Now

$$\begin{aligned} Z(x) &= \sum_{n\le x} v(n) \sum_{d\le x/n} \log d \\ &= x\sum_{n\le x}\frac{v(n)}{n}\log x/n - x\sum_{n\le x}\frac{v(n)}{n} + O(\sum_{n\le x}\left|\frac{v(n)}{n}\right|\log(2x/n) \\ & (x\log x - x)\sum_{n\le x}\frac{v(n)}{n} - x\sum_{n\le x}\frac{v(n)}{n}\log n + O(\sum_{n\le x}\left|\frac{v(n)}{n}\right|\log(2x/n). \end{aligned}$$

A first easy choice is the function

$$v(n) = \begin{cases} 1 \text{ if } n = 1 \\ -2 \text{ if } n = 2 \\ 0 \text{ if } n > 2 \end{cases}$$

then

$$Z(x) = x\log 2 + O(\log x)$$

and

$$W(x) = [x] - 2\,[x/2] = \begin{cases} 0 \text{ if } [x] \text{ is even} \\ 1 \text{ if } [x] \text{ is odd} \end{cases},$$

so W satisfies our assumptions. Thus

$$\begin{aligned} \Psi(x) &\ge x\log 2 + O(\log x) \\ \Psi(x) - \Psi(x/2) &\le x\log 2 + O(\log x) \\ \Psi(x/2) - \Psi(x/4) &\le \frac{x}{2}\log 2 + O(\log x) \\ & .. \\ \Psi(x) &\le x\log 2\sum_{j=1}^{k} 2^{-j} + O(k\log x) \end{aligned}$$

where k satisfies the inequalities

$$x/2^{k+1} \le 1 < x/2^k$$

so $k = O(\log x)$ and finally

$$\Psi(x) \le 2x\log 2 + O(\log^2 x).$$

□

Remark 2. *A better choice (Chebyshev) is* $v(1) = 1, v(2) = v(3) = v(5) = -1$, $v(30) = 1$ *and for all other integers* $n, v(n) = 0$. *Clearly for* $x \ge 30$

$$\sum_{n\le x}\frac{v(n)}{n} = 0.$$

7.1. **Mertens's estimates.**

Theorem 6. *We have*

$$\begin{aligned} 1)\ \sum_{n\le x}\frac{\Lambda(n)}{n} &= \log x + O(1),\\ 2)\ \sum_{p\le x}\frac{\log p}{p} &= \log x + O(1),\\ 3)\quad \sum_{p\le x}\frac{1}{p} &= \log\log x + A + O(\frac{1}{\log x}),\\ 4)\ \prod_{p\le x}(1-\frac{1}{p}) &= \frac{c}{\log x}(1+O(\frac{1}{\log x}))\ (c=e^{-\gamma}). \end{aligned}$$

Proof. We have proved that

$$S(x)=\sum_{n\le x}\log n = x\log x - x + O(\log x)$$

the relation $\Lambda * 1 = \log$, leads to

$$\begin{aligned} S(x) &= \sum_{n\le x}\Lambda(n)\left[\frac{x}{n}\right] = x\sum_{n\le x}\frac{\Lambda(n)}{n} + O(\sum\Lambda(n)),\\ &= x\sum_{n\le x}\frac{\Lambda(n)}{n} + O(\Psi(x)) = x\sum_{n\le x}\frac{\Lambda(n)}{n} + O(x). \end{aligned}$$

by Chebychev's estimate and this proves 1). From it we deduce 2) as the serie $\sum_{p^k,k\ge 2}(\log p/p^k)$ is convergent.

For the proof of 3) let us define

$$\begin{aligned} L(t) &:= \sum_{p\le t}\frac{\log p}{p}\\ R(t) &:= \sum_{p\le t}\frac{\log p}{p} - \log t; \end{aligned}$$

we have $R(t)=O(1)$ by 1).

$$\begin{aligned} \sum_{p\le x}\frac{1}{p} &= \frac{L(x)}{\log x} + \int_2^x \frac{L(t)}{t\log^2 t}dt\\ &= 1 + \frac{R(x)}{\log x} + \int_2^x \frac{1}{t\log t}dt + \int_2^x \frac{R(t)}{t\log^2 t}dt\\ &= 1 + O(\frac{1}{\log x}) + \log\log x - \log\log 2 + I(x) \end{aligned}$$

where

$$\begin{aligned} I(x) &= \int_2^x \frac{R(t)}{t\log^2 t}dt = \int_2^\infty \frac{R(t)}{t\log^2 t}dt - \int_x^\infty \frac{R(t)}{t\log^2 t}dt\\ &= c_2 + O(\frac{1}{\log x}), \end{aligned}$$

the integral

$$\int_2^\infty \frac{R(t)}{t\log^2 t}dt$$

being convergent.

Clearly

$$\prod_{p\le x}(1-\frac{1}{p}) = \exp(\sum_{p\le x}\log(1-\frac{1}{p})),$$

we use the classical expansion

$$-\log(1-x) = \sum_{n=1}^\infty \frac{x^n}{n}, |x|<1,$$

$$\begin{aligned}
-\sum_{p\le x}\log(1-\frac{1}{p}) &= \sum_{p\le x}\frac{1}{p} + \sum_{p\le x}\sum_{m=2}^\infty \frac{1}{mp^m} \\
&= \sum_{p\le x}\frac{1}{p} + \sum_{p}\sum_{m=2}^\infty \frac{1}{mp^m} - \sum_{p>x}\sum_{m=2}^\infty \frac{1}{mp^m} \\
&= \log\log x + A + B + O(\frac{1}{\log x}),
\end{aligned}$$

where B is the convergent serie $\sum_p \sum_{m=2}^\infty \frac{1}{mp^m}$. Finally

$$\begin{aligned}
\prod_{p\le x}(1-\frac{1}{p}) &= \exp(\sum_{p\le x}\log(1-\frac{1}{p})) = \frac{c}{\log x}\exp(O(\frac{1}{\log x})) \\
&= \frac{c}{\log x}(1+O(\frac{1}{\log x})).
\end{aligned}$$

□

8. The Prime Number Theorem

8.1. Functions π, ψ, θ.

We define the following functions

$$\pi(x) = \sum_{p\le x} 1,\ \Psi(x) = \sum_{p^k\le x}\log p = \sum_{n\le x}\Lambda(n), \theta(x) = \sum_{p\le x}\log p.$$

Lemma 17. *We have*

$$\begin{aligned}
\theta(x) &= \Psi(x) + O(\frac{x}{\log^2 x}) \\
\pi(x) &= \frac{\Psi(x)}{\log x} + O(\frac{x}{\log^2 x})
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\Psi(x) &= \theta(x) + \sum_{p\le\sqrt{x}}\log p \sum_{2\le m\le(\log x/\log p)} 1 \\
&= \theta(x) + O(\sum_{p\le\sqrt{x}}\log x) = \theta(x) + O(\sqrt{x}\log x).
\end{aligned}$$

For π, we apply Abel's summation and get

$$\pi(x) = \sum_{p\leq x} \frac{\log p}{\log p} = \frac{\theta(x)}{\log x} - \int_2^x \theta(t)(\frac{-dt}{t\log^2 t}),$$

as $\theta(x) \leq \Psi(x) = O(x)$, the integral $\int_2^x(\theta(t)/t\log^2 t)dt = O(\int_2^x dt/\log^2 t)$. Now, write

$$\begin{aligned}\int_2^x \frac{dt}{log^2 t} &= \int_2^{\sqrt{x}} \frac{dt}{log^2 t} + \int_{\sqrt{x}}^x \frac{dt}{log^2 t} \\ &\leq \frac{\sqrt{x}}{\log^2 2} + \frac{x}{(\log\sqrt{x})^2} = O(\frac{x}{\log^2 x}),\end{aligned}$$

so

$$\pi(x) = \frac{\theta(x)}{\log x} + O(\frac{x}{\log^2 x}).$$

□

Theorem 7. *The following assertions are equivalent*

$$i)\ \pi(x) \sim \frac{x}{\log x}\ \ when\ x \to \infty,$$

$$ii)\ \Psi(x) \sim x,\ \ when\ x \to \infty,$$

We have

$$iii)\ \frac{M(x)}{x} \to 0,\ \ when\ x \to \infty.$$

Proof. The first two assertions follows from the preceeding lemma, the last assertion is Landau's theorem. □

So in order to get the Prime Number Theorem

$$\pi(x) \sim \frac{x}{\log x} \text{ when } x \to \infty,$$

we need only to show that $M(x)/x$ tends to zero when x tends to infinity

9. Proof of the assumption $M(x)/x \to 0$, when $x \to +\infty$.

9.1. **Auxillary functions, first upper bound.**

Definition 5. *Let $y \geq 3$, and define the following completely multiplicative functions*

$$\begin{aligned} u_y(p^r) &= \begin{cases} 1 \text{ if } p > y \\ 0 \text{ if } p \leq y \end{cases}, r \geq 1 \\ v_y(p^r) &= \begin{cases} 1 \text{ if } p \leq y \\ 0 \text{ if } p > y \end{cases}, r \geq 1\ . \end{aligned}$$

Clearly

$$u_y * v_y = 1.$$

Lemma 18. *For all $y \geq 2$*

$$\limsup_{x\to\infty} \left|\frac{M(x)}{x}\right| \leq (\prod_{p\leq y}(1-\frac{1}{p}))(\int_1^\infty \left|\sum_{n\leq t} v_y(n)\mu(n)\right| \frac{dt}{t^2}.$$

Proof. Call

$$V_y(t) = \sum_{n,\leq t} v_y(n)\mu(n),$$
$$V_y^*(t) = \sum_{n,\leq t} v_y(n).$$

Let $1 = d_1 < d_2 < ...d_q$ the (finite) set of squarefree integers having all their prime factors $\leq y$. (For example with $y = 3, 1 = d_1 < 2 = d_2 < 3 = d_3 < 6 = d_4$).

Clearly

$$\mu = u_y\mu * v_y\mu,$$

so

$$M(x) = \sum_{n\leq x} u_y(n)\mu(n)V_y(\frac{x}{n});$$

but when n satisfies

$$\frac{x}{d_{j+1}} < n \leq \frac{x}{d_j}, j = 1, 2, ...q - 1$$

the sum $V_y(x/n)$ is constant and is equal to $V_y(d_j)$. So

$$M(x) = \sum_{j=1}^{q-1} V_y(d_j) \sum_{\frac{x}{d_{j+1}}<n\leq\frac{x}{d_j}} u_y(n)\mu(n) + V_y(d_q) \sum_{n\leq x/d_q} u_y(n)\mu(n).$$

This gives

$$\left|\frac{M(x)}{x}\right| \leq \sum_{j=1}^{q-1} |V_y(d_j)| \frac{1}{x} \sum_{\frac{x}{d_{j+1}}<n\leq\frac{x}{d_j}} u_y(n) + |V_y(d_q)| \frac{1}{x} \sum_{n\leq x/d_q} u_y(n).$$

Now it is easy to see that

$$\frac{1}{x}\sum_{n\leq x} u_y(n) \to \prod_{p\leq y}(1 - \frac{1}{p}) \text{ when } x \to \infty,$$

in fact

$$u_y = v_y\mu * 1$$

so

$$\frac{1}{x}\sum_{n\leq x} u_y(n) = \sum_{n\leq x} v_y(n)\mu(n)\frac{1}{x} \sum_{d\leq x/n} 1$$

the sum over the $n's$ is a finite sum and the sum over the $d's$ tend to $1/d$.

Consequently

$$\limsup \left|\frac{M(x)}{x}\right| \leq \left\{\prod_{p\leq y}(1 - \frac{1}{p})\right\} \left\{\sum_{j=1}^{q-1} |V_y(d_j)| (\frac{1}{d_j} - \frac{1}{d_{j+1}}) + |V_y(d_q)| \frac{1}{d_q}\right\}.$$

This can be written

$$\limsup \left|\frac{M(x)}{x}\right| \leq \left\{\prod_{p\leq y}(1 - \frac{1}{p})\right\} \left\{\int_1^\infty \frac{|V_y(t)|}{t^2}dt\right\}.$$

□

9.2. **Dealing with the integral $\int_1^\infty \frac{|V_y(t)|}{t^2}dt$.**

Lemma 19. *Let*

$$\alpha = \limsup \left|\frac{M(x)}{x}\right|$$

and suppose that $\alpha > 0$. Take $\beta > \alpha$ then when $y \to \infty$

$$\int_y^\infty \frac{|V_y(t)|}{t^2}dts < \beta(c-1)\log y + o(\log y)$$

where

$$c(= e^\gamma) = \lim_{y\to\infty}(\frac{1}{\log y})\prod_{p\le y}(1-\frac{1}{p})^{-1}.$$

Lemma 20. *Let*

$$\alpha = \limsup \left|\frac{M(x)}{x}\right|$$

and suppose that $\alpha > 0$. There exists a constant $\delta > 1$ (depending only on α) such that for any $\beta, 2 > \beta > \alpha$,

$$\int_1^y \frac{|M(t)|}{t^2}dts < (\beta/\delta)\log y + o(\log y).$$

9.3. **Proof of the theorem, assuming the last two lemmas.** We have

$$\begin{aligned}\alpha &= \limsup \left|\frac{M(x)}{x}\right| \le \left\{\prod_{p\le y}(1-\frac{1}{p})\right\}\left\{\int_1^\infty \frac{|V_y(t)|}{t^2}dt\right\} \\ &\le \frac{1}{c+o(1)}\frac{1}{\log y}(\int_1^y \frac{|V_y(t)|}{t^2}dt + \int_y^\infty \frac{|V_y(t)|}{t^2}dt),\end{aligned}$$

where we use Merten's estimate

$$\prod_{p\le y}(1-\frac{1}{p})^{-1} = (c+o(1))\log y.$$

So the preceeding lemmas yield to

$$\alpha \le \frac{1}{c+o(1)}\frac{1}{\log y}(\frac{\beta}{\delta}\log y + \beta(c-1)\log y + o(\log y)),$$

letting $y \to \infty$

$$\alpha \le \frac{\beta}{c\delta} + 1 - \frac{1}{c} = \beta(1-\frac{1}{c}(1-\frac{1}{\delta})),$$

but the quantity $(1-\frac{1}{c}(1-\frac{1}{\delta})$ is strictly less than 1 and depends only on α, so letting $\beta \to \alpha$ leads to a contradiction of our assumptions $\alpha > 0$.

9.4. **A solution to a differential equation.** Let

$$f(x) = \int_0^x \frac{1-e^{-u}}{u}du$$

and define

$$k(s) = \int_0^\infty e^{-sx}e^{f(x)}dx.$$

k is clearly well defined $(f(x) = O(\log x)), k$ is decreasing and has a continuous derivative.

Lemma 21. *We have*

$$sk(s) - \int_s^{s+1} k(u)du = 1, \forall s > 0.$$

Proof.

$$\begin{aligned}
\int_s^{s+1} k(u)du &= \int_0^{\infty} e^{f(x)}(\int_s^{s+1} e^{-ux}du)dx \\
&= \int_0^{\infty} e^{f(x)}e^{-sx}(\frac{1-e^{-x}}{x})dx \\
&= \int_0^{\infty} e^{f(x)}f'(x)e^{-sx}dx \\
&= \left[e^{f(x)}e^{-sx}\right]_0^{\infty} + s\int_0^{\infty} e^{f(x)}e^{-sx}dx \\
&= -1 + sk(s).
\end{aligned}$$

□

Lemma 22. *Let h be a decreasing function defined on $[1, +\infty[$, $h \geq 0$ with a continuous derivative, then*

$$\forall t \geq y, \sum_{p\leq y} \frac{\log p}{p} h(pt) = \int_t^{yt} \frac{h(v)}{v}dv + O(h(y)),$$

and

$$\forall t \geq 1, \sum_{y/t<p\leq y} \frac{\log p}{p} h(pt) = \int_y^{yt} \frac{h(v)}{v}dv + O(h(y)).$$

Proof. We give a detailed proof of the first assertion, the proof of the last one is similar. We use the elementary Mertens's estimate

$$\sum_{p\leq y} \frac{\log p}{p} = \log y + O(1).$$

$$\begin{aligned}
\sum_{p\leq y} \frac{\log p}{p} h(pt) &= h(yt)\sum_{p\leq y} \frac{\log p}{p} - \int_1^y (\sum_{p\leq u} \frac{\log p}{p})th'(ut)du \\
&= h(yt)(\log y + O(1) - \int_1^y (\log u + O(1))th'(ut)du \\
&= h(yt)\log y - [(\log u)h(ut)]_1^y + \int_1^y \frac{h(ut)}{u}du \\
&\quad + O(h(yt) + O(\int_1^y t\left|h'(ut)\right|du) \\
&= \int_1^y \frac{h(ut)}{u}du + O(h(y)) + O(\left|\int_1^y th'ut)du\right| \text{ (as } h' \text{ has a constant sign)}, \\
&= \int_1^y \frac{h(ut)}{u}du + O(h(y)) + O(h(t) + h(ty)) \\
&= \int_1^y \frac{h(ut)}{u}du + O(h(y)) \text{ as } h \text{ is decreasing and } t \geq y.
\end{aligned}$$

□

Lemma 23. *Let* $c = \lim_{y\to\infty} \frac{1}{\log y}\prod_{p\le y}(1-1/p)^{-1}$, *then*

$$\int_1^2 k(u)(2-u)du = c-1.$$

Proof. We give a usefull arithmetic proof . From the well known convolution relation $\log = \Lambda * 1$, we get

$$v_y \log = v_y\Lambda * v_y$$

so

$$\sum_{n\le x} v_y(n)\log n = \sum_{n\le y} v_y(n)\Lambda(n)\sum_{m\le x/n} v_y(m).$$

Denote by

$$V_y^*(t) = \sum_{n\le t} v_y(n),$$

and write

$$\log n = \log x - \log x/n,$$

we get

$$V_y^*(x)\log x - \sum_{n\le x} v_y(n)\log x/n = \sum_{n\le x} v_y(n)\Lambda(n)V_y^*(x/n).$$

We multiply by $h(x)/x^2$ and integrate from y to ∞

$$\begin{aligned}\int_y^\infty V_y^*(x)\log x\frac{h(x)}{x^2}dx &= \int_y^\infty \sum_{n\le x} v_y(n)\Lambda(n)V_y^*(x/n)\frac{h(x)}{x^2}dx + \int_y^\infty \sum_{n\le x} v_y(n)\log x/n\frac{h(x)}{x^2}dx\\ &= \int_y^\infty \sum_{n\le x} v_y(n)\Lambda(n)V_y^*(x/n)\frac{h(x)}{x^2}dx + E_1,\end{aligned}$$

where

$$E_1 = \int_y^\infty \sum_{n\le x} v_y(n)\log x/n\frac{h(x)}{x^2}dx$$

as h is decreasing

$$\begin{aligned}E_1 &\le h(y)\sum_n v_y(n)\int_{\max(y,n)}\log x/n\frac{dx}{x^2}\\ &\le h(y)\sum_n \frac{v_y(n)}{n}\int_1^\infty \frac{\log t}{t^2}dt\\ &= O(h(y)\log y).\end{aligned}$$

So

$$\begin{aligned}\int_y^\infty V_y^*(x)\log x\frac{h(x)}{x^2}dx &= \int_y^\infty \sum_{n\le x} v_y(n)\Lambda(n)V_y^*(x/n)\frac{h(x)}{x^2}dx + O(h(y)\log y)\\ &= \int_y^\infty \sum_{p\le y}\log p V_y^*(x/p)\frac{h(x)}{x^2}dx + O(h(y)\log y) + E_2,\end{aligned}$$

where

$$\begin{aligned} E_2 &= \int_y^\infty \sum_{p\le y, p^r\le x, r\ge 2} \log p V_y^*(x/p^r)\frac{h(x)}{x^2}dx \\ &\le h(y)(\sum_{p\le y, p^r\le x, r\ge 2} \frac{\log p}{p^r})(\int_1^\infty \frac{V_y^*(t)}{t^2}dt) \\ &= O(h(y)(\int_1^\infty \frac{\sum_{n\le t} v_y(n)}{t^2}dt)) \\ &= O(h(y)\sum \frac{v_y(n)}{n}) \\ & \quad O(h(y)\log y). \end{aligned}$$

Finally

$$\begin{aligned} \int_y^\infty V_y^*(x)\log x\frac{h(x)}{x^2}dx &= \sum_{p\le y}\frac{\log p}{p}\int_{y/p}^\infty \frac{V_y^*(u)}{u^2}h(pu)du + O(h(y)\log y) \\ &= \sum_{p\le y}\frac{\log p}{p}\int_{y/p}^y \frac{V_y^*(u)}{u^2}h(pu)du \\ & \quad +\sum_{p\le y}\frac{\log p}{p}\int_y^\infty \frac{V_y^*(u)}{u^2}h(pu)du + O(h(y)\log y) \end{aligned}$$

Clearly

$$\begin{aligned} \sum_{p\le y}\frac{\log p}{p}\int_{y/p}^y \frac{V_y^*(u)}{u^2}h(pu)du &= \int_1^y \frac{V_y^*(u)}{u^2}\sum_{y/u<p\le y}\frac{\log p}{p}h(pu)du \\ &= \int_1^y \frac{V_y^*(u)}{u^2}(\int_u^{u.y}\frac{h(v)}{v}dv)du + O(h(y)\log y) \end{aligned}$$

by the lemma.

Similarly

$$\sum_{p\le y}\frac{\log p}{p}\int_y^\infty \frac{V_y^*(u)}{u^2}h(pu)du = \int_y^\infty \frac{V_y^*(u)}{u^2}(\int_y^{u.y}\frac{h(v)}{v}dv)du + O(h(y)\log y).$$

Thus

$$\begin{aligned} \int_y^\infty V_y^*(x)\log x\frac{h(x)}{x^2}dx &= \int_1^y \frac{V_y^*(u)}{u^2}(\int_u^{u.y}\frac{h(v)}{v}dv)du \\ +\int_y^\infty \frac{V_y^*(u)}{u^2}(\int_y^{u.y}\frac{h(v)}{v}dv)du &+ O(h(y)\log y), \end{aligned}$$

this gives finally

$$\begin{aligned} & \int_y^\infty V_y^*(x)\left\{\log x\frac{h(x)}{x^2}dx - (\int_x^{x.y}\frac{h(v)}{v}dv)\right\}dx \\ = & \int_1^y \frac{V_y^*(u)}{u^2}(\int_y^{u.y}\frac{h(v)}{v}dv)du + O(h(y)\log y). \end{aligned}$$

We choose

$$h(t) = \frac{1}{\log y} k(\frac{\log t}{\log y})$$

where k is the function already defined. From the lemma on k we deduce that

$$h(x) \log x - \int_x^{xy} \frac{h(v)}{v} dv = 1,$$

so

$$\int_y^\infty \frac{V_y^*(x)}{x^2} dx = \int_1^y \frac{V_y^*(u)}{u^2} (\int_y^{u.y} \frac{h(v)}{v} dv) du + O(1).$$

When $u \leq y$, we have trivially that $V_y^*(u) = \sum_{n \leq u} v_y(n) = \sum_{n \leq u} 1 = [u] + O(1)$, so

$$\begin{aligned}
\int_y^\infty \frac{V_y^*(x)}{x^2} dx &= \int_1^y \frac{1}{u} (\int_y^{u.y} \frac{h(v)}{v} dv) du + O(1) \\
&= \int_y^{y^2} \frac{h(v)}{v} \log \frac{y^2}{v} dv + O(1) \\
&= \frac{1}{\log y} \int_y^{y^2} k(\frac{\log v}{\log y}) \log \frac{y^2}{v} \frac{dv}{v} + O(1) \\
&= (\int_1^2 k(u)(2-u) du) \log y + O(1).
\end{aligned}$$

But we have also

$$\int_y^\infty \frac{V_y^*(x)}{x^2} dx = \int_1^\infty \frac{V_y^*(x)}{x^2} dx - \int_1^y \frac{V_y^*(x)}{x^2} dx$$

and, when $u \leq y, V_y^*(u) = [u] + O(1)$, so

$$\begin{aligned}
\int_y^\infty \frac{V_y^*(x)}{x^2} dx &= \sum_n \frac{v_y(n)}{n} - \log y + O(1) \\
&= \prod_{p \leq y} (1 - \frac{1}{p})^{-1} - \log y + O(1) \\
&= (c-1) \log y + O(1) + o(\log y),
\end{aligned}$$

and finally

$$(c-1) \log y + O(1) + o(\log y) = (\int_1^2 k(u)(2-u) du) \log y + O(1)$$

and the lemma after dividing by $\log y$ and letting $y \to \infty$. □

9.5. **Proof of the lemmas.**

Proof of th lemma 19. From $\Lambda = \mu * \log$, and Mobius formula,we get $\Lambda = -\mu \log * 1$, so $\mu \log = -\mu * \Lambda$. This gives

$$-v_y \mu \log = v_y \mu * v_y.$$

Following the same method as in the proof of the last lemma, we have

$$|V_y(t) \log t| \leq \sum_{n \leq t} v_y(n) \Lambda(n) |V_y(t)| + \sum_{n \leq t} v_y(n) \mu^2(n) \log t/n;$$

So

$$\int_y^\infty \left|\frac{V_y(t)}{t^2}\right| \left\{\log th(t) - \int_y^{yt} \frac{h(v)}{v} dv\right\} dt \le \int_1^y \left|\frac{V_y(t)}{t^2}\right| (\int_y^{yt} \frac{h(v)}{v} dv) dt + O(1).$$

With the same choice of the function h we get

$$\int_y^\infty \left|\frac{V_y(t)}{t^2}\right| dt \le \int_1^y \left|\frac{M(t)}{t^2}\right| (\int_y^{yt} \frac{h(v)}{v} dv) dt + O(1)$$

where we replaced $V_y(t)$ by $M(t)$ in the range $1 \le t \le y$.

Let $\beta > \alpha$, then for t large enough (say $t > t_\beta$), $|M(t)| \le \beta t$. So

$$\begin{aligned}\int_y^\infty \left|\frac{V_y(t)}{t^2}\right| dt &\le \beta \int_1^y \frac{1}{t} (\int_y^{yt} \frac{h(v)}{v} dv) dt + O(1)\\ &\le \beta(\int_1^2 k(u)(2-u)du) \log y + o(logy)\\ &\le \beta(c-1)\log y + o(\log y).\end{aligned}$$

□

Proof of the lemma 20. We recall the well known result (see corollary 2) (M can be choosen$= 4$)

$$\left|\int_a^b \frac{M(t)}{t^2} dt\right| \le M.$$

If he function $M(t)$ does not change sign, then the lemma is trivially true by applying this upper bound.

Consider $\beta, \alpha < \beta < 2$ and let t_β be sufficiently large such that $|M(t)| \le \beta t$ if $t > t_\beta$. Consider two different zeroes $t_1 < t_2$ of $M(t)$. We shall prove that

$$\left|\int_{t_1}^{t_2} \frac{M(t)}{t^2}\right| dt \le \frac{\beta}{\delta}. \log \frac{t_2}{t_1}$$

where $\delta = \min(2, 1 + \frac{\alpha^2}{4M})$ and consequently the lemma.

We distinguish three cases according to the size of t_2/t_1.

(i) $\log \frac{t_2}{t_1} > M\delta/\beta$. in this case this follows from the definition of M;

(ii) $\log \frac{t_2}{t_1} \le M\delta/\beta$ and $t_2/t_1 \le 1/(1-\beta/2)$. As $M(t_1) = 0, |M(t)| \le t - t_1 \le t\beta/2$ for all t in $[t_1, t_2]$ by the assumption $t_2/t_1 < 1/(1-\beta/2)$ and inequality holds. □

(iii) $\log \frac{t_2}{t_1} < M\delta/\beta$ and $t_2/t_1 > 1/(1-\beta/2)$. we apply the previous inequality in the range $\left[t_1, \frac{t_1}{1-\beta/2}\right]$ and the inequality $|M(t)| \le \beta t$ in the range $\left[\frac{t_1}{1-\beta/2}, t_2\right]$. so

$$\begin{aligned}\left|\int_{t_1}^{t_2} \frac{M(t)}{t^2}\right| dt &\le \frac{\beta}{2}\log(\frac{1}{1-\beta/2}) + \beta\log(\frac{t_2}{t_1}(1-\frac{\beta}{2})\\ &\le \beta\log\frac{t_2}{t_1} + \frac{\beta}{2}\log(1-\frac{\beta}{2}).\end{aligned}$$

We apply the inequality

$$\begin{aligned}\frac{\beta}{2}\log(1-\frac{\beta}{2}) &\le -\frac{\beta^2}{4} \\ &\le -M(\delta-1)\end{aligned}$$

so

$$\left|\int_{t_1}^{t_2}\frac{M(t)}{t^2}\right| dt \le \beta\log\frac{t_2}{t_1} - M(\delta-1);$$

but we are in the case $\log\frac{t_2}{t_1} < M\delta/\beta$, so

$$\left|\int_{t_1}^{t_2}\frac{M(t)}{t^2}\right| dt \le \frac{\beta}{\delta}\log\frac{t_2}{t_1}.$$

10. Selberg' Formula

The first elementary proof of the Prime Number Theorem is build on Selberg's formula

Theorem 8 (Selberg)**.**

$$\sum_{n\le x}\Lambda(n)\log n + \sum_{n\le x}\Lambda(n)\sum_{m\le x/n}\Lambda(m) = 2x\log x + O(x).$$

Proof. We have

$$\begin{aligned}\log^2 n &= (\Lambda * 1)(n)\log n \\ &= (\Lambda * \log)(n) + (\Lambda\log * 1)(n)\end{aligned}$$

so

$$\begin{aligned}(\log^2 * \mu)(n) &= (\Lambda * \log * \mu)(n) + \Lambda(n)\log n \\ &= (\Lambda * \Lambda)(n) + \Lambda(n)\log n,\end{aligned}$$

and taking the sum

$$\sum_{n\le x}(\log^2 * \mu)(n) = \sum_{n\le x}\Lambda(n)\log n + \sum_{n\le x}\Lambda(n)\sum_{m\le x/n}\Lambda(m),$$

so we need to show that

$$\sum_{n\le x}(\log^2 * \mu)(n) = 2x\log x + O(x).$$

As in Landau's proof, we approximate the function $\log^2$ by some nice arithmetic functions.

As easily seen

$$\sum_{n\le x}\log^2 n = x\log^2 x - 2x\log x + 2x + O(\log^2 x),$$

we have proved that

$$\sum_{n\le x}(1*1)(n) = x\log x + c_1 x + O(\sqrt{x}),$$

similarly (using the hyperbola method)

$$\sum_{n\le x}(1*1*1)(n) = \frac{1}{2}x\log^2 x + c_2 x\log x + c_3 x + O(x^{2/3+\varepsilon})$$

so we take the arithmetic function

$$g(n) = 2(1 * 1 * 1)(n) + a(1 * 1)(n) + b$$

then

$$\sum_{n \le x} \log^2 n - \sum_{ns < x} g(n) = O(x^{3/4})$$

say, when choosing $a = -2 - 2c_2$ and $b = 2 - 2c_3 - ac_1$.

We write

$$\sum_{n \le x} (\log^2 * \mu)(n) = \sum_{ns < x} (\mu * g)(n) + \sum_{n \le x} (\mu * (\log^2 - g)(n).$$

The first sum on the right is equal to

$$\begin{aligned} & 2 \sum_{n \le x} (\mu * 1 * 1 * 1)(n) + a \sum_{n \le x} (\mu * 1 * 1)(n) + b \sum_{n \le x} (\mu * 1)(n) \\ = \quad & 2x \log x + O(x), \end{aligned}$$

the secon sum is less than

$$\begin{aligned} & \sum_{n \le x} |\mu(n)| \left| \sum_{m \le x/n} \log^2 n - \sum_{m < x/n} g(n) \right| \\ = \quad & O(\sum_{n \le x} \frac{x^{3/4}}{n^{3/4}}) = O(x). \end{aligned}$$

□