‘Modular Completions’ as Non-holomorphic Eisenstein-like Series

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Main references:

- Eguchi-Y.S , JHEP 1103(2011)107, (arXiv:1012.5721[hep-th])

- Y.S, JHEP 1201(2012)098, (arXiv:1109.3365[hep-th])

+ Work in progress

Introduction

SL(2)/U(1) supercoset (N=2 Liouville theory)

- Simplest non-rational N=2 SCFT
- Describes a curved, non-compact background ('cigar geometry')
- Elliptic genus (supersymmetric index) is modular, but non-holomorphic.

[Troost 2010] , [Eguchi-Y.S 2010]

Elliptic genus is expanded by the 'modular completions' of characters.

Introduction

Building blocks :

"extended"
(spectral flow sum)

[Eguchi-Taormina 1988],
[Odake 1989]
Eguchi-Y.S. 2003]

"continuous character" (non-BPS)

$$\chi_{\mathbf{con}}(p,m;\tau,z) \;=\; q^{\frac{p^2}{4NK}}\Theta_{m,NK}\left(\tau,\frac{2z}{N}\right)\frac{\theta_1(\tau,z)}{i\eta(\tau)^3}.$$

"discrete character" (BPS)

$$\chi_{\mathbf{dis}}(v,a;\tau,z) = \sum_{n\in\mathbb{Z}}\frac{(yq^{Nn+a})^{\frac{v}{N}}}{1-yq^{Nn+a}}\,y^{2K\left(n+\frac{a}{N}\right)}q^{NK\left(n+\frac{a}{N}\right)^2}\,\frac{\theta_1(\tau,z)}{i\eta(\tau)^3}.$$

"mock modular form"

Introduction

Modular transformation

(schematically written as…)

$$\chi_{con}\left(p, m; -\frac{1}{\tau}\right) = \sum_{m} \int dp'\, S(p, m|p', m')\, \chi_{con}(p', m'; \tau)$$

$$\chi_{dis}\left(v, a; -\frac{1}{\tau}\right) = \sum_{v', a'} A(v, a|v', a')\, \chi_{dis}(v', a'; \tau)$$

$$+ \sum_{m'} \int dp'\, B(v, a|p', m')\, \chi_{con}(p', m'; \tau, z)$$

(typical for mock modular forms)

“mixing term” (Mordell integral)

Introduction

Existence of mixing term

Difficulty in construction of objects with good modular property !

(Would be typical for non-compact, curved target space)

Introduction

"Modular completion"

[Eguchi-Y.S 2010]

$$\widehat{\chi}_{\mathbf{dis}}(v,a;\tau,z) = \chi_{\mathbf{dis}}(v,a;\tau,z) + [\text{non-hol. correction terms}]$$

$$\left(\chi_{\mathbf{dis}}(v,a;\tau,z) = \sum_{n\in\mathbb{Z}} \frac{(yq^{Nn+a})^{\frac{v}{N}}}{1-yq^{Nn+a}} y^{2K\left(n+\frac{a}{N}\right)} q^{NK\left(n+\frac{a}{N}\right)^2} \frac{\theta_1(\tau,z)}{i\eta(\tau)^3}. \right)$$

(closely related with [Zwegers 2002], [Troost 2010])

Introduction

We schematically define :

$\widehat{\chi}_{dis}(v, a; \tau)$ is 'modular completion' of $\chi_{dis}(v, a; \tau)$

$$\Longleftrightarrow$$

$$\widehat{\chi}_{dis}\left(v, a; -\frac{1}{\tau}\right) = \sum_{v', a'} A(v, a|v', a')\, \widehat{\chi}_{dis}(v', a'; \tau).$$

(no mixing terms)

In this talk, I would like to discuss

•Simpler expression of the modular completions, based on the path-integration in the SL(2)/U(1) supergauged WZW (with arbitrary level).

→ Non-hol. Eisenstein-like series

•Application to the Gepner-like orbifolds for non-compact CY model.

Contents

‘Modular Completions’

Modular Completions

Instead of treating the extended characters,
start with the closely related function ;

‘Appell function’ (Appell-Lerch sum)

$$f_u^{(k)}(\tau, z) := \sum_{n\in\mathbb{Z}} \frac{q^{kn^2} y^{2kn}}{1 - yw^{-1}q^n}$$

$$(q \equiv e^{2\pi i\tau},\ y \equiv e^{2\pi iz},\ w \equiv e^{2\pi iu}, \quad k \in \mathbb{Z}_{>0})$$

(~ a typical example of mock modular forms)

Modular Completions

'modular completion' of Appell function: [Zwegers 2002]

$$\widehat{f}_u^{(k)}(\tau, z) := f_u^{(k)}(\tau, z) - \frac{1}{2} \underbrace{\sum_{m \in \mathbb{Z}_{2k}} R_{m,k}(\tau, u)\Theta_{m,k}(\tau, 2z)}_{\text{correction term}}$$

(~ harmonic Maass form)

Modular Completions

where we set

looks level –k theta function

$$R_{m,k}(\tau,u) := \sum_{\nu\in m+2k\mathbb{Z}} \left\{ \mathrm{sgn}(\nu+0) - \mathrm{Erf}\left(\sqrt{\frac{\pi\tau_2}{k}}(\nu+2k\alpha)\right)\right\} w^{-\nu} q^{-\frac{\nu^2}{4k}}$$

non-holomorphic

(Error fn $\mathrm{Erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt, \qquad (x\in\mathbb{R})$

$u \equiv \alpha + \beta\tau$)

Modular Completions

$\widehat{f}_u^{(k)}(\tau, z)$ is a non-holomorphic weak Jacobi form of weight 1 , index $(k, -k)$

$$\widehat{f}_u^{(k)}(\tau+1, z) = \widehat{f}_u^{(k)}(\tau, z)$$

$$\widehat{f}_{\frac{u}{\tau}}^{(k)}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \tau e^{2\pi i k \frac{z^2-u^2}{\tau}}\, \widehat{f}_u^{(k)}(\tau, z)$$

$$\widehat{f}_u^{(k)}(\tau, z+m\tau) = q^{-km^2} y^{-2km}\, \widehat{f}_u^{(k)}(\tau, z)$$

……

Modular Completions

We note :

- The function $\widehat{f}_u^{(k)}(\tau, z)$ naturally appears in the path-integral evaluation of the elliptic genus of SL(2)/U(1). [Troost 2010]

- The modular completion $\widehat{\chi}_{\mathbf{dis}}(v, a; \tau, z)$ was defined as its 'Fourier transform'. They are naturally read off from the torus partition function as well as the elliptic genus.
 [Eguchi-Y.S 2010]

Modular Completions

explicitly written as

$$\widehat{\chi}_{\mathbf{dis}}(v,a;\tau,z) := \frac{1}{N}\sum_{b\in\mathbb{Z}_N} e^{-2\pi i\frac{vb}{N}} q^{\frac{K}{N}a^2} y^{\frac{2K}{N}a}\, \widehat{f}_0^{(2NK)}\left(\tau, \frac{z+a\tau+b}{N}\right)\frac{\theta_1(\tau,z)}{i\eta(\tau)^3}$$

$$\equiv \chi_{\mathbf{dis}}(v,a;\tau,z) - \frac{1}{2}\sum_{j\in\mathbb{Z}_{2K}} R_{v+Nj,NK}(\tau)\Theta_{v+Nj+2Ka,NK}\left(\tau,\frac{2z}{N}\right)\frac{\theta_1(\tau,z)}{i\eta(\tau)^3}$$

$$\equiv \frac{\theta_1(\tau,z)}{2\pi\eta(\tau)^3}\sum_{n,r\in\mathbb{Z}}\left\{\int_{\mathbb{R}+i(N-0)} dp - \int_{\mathbb{R}-i0} dp\,\left(yq^{Nn+a}\right)\right\} \times \frac{e^{-\pi\tau_2\frac{p^2+(v+Nr)^2}{NK}}}{p-i(v+Nr)}\frac{\left(yq^{Nn+a}\right)^{r+\frac{v}{N}}}{1-yq^{Nn+a}}\, y^{2K\left(n+\frac{a}{N}\right)}\, q^{NK\left(n+\frac{a}{N}\right)^2}$$

Natural for path-integral

Modular Completions

We further note :

- 'Twisted elliptic genus' (inclusion of 'u-variable') [Ashok-Troost 2011]

- Calculation of the elliptic genus based on the GLSM [Ashok-Troost 2013], [Murthy 2013], [Ashok-Doroud-2013]

Elliptic Genus of SL(2)/U(1) Supercoset & "Non-holomorphic Eisenstein-like Series"

SL(2)/U(1) Supercoset (Gauged WZW)

non-rational (non-compact) N=2 SCFT with

$$\hat{c}\left(\equiv \frac{c}{3}\right) = 1 + \frac{2}{k}, \quad (k \equiv \kappa - 2 \in \mathbb{R}_{>0})$$

not assumed to be rational

$$\begin{aligned}
S(g, A, \psi^{\pm}, \tilde{\psi}^{\pm}) &= \kappa S_{\mathrm{gWZW}}(g, A) + S_{\psi}(\psi^{\pm}, \tilde{\psi}^{\pm}, A), \\
\kappa S_{\mathrm{gWZW}}(g, A) &= \kappa S_{\mathrm{WZW}}^{SL(2,\mathbb{R})}(g) + \frac{\kappa}{\pi} \int_{\Sigma} d^2 v \left\{ \mathrm{Tr}\left(\frac{\sigma_2}{2} g^{-1} \partial_{\bar{v}} g\right) A_v + \mathrm{Tr}\left(\frac{\sigma_2}{2} \partial_v g g^{-1}\right) A_{\bar{v}} \right. \\
&\quad \left. + \mathrm{Tr}\left(\frac{\sigma_2}{2} g \frac{\sigma_2}{2} g^{-1}\right) A_{\bar{v}} A_v + \frac{1}{2} A_{\bar{v}} A_v \right\}, \\
S_{\mathrm{WZW}}^{SL(2,\mathbb{R})}(g) &= -\frac{1}{8\pi} \int_{\Sigma} d^2 v\, \mathrm{Tr}\left(\partial_\alpha g^{-1} \partial_\alpha g\right) + \frac{i}{12\pi} \int_B \mathrm{Tr}\left((g^{-1} dg)^3\right), \\
S_{\psi}(\psi^{\pm}, \tilde{\psi}^{\pm}, A) &= \frac{1}{2\pi} \int d^2 v \left\{ \psi^+ (\partial_{\bar{v}} + A_{\bar{v}}) \psi^- + \psi^- (\partial_{\bar{v}} - A_{\bar{v}}) \psi^+ \right. \\
&\quad \left. + \tilde{\psi}^+ (\partial_v + A_v) \tilde{\psi}^- + \tilde{\psi}^- (\partial_v - A_v) \tilde{\psi}^+ \right\},
\end{aligned}$$

Revisit to Elliptic Genus of SL(2)/U(1)

Torus partition function
(regularized)

[Eguchi-Y.S. 2010]

regularization

$$Z_{\mathrm{reg}}(\tau, z, \bar{z}; \epsilon) = k e^{-2\pi \frac{k+4}{k\tau_2} z_2^2} \int_{\mathbb{C}} \frac{d^2 u}{\tau_2} \sigma(u, z, \bar{z}; \epsilon) \times \left| \frac{\theta_1\left(\tau, u + \frac{k+2}{k} z\right)}{\theta_1\left(\tau, u + \frac{2}{k} z\right)} \right|^2 e^{-4\pi z_2 \frac{u_2}{\tau_2}} e^{-\frac{\pi k}{\tau_2} |u|^2}$$

bosonic & fermionic determinants

twisted boson
('winding modes')

Revisit to Elliptic Genus of SL(2)/U(1)

Factor of IR-regularization:

$$\sigma(u, z, \bar{z}; \epsilon) := 1 - \sum_{m_1, m_2 \in \mathbb{Z}} e^{-\frac{1}{\epsilon \tau_2}\left\{(s_1+m_1)\tau+(s_2+m_2)+\frac{2z}{k}\right\}\left\{(s_1+m_1)\bar{\tau}+(s_2+m_2)+\frac{2\bar{z}}{k}\right\}}$$

$\Rightarrow$ Removes the singularities of integrand

Revisit to Elliptic Genus of SL(2)/U(1)

Elliptic genus

$$
\begin{aligned}
\mathcal{Z}(\tau, z) &= \lim_{\epsilon \to +0} e^{-\frac{\pi}{2}\frac{\hat{c}}{\tau_2} z^2} Z_{\mathrm{reg}}(\tau, z, \bar{z} = 0; \epsilon) \\
&= \lim_{\epsilon \to +0} k e^{\frac{\pi z^2}{k \tau_2}} \int_{\mathbb{C}} \frac{d^2 u}{\tau_2} \sigma(u, z, 0; \epsilon) \frac{\theta_1\left(\tau, u + \frac{k+2}{k} z\right)}{\theta_1\left(\tau, u + \frac{2}{k} z\right)} e^{2\pi i z \frac{u_2}{\tau_2}} e^{-\frac{\pi k}{\tau_2}|u|^2} \\
&= \lim_{\epsilon \to +0} k e^{\frac{\pi z^2}{k \tau_2}} \sum_{m_1, m_2 \in \mathbb{Z}} \int_0^1 ds_1 \int_0^1 ds_2 \, \sigma(s_1 \tau + s_2, z, 0; \epsilon) \\
&\quad \times \frac{\theta_1\left(\tau, s_1 \tau + s_2 + \frac{k+2}{k} z\right)}{\theta_1\left(\tau, s_1 \tau + s_2 + \frac{2}{k} z\right)} e^{2\pi i z s_1} \underline{e^{-\frac{\pi k}{\tau_2}|(s_1 + m_1)\tau + (s_2 + m_2)|^2}}
\end{aligned}
$$

Origin of complication…

Revisit to Elliptic Genus of SL(2)/U(1)

Useful rewriting : [Y.S 2011]

$$\mathcal{Z}(\tau, z) = \sum_{n_1, n_2 \in \mathbb{Z}} s_{(n_1, n_2)} \cdot \mathcal{Z}^{(\infty)}(\tau, z)$$

Spectral flow operator

" $\mathbb{Z}_\infty$ -orbifold " of cigar ($\cong$ universal cover of trumpet)

(winding modes decouple)

Spectral flow operator

$$s_{(m,n)} \cdot f(\tau, z) := (-1)^{m+n} q^{\frac{\hat{c}}{2} m^2} y^{\hat{c} m} e^{2\pi i \frac{mn}{k}} f(\tau, z + m\tau + n)$$

→ defined with keeping the modular covariance

We can explicitly evaluate $\mathcal{Z}^{(\infty)}(\tau, z)$ as

no winding modes!

$$\mathcal{Z}^{(\infty)}(\tau, z) = e^{-\frac{\pi z^2}{k\tau_2}} \int_{\Sigma} \frac{d^2 u}{\tau_2} \frac{\theta_1(\tau, u+z)}{\theta_1(\tau, u)} e^{2\pi i z \frac{u_2}{\tau_2}}$$

$$= e^{-\frac{\pi z^2}{k\tau_2}} \frac{\theta_1(\tau, z)}{2\pi z \eta(\tau)^3}$$

Namely,

$$\mathcal{Z}(\tau, z) = \sum_{m,n\in\mathbb{Z}} s_{(m,n)} \cdot \left[e^{-\frac{\pi z^2}{k\tau_2}} \frac{\theta_1(\tau, z)}{2\pi z \eta(\tau)^3}\right]$$

$$\equiv \frac{\theta_1(\tau, z)}{2\pi\eta(\tau)^3} \sum_{\lambda\in\Lambda} \frac{\rho_{1/k}(\lambda, z)}{z+\lambda}$$

~ "non-holomorphic Eisenstein-like series"

$$\left(\rho_\kappa(\lambda, z) := e^{-\kappa\frac{\pi}{\tau_2}\{|\lambda|^2 + 2\bar{\lambda}z + z^2\}} , \quad \Lambda := \mathbb{Z}\tau + \mathbb{Z} \right)$$

‘Non-holomorphic Eisenstein Series’

$$\mathcal{Z}(\tau, z) = \frac{\theta_1(\tau, z)}{2\pi\eta(\tau)^3} \sum_{\lambda \in \Lambda} \frac{\rho_{1/k}(\lambda, z)}{z + \lambda}$$

Simplest functional form

Modular and spectral flow properties are manifest. (non-holomorphic Jacobi form)

Another Derivation

$$\mathcal{Z}^{(\infty)}(\tau, z) = \frac{1}{k}\int_0^k d\lambda\, \widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, 0; \tau, z)$$

[Y.S 2011]

$$\begin{aligned}
\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, n; \tau, z) \; &:= \; \frac{\theta_1(\tau,z)}{2\pi\eta(\tau)^3} \sum_{\nu\in\lambda+k\mathbb{Z}} \left\{ \int_{\mathbb{R}+i(k-0)} dp - \int_{\mathbb{R}-i0} dp\; (yq^n) \right\} \\
&\qquad \times \frac{e^{-\pi\tau_2 \frac{p^2+\nu^2}{k}}}{p - i\nu} \frac{(yq^n)^{\frac{\nu}{k}}}{1-yq^n} y^{\frac{2n}{k}} q^{\frac{n^2}{k}} \\
&= \frac{\theta_1(\tau,z)}{i\eta(\tau)^3} \frac{(yq^n)^{\frac{\lambda}{k}}}{1-yq^n} y^{\frac{2n}{k}} q^{\frac{n^2}{k}} \\
&\quad + \frac{\theta_1(\tau,z)}{2\pi\eta(\tau)^3} \sum_{\nu\in\lambda+k\mathbb{Z}} \int_{\mathbb{R}+i(k-0)} dp\, \frac{e^{-\pi\tau_2 \frac{p^2+\nu^2}{k}}\, (yq^n)^{\frac{\nu}{k}}}{p-i\nu} y^{\frac{2n}{k}} q^{\frac{n^2}{k}} \\
&\equiv \mathrm{ch}_{\mathbf{dis}}(\lambda, n; \tau, z) + [\text{non-hol. correction terms}]
\end{aligned}$$

Modular completion of irreducible discrete ch.

Another Derivation

Then, we again achieve the same result :

$$\begin{aligned}
\mathcal{Z}^{(\infty)}(\tau, z) &= \frac{1}{k}\frac{\theta_1(\tau,z)}{i\eta(\tau)^3}\int_0^k d\lambda\, \frac{y^{\frac{\lambda}{k}}}{1-y} \\
&\quad + \frac{1}{k}\frac{\theta_1(\tau,z)}{2\pi\eta(\tau)^3}\int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} dp\, \frac{y^{\frac{\nu}{k}} e^{-\pi\tau_2\frac{p^2+\nu^2}{k}}}{p-i\nu} \\
&= \frac{\theta_1(\tau,z)}{2\pi z\eta(\tau)^3} + \frac{\theta_1(\tau,z)}{2\pi z\eta(\tau)^3}\left(e^{-\frac{\pi z^2}{k\tau_2}} - 1\right) \\
&= e^{-\frac{\pi z^2}{k\tau_2}}\frac{\theta_1(\tau,z)}{2\pi z\eta(\tau)^3}
\end{aligned}$$

Relation to the previous works

In the case of $k = N/K, \, (\hat{c} = 1 + \frac{2K}{N})$

[Eguchi-Y.S 2010] (see also[Troost 2010], [Ashok-Troost 2013], [Murthy 2013], [Ashok-Doroud-Troost 2013])

'modular completion' of discrete character

$$
\begin{aligned}
\mathcal{Z}(\tau, z) &= \sum_{v \in \mathbb{Z}_N} \sum_{\substack{a \in \mathbb{Z}_N \\ v + Ka \in N\mathbb{Z}}} \widehat{\chi}_{\mathbf{dis}}(v, a; \tau, z) \\
&\equiv \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \frac{1}{N} \sum_{a,b \in \mathbb{Z}_N} q^{\frac{K}{N}a^2} y^{\frac{2K}{N}a} e^{2\pi i \frac{K}{N}ab} \, \widehat{f}^{(NK)}\left(\tau, \frac{z + a\tau + b}{N}\right)
\end{aligned}
$$

Zwegers' function

Relation to the previous works

Combine these formulas with the new calculation presented above.

Modular completions are also expressible in terms of the non-holomorphic Eisenstein series.

We especially obtain a very simple formula :

$$
\begin{aligned}
\widehat{f}^{(k)}(\tau, z) &\equiv \widehat{f}^{(k)}_{u=0}(\tau, z) \\
&= \frac{i}{2\pi} \sum_{\lambda \in \Lambda} \frac{\rho_k(\lambda, z)}{z + \lambda}
\end{aligned}
$$

$$
\rho_\kappa(\lambda, z) := e^{-\kappa \frac{\pi}{\tau_2} \left\{ |\lambda|^2 + 2\bar{\lambda} z + z^2 \right\}}
$$

※R.H.S is well-defined for an arbitrary level $k \in \mathbb{R}_{>0}$

(parametrical extension of the Zwegers' function)

We also obtain

$$\widehat{\chi}_{\mathbf{dis}}(v, a; \tau, z) \;=\; \frac{\theta_1(\tau, z)}{2\pi\eta(\tau)^3} \sum_{b\in\mathbb{Z}_N} \sum_{\lambda\in a\tau+b+N\Lambda} e^{-2\pi i \frac{b}{N}(v+Ka)} \frac{\rho_{\frac{K}{N}}(\lambda, z)}{z+\lambda}$$

Again, modular & spectral flow properties are easily shown based on this formula.

Application :
Gepner-like Orbifolds
for
Non-compact CY

Gepner-like Orbifolds

$$M_{\text{Gepner}} = \bigotimes_{i=1}^{r} M_{k_i} \bigotimes_{j=1}^{r'} L_{N_j,K_j} \Big|_{\mathbb{Z}_N-\text{orbifold}}$$

[Eguchi-Y.S 2004]

Expected to describe a non-compact CY-background

Non-compact

$$\hat{c} = \sum_{i=1}^{r} \frac{k_i}{k_i+2} + \sum_{j=1}^{r'} \left(1 + \frac{2K_j}{N_j}\right) \in \mathbb{Z}_{>0}, \qquad N := \text{LCM}\,\{k_i+2,\ N_j\}$$

$$M_k \cong [\mathcal{N} = 2 \text{ minimal model } (SU(2)/U(1)) \text{ of level } k]$$

$$L_{N,K} \cong [\text{supercoset } SL(2)/U(1) \text{ of level } N/K\ ]$$

Gepner-like Orbifolds

We especially focus on the elliptic genus

- Weak Jacobi form of weight 0, index $\hat{c}/2$ (good modular & spectral flow properties) .
- Stable under marginal deformations.

Elliptic Genus of Each Sector

Elliptic genus of N=2 minimal model

$$Z^{(k)}_{\mathrm{min}}(\tau, z) = \sum_{\ell=0}^{k} \mathrm{ch}^{(\widetilde{\mathrm{R}}),k}_{\ell,\ell+1}(\tau, z) \equiv \frac{\theta_1\left(\tau, \frac{k+1}{k+2} z\right)}{\theta_1\left(\tau, \frac{z}{k+2}\right)}.$$

[Witten 93, Henningson 93]

Elliptic Genus of Each Sector

Elliptic genus of SL(2)/U(1) model

[Troost 2010, Eguchi-Y.S 2010]

As we observed above, it is rewritten in the simple form :

$$\mathcal{Z}(\tau, z) = \frac{\theta_1(\tau, z)}{2\pi\eta(\tau)^3} \sum_{\lambda \in \Lambda} \frac{\rho_{1/k}(\lambda, z)}{z + \lambda}$$

Probably, easier to calculate

Gepner-like Orbifolds

$Z_{\text{Gepner}}(\tau, z)$?

reconsidered based on the
modular completions

(closely related work [Ashok-Troost 2012]

calculable in principle
(as in the compact Gepner models [EOTY 89, KYY 93])

'character expansion' looks difficult…
(due to the non-holomorphic corrections)

What is universal functional form ?

Odd Dimensional Non-compact CY

$$\hat{c} = 2L + 3$$

$$Z_{\text{Gepner}}(\tau, z) \quad = \quad \frac{1}{2}\frac{\vartheta(\tau, 2z)}{\vartheta(\tau, z)}\,\widetilde{Z}_{\text{Gepner}}(\tau, z)$$

Elliptic genus in the case of $\hat{c} = 2L$

(~ 'non-holomorphic version of Gritsenko's theorem')

Odd Dimensional Non-compact CY

It is enough to only consider
the even $\hat{c}$ cases

Note : Z_{Gepner} is holomorphic in the case of $\hat{c} = 3$

(just same form as elliptic genera of compact CY3)

Even Dimensional Non-compact CY

$$\hat{c} = 2L$$

A reasonable ansatz (not based on ch. expansion) :

$$\mathcal{Z}_{\text{Gepner}}(\tau, z) = \vartheta(\tau, z)^{2L} \left[\chi \wp(\tau, z)^{L} + \sum_{s=1}^{L} f_s(\tau) \wp(\tau, z)^{L-s} \right]$$

Non-holomorphic modular form of weight 2s

$$\vartheta(\tau, z) := \frac{\theta_1(\tau, z)}{\partial_z \theta_1(\tau, 0)} \equiv \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3}$$

Even Dimensional Non-compact CY

How to compute $f_s(\tau)$?

- Compute with keeping the properties as weak Jacobi form manifest.
- Make use of the previous formulas of non-holomorphic Eisenstein series for the modular completions.
- Holomorphic contributions yield the Eisenstein series in the usual sense.

$f_s(\tau)$ is again schematically expressible as the non-holomorphic Eisenstein-like series ;

$$f_s(\tau) \;=\; \text{linear combination of}$$

$$\sum_{(\lambda_1,\ldots,\lambda_s)\in\Lambda}{}' \prod_{i=1}^{2s} \left[\frac{1}{\lambda_i} P_i \left(\frac{\pi}{\tau_2} |\lambda_i|^2 \right) e^{-\kappa_i \frac{\pi}{\tau_2} |\lambda_i|^2} \right]$$

Good modular behavior

$$\Lambda \;:\; SL(2,\mathbb{Z})\text{-inv. sub-lattice of } (\mathbb{Z}\tau \oplus \mathbb{Z})^s$$

$$\kappa_i \in \left\{ +0,\ \frac{K_1}{N_1}, \frac{K_2}{N_2} \ldots \right\} \text{(associated to } L_{N_1,K_1} \otimes L_{N_2,K_2} \otimes \cdots \text{)}$$

An Example :

Simplest case : $\mathrm{ALE}(A_{N-1})$

$$[M_{N-2} \otimes L_{N,1}]|_{\mathbb{Z}_N-\text{orbifold}}$$

$$\begin{aligned}
\mathcal{Z}_{\mathrm{ALE}(A_{N-1})}(\tau, z) &= \frac{1}{N} \sum_{a,b \in \mathbb{Z}_N} \mathcal{Z}^{(N-2)}_{[a,b]}(\tau, z)\, \mathcal{Z}^{(N,1)}_{[a,b]}(\tau, z) \\
&\equiv \vartheta(\tau, z)^2 \left[(N-1)\wp(\tau, z) + f_1^{(N)}(\tau) \right],
\end{aligned}$$

An Example :

$$f_1^{(N)}(\tau) \;=\; N\sum_{\lambda_1\in\Lambda}{}' \sum_{\lambda_2\in\lambda_1+N\Lambda} \frac{\rho_{+0}(\lambda_1)\rho_{\frac{1}{N}}(\lambda_2)}{\lambda_1\lambda_2} + \sum_{\lambda\in\Lambda}{}' \frac{\rho_{\frac{1}{N}}(\lambda)}{\lambda^2}\left\{1+\frac{2\pi}{N}\frac{|\lambda|^2}{\tau_2}\right\}$$

$$\rho_\kappa(\lambda) \;:=\; \rho_\kappa(\lambda,0) \equiv e^{-\kappa\frac{\pi}{\tau_2}|\lambda|^2}$$

Note :

$$\sum_{\lambda\in\Lambda} \rho_{+0}(\lambda) F(\lambda) := \lim_{\epsilon\to+0} \sum_{\lambda\in\Lambda} \rho_{\epsilon}(\lambda) F(\lambda)$$

$$\sum_{\lambda\in\Lambda}{}' \frac{\rho_{+0}(\lambda)}{\lambda^2} = \widehat{G}_2(\tau) \equiv G_2(\tau) - \frac{\pi}{\tau_2}$$

‘modular completion’ of $G_2(\tau)$

Summary

Summary

Modular completions

$$\widehat{f}_u^{(k)}(\tau, z) := f_u^{(k)}(\tau, z) - \frac{1}{2} \sum_{m \in \mathbb{Z}_{2k}} R_{m,k}(\tau, u) \Theta_{m,k}(\tau, 2z)$$

$$\widehat{\chi}_{\mathbf{dis}}(v, a; \tau, z) = \chi_{\mathbf{dis}}(v, a; \tau, z) + [\text{non-hol. correction terms}]$$

$\cdots$

→ expressible in terms of the
'non-holomorphic Eisenstein series'

Summary

In other words,

Eisenstein-like series with a gaussian damping factor

- Modular and spectral flow properties are manifest.
- Expect to play complementary roles to the approach of representation theory.

Summary

Elliptic genera of the non-compact
Gepner-like orbifolds

(based on the modular completion)

- The 'character expansion' is very complicated.
- A simpler expression is achieved by means of the 'non-holomorphic Eisenstein series' (except for CY3 case).

Thank you very much for your attention!