Bounds on edge colorings with restrictions on the union of color classes

N.R. Aravind and C.R. Subramanian

The Institute of Mathematical Sciences, Chennai - 600 113, India.

email: \{nraravind, crs\}@imsc.res.in

Abstract

We consider constrained proper edge colorings of the following type: Given a positive integer $j$ and a family $\mathcal{F}$ of connected graphs on 3 or more vertices, we require that the subgraph formed by the union of any $j$ color classes has no copy of any member of $\mathcal{F}$. This generalizes some well-known types of colorings such as acyclic edge colorings, distance-2 edge colorings, low treewidth edge colorings, etc.

For such a generalization of restricted colorings, we obtain an upper bound of $O(d^{\max(\theta, 1)})$ on the minimum number of colors sufficient for such a coloring. Here, $d$ refers to the maximum degree of the graph and $\theta$ is a parameter defined by $\theta = \theta(j, \mathcal{F}) = \max_{H \in \mathcal{F}} \left( \frac{|V(H)| - 2}{|E(H)| - j} \right)$. Our proof is based on probabilistic arguments. In particular, we obtain $O(d)$ upper bounds for proper edge colorings with various interesting restrictions placed on the union of color classes. For example, we obtain $O(d)$ upper bounds on edge colorings with restrictions such as (i) the union of any 3 color classes should be an outerplanar graph, (ii) the union of any 4 color classes should have treewidth at most 2, (iii) the union of any 5 color classes should be planar, (iv) the union of any 16 color classes should be 5-degenerate, etc.

We also consider generalizations where we require simultaneously for several pairs $(j_i, \mathcal{F}_i)$ ($i = 1, \ldots, s$) that the union of any $j_i$ color classes has no copy of any member of $\mathcal{F}_i$ and obtain upper bounds on the corresponding chromatic indices. As a corollary, we obtain that each of the four restrictions above can be satisfied simultaneously using $O(d)$ colors.

Proposed running head: Bounds on restricted edge colorings.
1 Introduction

All graphs considered here are simple and undirected. A proper edge coloring is a labeling of the edges of a graph such that touching edges (i.e. edges sharing a common endpoint) do not get the same color. The minimum number of colors sufficient for a proper edge coloring of a graph $G$ is called the chromatic index and is denoted by $\chi'(G)$. This is a well-studied parameter and it is known by a theorem of Vizing [11] that $\chi'(G)$ is always at most $\Delta + 1$ where $\Delta$ denotes the maximum degree of a vertex in $G$.

Several variants of edge colorings have been studied by imposing additional restrictions on the colorings. An interesting example is the acyclic edge coloring which is a proper coloring of the edges of a graph such that there are no bichromatic cycles in the coloring, equivalently, the union of any two color classes must form a forest. Alon, McDiarmid and Reed ([1]) showed that if $G$ has maximum degree $d$, then the acyclic chromatic index is $O(d)$. A distance-2 edge coloring or a strong edge coloring is a proper edge coloring in which edges adjacent to a common edge must also get distinct colors and it is always possible to obtain such a coloring using $O(d^2)$ colors.

Recently, a general notion of a restricted vertex coloring in which we place some restrictions on the union of color classes was considered by Nesetril and Ossona de Mendez in [8], and in [3], the present authors obtained bounds for these types of colorings in terms of the maximum degree of the graph. In this paper, we consider the edge analogues of such restricted colorings and obtain similar upper bounds.

In the case of acyclic coloring, it turns out (see [1]) that any acyclic vertex coloring requires $\Omega(d^{4/3})$ colors for some graphs while, as mentioned before, an acyclic edge coloring is always possible using only $O(d)$ colors. We might expect that restricted edge colorings of such type, in general, require fewer number of colors than their vertex analogues and in this paper we show that this is indeed true for several types of edge colorings. In fact, we show that
for several such edge colorings (like those mentioned in the abstract), the upper bound is simply $O(d)$.

First, we formally define a general notion of a restricted edge coloring.

**Definition 1.1** Let $F$ be a family of connected graphs on 3 or more vertices and $j$ be a positive integer such that $j < \min_{H \in F}(|E(H)|)$. We define a $(j, F)$ edge coloring to be a proper coloring of the edges of a graph $G$ so that the subgraph of $G$ induced by the union of any $j$ color classes does not contain an isomorphic copy of $H$ as a subgraph, for each $H \in F$. We denote by $\chi'_{j,F}(G)$ the minimum number of colors sufficient for a $(j, F)$ edge coloring of $G$.

**Remark:** We require $j < |E(H)|$ for each $H \in F$ because otherwise if $G$ contains a copy of $H$ such that $j \geq |E(H)|$, no proper coloring of $E(G)$ would be a $(j, F)$ edge coloring. Also if $j < |E(H)|$ for each $H \in F$, we are guaranteed at least one $(j, F)$ edge coloring, namely the trivial coloring in which each edge gets a distinct color.

**Notation:** For a positive integer $j$ and a family $F$ of graphs such that $j < E(H)$ for each $H \in F$, we define and use $\theta(j, F)$ to denote the expression below:

$$\theta(j, F) = \max_{H \in F} \left( \frac{|V(H)| - 2}{|E(H)| - j} \right)$$

The main result of this paper is the following theorem which provides upper bounds on the optimal number of colors used in such a coloring.

**Theorem 1.2** Let $F$ be a family of connected graphs on 3 or more vertices and let $j$ be a positive integer such that $j < \min_{H \in F}(|E(H)|)$. Let $\theta = \theta(j, F)$. Then there exists a constant $C = C(j, F)$ such that for any graph $G$ of maximum degree $d$, $\chi'_{j,F}(G) \leq \lceil Cd^{\max(\theta,1)} \rceil$.

As mentioned before, the acyclic chromatic index of any graph of maximum degree $d$ is at most $O(d)$. This naturally leads to the general question of determining those $(j, F)$ pairs for which $\chi'_{j,F}(G) = O(d)$. The following corollary of the previous theorem provides an answer to this question.
Corollary 1.3  Let $\mathcal{F}$ be a family of connected graphs on 3 or more vertices and let $D = D(\mathcal{F}) = \min_{H \in \mathcal{F}}(|E(H)| - |V(H)|)$. Let $j$ be any positive integer such that $j < \min_{H \in \mathcal{F}}(|E(H)|)$ and $j \leq D + 2$. Then there exists a constant $C = C(\mathcal{F})$ such that for any graph $G$ of maximum degree $d$, $\chi'_{j,\mathcal{F}}(G) \leq \lceil Cd \rceil$.

Proof: Follows from Theorem 1.2 after applying the easy to verify observation that $\theta \leq 1$ if and only if $j \leq D + 2$.

2 Proof of results

To prove Theorem 1.2, we need the following non-symmetric form of Erdős-Lovász local lemma (see [2]).

Lemma 2.1  Let $\{A_1, A_2, ..., A_n\}$ be a family of events in an arbitrary probability space. Let the graph $H = (V, E)$ on the nodes $1, 2, ..., n$ be a dependency digraph for the events $A_i$; that is, assume that for each $i$, $A_i$ is mutually independent of the family of events $\{A_j : (i, j) \notin E\}$.

If there are reals $0 \leq y_i < 1$ such that for all $i$,

$$Pr(A_i) \leq y_i \prod_{(i, j) \in E} (1 - y_j)$$

then

$$Pr(\cap \overline{A_i}) \geq \prod_{i=1}^{n} (1 - y_i) > 0$$

so that with positive probability no event $A_i$ occurs.

We prove the following explicit version of Theorem 1.2, wherein we have not attempted to optimize the constant $C(j, \mathcal{F})$.

Proposition 2.2  Let $\mathcal{F}$ be a family of connected graphs on 3 or more vertices and $j$ be a positive integer as in Theorem 1.2. Let

$$\theta = \theta(j, \mathcal{F}) = \max_{H \in \mathcal{F}} \frac{|V(H)| - 2}{|E(H)| - j},$$

$$D = D(\mathcal{F}) = \min_{H \in \mathcal{F}}(|E(H)| - |V(H)|),$$

$$C = C(j, \mathcal{F}) = 200 \cdot 2^{6j+6D} \cdot (3j)^{2j}.$$ Then, for any graph $G$ of maximum degree $d$, $\chi'_{j,\mathcal{F}}(G) \leq \lceil (Cd)^{\max(\theta, 1)} \rceil$. 

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Proof of Proposition 2.2: Let $G = (V, E)$ be the given graph. Without loss of generality, we assume that $j \geq 2$. When $j = 1$, any $(j, \mathcal{F})$ coloring is the same as a proper edge coloring of $G$ which always exists with $d + 1$ colors (by Vizing’s theorem, see [12] pages-277-279). Henceforth, we assume that $j \geq 2$.

Put $x = \lceil (Cd)^{\max(\theta, 1)} \rceil$ where $C = 200 \cdot 2^{6j+6D} \cdot (3^j)^2$.

Let $f : E \rightarrow \{1, 2, ..., x\}$ be a random edge coloring of $G$, where for each edge $e \in E$ independently, the color $f(e) \in \{1, 2, ..., x\}$ is chosen uniformly at random. It suffices to prove that with positive probability, $f$ is a $(j, \mathcal{F})$ edge coloring of $G$. To this end, we define a family of bad events whose absence implies that the random coloring is a $(j, \mathcal{F})$ edge coloring and use the Lovasz local lemma to show that with positive probability none of these events occur. The events we consider are of the following two types.

a) **Type I**: For each pair of touching edges $e_1 = (u, v)$ and $e_2 = (u, w)$, let $A_{e_1,e_2}$ be the event that $f(e_1) = f(e_2)$.

We define $\alpha = \frac{1}{j}$. The definition of the Type II event depends on whether $\alpha < 1$ or $\alpha \geq 1$.

Case $\alpha < 1$:

b) **Type II**: For each connected subgraph $L$ of $V(G)$ such that $|V(L)| \geq 3$ and $|E(L)| = max\{|V(L)|-1, [\alpha(|V(L)|-2)+j]\}$, let $B_L$ be the event that the edges in $L$ are colored using at most $j$ colors in the coloring by $f$.

Note that for each $H \in \mathcal{F}$, we have $|E(H)| \geq |V(H)|-1$ and $|E(H)| \geq [\alpha(|V(H)|-2)+j]$ and hence the absence of type II events in this case ensures that the union of $j$ color classes cannot have a copy of any member of $\mathcal{F}$.

Case $\alpha \geq 1$:

b) **Type II**: For each connected subgraph $L$ of $V(G)$ such that $|V(L)| \geq 3$ and $|E(L)| = |V(L)|+D$, let $B_L$ be the event that the edges in $L$ are colored using at most $j$ colors in the coloring by $f$. Since $j \geq 2$, in this case, $D \geq 0$. Also, for each $H \in \mathcal{F}$, we have $|E(H)| \geq |V(H)|+D$ and thus the absence of type II events in this case ensures that the union of $j$ color classes cannot have a copy of any member of $\mathcal{F}$.
Thus we see that if none of the events of the two types above occurs, then $f$ is a $(j, \mathcal{F})$ edge coloring.

It remains to show that with positive probability none of these events happen. To prove this we apply the local lemma. Note that any event of either of the two types is mutually independent of all events that do not share an edge in common with the given event.

We need to estimate the number of events of each type possibly influencing any given event. This estimate follows from the following two simple lemmas.

**Lemma 2.4** Let $e = (u, v)$ be an arbitrary edge of the graph $G = (V, E)$. Then the following two statements hold.

(i) $e$ touches at most $2d$ edges in $G$.

(ii) $e$ belongs to at most $2k^{2j+2D+1}4^kd^{k-2}$ subgraphs of $V(G)$ on $k$ vertices which are as in a Type II event.

**Proof** Part (i) follows from the fact that $\Delta(G) = d$.

Part (ii) can be seen as follows: If $\alpha < 1$, let $G(e, k)$ be the set of connected subgraphs (containing $e$) in $G$ on $k$ vertices and having $\max\{k - 1, \lceil \alpha(k-2) + j \rceil \}$ edges. If $\alpha \geq 1$, let $G(e, k)$ be the set of connected subgraphs (containing $e$) in $G$ on $k$ vertices and having $k + D$ edges. Let $T(e, k)$ be the set of $k$-vertex trees in $G$ containing $e$ with some arbitrary linear order imposed on them.

If $\alpha < 1$, each tree in $T(e, k)$ can be identified with at most

$$\left( \max\{0, \lceil \alpha(k-2) + j \rceil - (k-1) \} \right) \leq k^{2j-2}$$

connected subgraphs in $G(e, k)$ on the same set of vertices. If $\alpha \geq 1$, each tree in $T(e, k)$ can be identified with at most $(\binom{k}{D+1}) \leq k^{2D+2}$ connected subgraphs in $G(e, k)$ on the same set of vertices. Each connected subgraph $H$ in $G(e, k)$ has at least one tree in $T(e, k)$ the smallest (with respect to the assumed linear ordering) of which is identified with $H$. Thus $|G(e, k)| \leq k^{2j+2D}|T(e, k)|$, irrespective of whether $\alpha < 1$ or $\alpha \geq 1$. 

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We now find an upper bound for $|T(e, k)|$. Since there are at most $4^k$ unlabeled trees on $k$ vertices (see Chapter 8 of [7]), there are at most $4^k$ choices for choosing the unlabeled structure of a tree in $T(e, k)$. Once this unlabeled structure is fixed, we now have to embed this unlabeled tree in $G$. The number of choices for the position of the edge $e$ in the unlabeled tree is at most $2^{k-1}$. Now the remaining vertices in the unlabeled tree can be embedded in at most $d \times (k-2)$ ways. To see this, we observe that there are $d$ choices for each neighbor of $v$ in the chosen unlabeled tree. Once these are fixed, the number of choices for the neighbor of each first neighbor is again $d$. Repeating this process, we can see that the number of choices for embedding all the vertices (other than $u, v$) is at most $d \times (k-2)$. This proves (ii).

**Lemma 2.5** For $\{(i, j) \in \{I, II\} \}$ the $(i, j)$-th entry of the matrix $M$ given below is an upper bound on the number of events of type $j$ in the dependency graph $H$ which can possibly influence an event of type $i$ in $H$.

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II($B_L'$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>4$d$</td>
<td>$4^j \times 2^j + 14^j \times d^{k-2}$</td>
</tr>
<tr>
<td>II($B_L$)</td>
<td>2$md$</td>
<td>$2m^2 \times 2^j + 14^j \times d^{k-2}$</td>
</tr>
</tbody>
</table>

Here, $m$ is the number of edges in $L$ and $l$ is the number of vertices in $L'$. The lemma follows from Lemma 2.4 and the fact that any event is mutually independent of all other events which do not share any edge with the given event. We now estimate the probability of occurrence of each type of event.

**Fact 2.6** (i) For each type I event $A$, $Pr(A) = \frac{1}{x}$.

(ii) For each type II event $B_L$, $Pr(B_L) \leq \frac{m^j}{x^{m-j}}$, where $m = |E(L)|$.

The number of ways in which $m$ edges can be colored using at most $j$ colors from $\{1, 2, ..., x\}$ is at most $\binom{x}{j} j^m \leq x^j j^m$. This proves (ii).

We now define the constants $y_i$ to enable us to apply the local lemma.

For an event $A$ of type I, we define $y_A = \frac{9}{x}$. For an event $B_L$ of type II, we define $y_{B_L} = \frac{(3j)^m}{x^{m-j}}$, where $m = |E(L)|$.

If $\alpha < 1$, $|E(L)| - j \geq \alpha(|V(L)| - 2)$ for each forbidden $j$-colored graph $L$ and using $x > 3j$, we note that $y_{B_L} \leq \frac{(3j)^{\alpha}}{x^{\alpha(k-2)}}$ where $k = |V(L)|$. 7
If $\alpha \geq 1$, then $|E(L)| - j \geq |V(L)| - 2$ for each forbidden $j$-colored graph $L$ and hence $y_{B_L} = \frac{(3j)^{k+L}}{(3j)^{k+L-2} + D - j + 2} \leq \frac{(3j)^{k+L}}{(3j)^{k+L-2} + 2}$, where $k = |V(L)|$. Here we used $x > 3j$ and also the fact that $D \geq j - 2$ whenever $\alpha \geq 1$.

In either case, by substituting $x = (Cd)^{max(\theta, 1)}$, we find that $y_{B_L} \leq \frac{(3j)^{k+L}}{(3j)^{k+L-2}}$ and hence $(1 - y_{B_L}) \geq 1 - \frac{(3j)^{j+k-2}}{(3j)^{k-2}}$.

By Lemma 2.1, Lemma 2.5 and Fact 2.6, it thus suffices to verify the following two inequalities.

$$\frac{1}{x} \leq \frac{9}{x} (1 - \frac{9}{x}) \prod_{l \geq 3} (1 - y_{B_L})^{4l^2j+2D+1} l^{d-2}$$

- (1)

$$\frac{j^m}{x^{m-j}} \leq \frac{(3j)^m}{x^{m-j}} (1 - \frac{9}{x}) \frac{2md}{l \geq 3} (1 - y_{B_L})^{2m(j+l-2)+1} l^{d-2}, \forall m \geq 3$$

- (2)

We see that (2) is equivalent to (1). Thus it is sufficient to prove (1).

In (1), we substitute $x = (Cd)^{max(\theta, 1)}$ where $C = 200 \cdot (2)^{6j+6D} \cdot (3j)^{2j}$ and using the fact that $(1 - \frac{1}{z})^z \geq 1/4$ for all $z \geq 2$, as well as the fact that $(1 - y_{B_L}) \geq 1 - \frac{(3j)^{j+l-2}}{(3j)^{k-2}}$ we see that it is sufficient to prove:

$$\frac{1}{9} \leq 4^{\frac{3md}{l}} 4^{-S}$$

where

$$S = \sum_{l \geq 3} \frac{(3j)^{j+l-2} \cdot A^l+1 \cdot l^{2j+2D+1}}{200^{l-2} \cdot (6j+6D)(l-2) \cdot (3j)(2j)(l-2)}$$

Using the fact that $j + l - 2 \leq 2j(l - 2), \forall j \geq 2, l \geq 3,$ and also the fact that $l^{2j+2D+1} < 2^{(2j+2D)} l \leq 2^{(6j+6D)(l-2)}, \forall j \geq 2, l \geq 3, D \geq -1,$
we get
\[ S \leq \sum_{t \geq 3} \frac{4^{t+1}}{200^t - 2} = \frac{64}{49} < \frac{4}{3}. \]

We thus find that it is sufficient to prove:
\[ \frac{1}{9} \leq 4^{-\frac{36d}{x}} 4^{-\frac{4}{3}}. \]

Since \( x \geq 216d \), the above inequality is true.

Thus by Erdős-Lovász Local Lemma, with positive probability, none of the bad events occurs and hence there exists a \((j, F)\) edge coloring using \(O(d_{\max}(\theta, 1))\) colors. This completes the proof of Proposition 2.2 and hence of Theorem 1.2.

3 Consequences

We now apply Theorem 1.2 to some interesting families of graphs to obtain the results in the following table. Throughout, we assume that \( \Delta(G) = d \).
<table>
<thead>
<tr>
<th>Restriction on the union of color classes</th>
<th>j</th>
<th>$\mathcal{F}$</th>
<th>$\theta(j, \mathcal{F})$</th>
<th>Bound on $\chi'_{j,\mathcal{F}}(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Planar</td>
<td>5</td>
<td>Subdivisions of $K_{3,3}$ and $K_5$</td>
<td>1</td>
<td>$O(d)$</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>&quot;&quot;</td>
<td>4/3</td>
<td>$O(d^{4/3})$</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>&quot;&quot;</td>
<td>2</td>
<td>$O(d^2)$</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>&quot;&quot;</td>
<td>3</td>
<td>$O(d^3)$</td>
</tr>
<tr>
<td>Outerplanar</td>
<td>3</td>
<td>Subdivisions of $K_4$ and $K_{2,3}$</td>
<td>1</td>
<td>$O(d)$</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>&quot;&quot;</td>
<td>3/2</td>
<td>$O(d^{3/2})$</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>&quot;&quot;</td>
<td>3</td>
<td>$O(d^3)$</td>
</tr>
<tr>
<td>Treewidth at most 2</td>
<td>4</td>
<td>Subdivisions of $K_4$</td>
<td>1</td>
<td>$O(d)$</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>&quot;&quot;</td>
<td>2</td>
<td>$O(d^2)$</td>
</tr>
<tr>
<td>Treewidth at most $k$ for $k \geq 2$</td>
<td>$k+2$</td>
<td>Edge minimal graphs of treewidth more than $k$</td>
<td>1</td>
<td>$O(d)$</td>
</tr>
<tr>
<td>$k$-degenerate graphs</td>
<td>$\frac{k^2+k+2}{2}$</td>
<td>Edge minimal graphs that are non-$k$-degenerate</td>
<td>1</td>
<td>$O(d)$</td>
</tr>
<tr>
<td>$k$-colorable graphs</td>
<td>$\frac{k^2-k+2}{2}$</td>
<td>Edge-critical $(k+1)$-chromatic graphs</td>
<td>1</td>
<td>$O(d)$</td>
</tr>
<tr>
<td>Genus at most $g$</td>
<td>$2g+3$</td>
<td>Edge minimal graphs of genus more than $g$</td>
<td>1</td>
<td>$O(d)$</td>
</tr>
</tbody>
</table>

**Justification for some entries:**

1. **Planarity restriction:**

Note that any subdivision of $K_5$ is a graph on $5+k$ vertices and $10+k$ edges for some $k \geq 0$. Similarly, any subdivision of $K_{3,3}$ is a graph on $6+l$ vertices and $9+l$ edges for some $l \geq 0$. Hence

$$\theta(j, \mathcal{F}) = \max_{k,l \geq 0} \left\{ \frac{3+k}{10-j+k}, \frac{4+l}{9-j+l} \right\}.$$

This value is at most 1 if $j \leq 5$ and is $4/3$ for $j = 6$ and is 2 for $j = 7$ and is 4 for $j = 8$. This proves the entries in the table.

2. **Outerplanarity restriction:**

Note that any subdivision of $K_4$ is a graph on $4+k$ vertices and $6+k$ edges for some $k \geq 0$. Similarly, any subdivision of $K_{2,3}$ is a graph on
5 + l vertices and 6 + l edges for some \( l \geq 0 \). Hence

\[
\theta(j, \mathcal{F}) = \max_{k, l \geq 0} \left\{ \frac{2 + k}{6 - j + k}, \frac{3 + l}{6 - j + l} \right\}.
\]

This value is at most 1 if \( j \leq 3 \) and is \( 3/2 \) for \( j = 4 \) and is 3 for \( j = 5 \). This proves the entries in the table.

3. \( k \)-degeneracy restriction:

Any connected minimal (with respect to edge deletion) graph of degeneracy \( k + 1 \) is a graph on \( v \) vertices for some \( v \geq k + 2 \) and has minimum degree \( k + 1 \) and hence has at least \( v(k+1)/2 \) edges. Thus,

\[
D \geq (k + 2)(k - 1)/2 \quad \text{and hence for } j \leq \frac{(k+2)(k-1)}{2} + 2 = \frac{k^2+k+2}{2}, \text{ we can apply Corollary 1.3 to deduce that } O(d) \text{ colors suffice.}
\]

4. \( k \)-colorability restriction:

Any connected minimal (with respect to edge deletion) graph of chromatic number \( k + 1 \) is a graph on \( v \) vertices for some \( v \geq k + 1 \) and has minimum degree at least \( k \) and hence has at least \( vk/2 \) edges. Thus,

\[
D \geq (k + 1)(k - 2)/2 \quad \text{and hence for } j \leq \frac{(k+1)(k-2)}{2} + 2 = \frac{k^2-k+2}{2}, \text{ we can apply Corollary 1.3 to deduce that } O(d) \text{ colors suffice.}
\]

5. treewidth at most \( k \):

It can be shown by a simple inductive argument that any graph of treewidth more than \( k \) contains at least \( v + k \) edges provided \( k \geq 2 \). This shows that for \( j \leq k + 2 \), \( \theta(\mathcal{F}) \leq 1 \).

6. Genus at most \( g \):

By Euler’s polyhedral formula, the number of edges in a graph of genus at least \( g + 1 \) and having \( v \) vertices is at least \( v + 2g + 1 \). Thus

\[
D(\mathcal{F}) = \min_{H \in \mathcal{F}}(|E(H)| - |V(H)|) \geq 2g + 1. \quad \text{Hence, by Corollary 1.3,}
\]

for \( j \leq 2g + 3 \), \( O(d) \) colors suffice.

4 Extensions to colorings with several families forbidden simultaneously

We can also extend our results to more restricted edge colorings where we require simultaneously for several pairs \((j_i, \mathcal{F}_i) (i = 1, \ldots, s)\) that the union of any \( j_i \) color classes has no copy of any member of \( \mathcal{F}_i \). The vertex versions of such colorings were considered by Nesetril and Ossona de Mendez in [8]
for families of $H$-minor-free graphs. A slightly relaxed notion (where we don’t insist on properness) was studied by DeVos, et. al. in [4] for families of $H$-minor-free graphs. However, we obtain bounds which work for any arbitrary graph $G$. We first formally define these colorings.

**Definition 4.1** Let $\mathcal{P} = \{(j_1, \mathcal{F}_1), \ldots, (j_s, \mathcal{F}_s)\}$ be a set of $s \geq 1$ pairs such that for each $i \leq s$, $j_i$ is a positive integer and $\mathcal{F}_i$ is a family of connected graphs on 3 or more vertices such that $j_i < |E(H)|$ for each $H \in \mathcal{F}_i$. We define a $\mathcal{P}$-edge coloring to be a proper edge coloring of $G$ so that, for each $i \leq s$, the union of any $j_i$ color classes does not contain an isomorphic copy of $H$ as a subgraph, for each $H \in \mathcal{F}_i$. We denote by $\chi'_\mathcal{P}(G)$ the minimum number of colors sufficient for a $\mathcal{P}$-edge coloring of $G$.

We now present the main result of this section.

**Theorem 4.2** Let $\mathcal{P} = \{(j_1, \mathcal{F}_1), \ldots, (j_s, \mathcal{F}_s)\}$ be a set of $s \geq 1$ pairs such that for each $i \leq s$, $j_i$ is a positive integer and $\mathcal{F}_i$ is a family of connected graphs on 3 or more vertices such that for each $j_i < |E(H)|$ for each $H \in \mathcal{F}_i$. Define

$$
\theta_i = \theta(j_i, \mathcal{F}_i) = \max_{H \in \mathcal{F}_i} \frac{|V(H)| - 2}{|E(H)| - j_i}, \ \forall i \leq s,
$$

$$
D_i = D(\mathcal{F}_i) = \min_{H \in \mathcal{F}_i} (|E(H) - |V(H)|), \ \forall i \leq s,
$$

$$
C_i = C(j_i, \mathcal{F}_i) = 200s \cdot 2^{6j_i + 6D_i} \cdot (3j_i)^{2j_i}, \ \forall i \leq s,
$$

$$
\theta = \max_{i \leq s} \theta_i, \ \ C = \max_{i \leq s} C_i.
$$

Then, for any graph $G$ of maximum degree $d$, $\chi'_\mathcal{P}(G) \leq \lceil (Cd)^{\max(\theta, 1)} \rceil$.

We skip the proof of the above theorem as it is based on an application of the Local Lemma and is similar to the proof of Theorem 1.2.

By setting $\mathcal{P}_s = \{(1, \mathcal{F}_1), \ldots, (s, \mathcal{F}_s)\}$ where $\mathcal{F}_i$ is the set of all $i$-edge colorable (usual edge coloring) graphs of treewidth $i + 1$, for each $i \leq s$, we can get upper bounds on the type of edge colorings studied by DeVos, et. al. in [4].
Corollary 4.3  For \( s \geq 1 \), let \( \chi'_{P_s}(G) \) denote the minimum number of colors sufficient to obtain a proper edge coloring of \( G \) so that the union of any \( j \leq s \) color classes forms a subgraph of treewidth at most \( j \). Then, there exists a constant \( C = C(s) \) such that for any graph of maximum degree \( d \), 
\[
\chi'_{P_s}(G) \leq C \cdot d.
\]

Generalized acyclic edge colorings:

This notion was introduced in [5] and is a generalization of the acyclic edge colorings. For any \( r \geq 3 \), the \( r \)-acyclic chromatic index \( a'_r(G) \) is the minimum number colors sufficient to properly color the edges of \( G \) so that every \( k \)-cycle uses at least \( \min\{r, k\} \) colors, for every \( k \geq 3 \). Note that this specializes to the standard acyclic chromatic index when \( r = 3 \). In [6], it is shown that for every fixed \( r \geq 4 \), \( a'_r(G) = O(\frac{d}{r/2}) \).

This result follows as a corollary of Theorem 4.2. Let \( l = \lceil \frac{r}{2} \rceil + 1 \). Let \( P \) be defined by
\[
P = \{ (2, P_3), (3, P_4), \ldots, (l - 1, P_l), (r - 1, \{ C_k : k > r \}) \}.
\]
Here, \( P_k \) denotes a path on \( k \) edges and \( C_k \) denotes a cycle on \( k \) edges. The first \( l - 2 \) pairs forbid any path having \( k \leq l \) edges being colored with fewer than \( k \) colors. This, in turn, implies that any cycle \( C_k \) on \( k \leq r \) edges is colored with \( k \) colors. The last pair takes care of the remaining cycles. Thus, every \( P \)-edge coloring is also a generalized \( r \)-acylic edge coloring. It is easy to see that
\[
\forall k, 3 \leq k \leq l, \quad \theta(k - 1, P_k) = k - 1 \leq \lfloor \frac{r}{2} \rfloor,
\]
\[
\theta(r - 1, \{ C_k : k > r \}) = \max_{k \geq 1} \frac{r + k - 2}{k + 1} = \frac{r - 1}{2} \leq \lfloor \frac{r}{2} \rfloor.
\]
Applying Theorem 4.2, for each fixed \( r \geq 3 \), we have \( a'_r(G) \leq \chi'_{P}(G) = O(d^{r/2}) \). The upper bound is tight up to a constant factor as shown in [6].

Note that if, instead of defining \( P \) as above, we had used the natural definition of
\[
P = \{ (2, C_3), (3, C_4), \ldots, (r - 1, \{ C_k : k \geq r \}) \},
\]
we would have only obtained a bound of \( O(d^{r-2}) \). In fact, our choice of \( P \) was motivated by the choice of bad events used in [6]. This shows that it will help to try to upper bound a more restrictive coloring. We formally apply this observation in the following subsection.
4.1 Improving some of the table entries:

For a connected graph $H$, let $dl(H)$ denote the diameter of the line graph of $H$. This means that any two edges in $H$ are part of a path in $H$ on at most $dl(H) + 1$ edges. Note that if a proper edge coloring of $G$ is such that any path in $G$ on $k$ (for each $k \leq dl(H) + 1$) edges uses exactly $k$ colors, then any copy of $H$ in $G$ must use at least $|E(H)|$ colors. Otherwise, there must be two edges in a copy of $H$ colored the same and since these are part of some path on $k \leq dl(H) + 1$ edges, this path must use at most $k - 1$ colors, a contradiction. This, in turn, implies that for any $j < |E(H)|$, any $j$ color classes of this coloring does not have a copy of $H$. This is a more restricted coloring than forbidding a copy of $H$ in any $j$ color classes. But, this may result in a better bound. By applying Theorem 4.2 to this observation, we get the following refinement of Theorem 1.2 whose proof can be easily worked out.

**Theorem 4.4** Let $F$ be a fixed family of connected graphs on 3 or more vertices and let $j$ be a positive integer such that $j < \min_{H \in F} \{ |E(H)| \}$. Let $F = F_1 \cup F_2$ be a fixed partition of $F$ where $F_1$ is finite. Let $\theta_2 = \theta(j, F_2)$ and $\theta_1 = \max_{H \in F_1} \min(dl(H), \theta(j, H))$ where $dl(H)$ is the diameter of the line graph of $H$. Then there exists a constant $C = C(j, F_1, F_2)$ such that for any graph $G$ of maximum degree $d$, $\chi'_{j,F}(G) \leq \lceil Cd^{\max(1,\theta_1,\theta_2)} \rceil$.

The motivation for this theorem is that for a suitable choice of the partition $F = F_1 \cup F_2$, it may be that $\max\{\theta_1, \theta_2\} < \theta(j, F)$ resulting in an asymptotic improvement of the bound. This is illustrated in the following two improvements on entries in Table 1 in the previous section.

1. For the planarity restriction with $j = 8$, we can improve the upper bound to $O(d^2)$ from the $O(d^4)$ presented before. Write $F = F_1 \cup F_2$ where $F_1$ is the set of all subdivisions of $K_{3,3}$ with at most one subdivision and $F_2 = F \setminus F_2$. $F_1$ has exactly two members and for each of them, the diameter of the corresponding line graph $L(H)$ is 2 and hence $\theta_1 = 2$. Also

$$\theta(8, F_2) = \max_{k \geq 0, l \geq 2} \left\{ \frac{3 + k}{10 - 8 + k}, \frac{4 + l}{9 - 8 + l} \right\} = 2.$$  

Thus, by Theorem 4.4, we can properly color the edges of a graph of maximum degree $d$ using $O(d^2)$ colors so that the union of any 8 color classes is planar.
2. For the outerplanarity restriction with \( j = 5 \), write \( \mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \) where \( \mathcal{F}_1 \) is the set of all subdivisions of \( K_{2,3} \) with at most one subdivision and \( \mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1 \). For each of the two members in \( \mathcal{F}_1 \), the diameter of the corresponding line graph \( L(H) \) is 2 and hence \( \theta_1 = 2 \). Also

\[
\theta(5, \mathcal{F}_2) = \max_{k \geq 0, l \geq 2} \left\{ \frac{2 + k}{6 - 5 + k}, \frac{3 + l}{6 - 5 + l} \right\} = 2.
\]

Thus, by Theorem 4.4, we can properly color the edges of a graph of maximum \( d \) using \( O(d^2) \) colors so that the union of any 5 color classes is outerplanar.

5 Conclusions and Open Problems

We considered a generalization of some known edge colorings like acyclic edge colorings and obtained upper bounds on the chromatic index in terms of the maximum degree \( d \). For several \((j, \mathcal{F})\) edge colorings, the bounds are actually \( O(d) \), thereby showing that imposing additional restrictions involving any few color classes does not necessarily increase the required number of colors asymptotically. Obviously, these bounds are tight within a constant factor for such colorings. It would be interesting to establish the tightness (at least within constant or polylog multiplicative factor) of other super linear upper bounds.

It would also be interesting to obtain constructive (that is, algorithmically efficiently realizable) bounds which match the bounds presented in this paper for some specific pairs \((j, \mathcal{F})\). For some colorings, there is an asymptotic gap between existential and constructive bounds. For example, acyclic chromatic index of any graph is at most \( 16d \) but the currently known constructive bound (see [10]) is only shown to be \( O(d \log d) \).

An interesting direction is to explore possible improvements in the bounds for random graphs or for random regular graphs. Such results have been obtained for acyclic edge coloring in [9] where it was shown that the acyclic chromatic index of a random \( d \)-regular graph is at most \( d+1 \).

References


