Generalized Mathieu Moonshine and Siegel Modular Forms

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Mock Modular Forms and Physics

IMSc, Chennai, April 18, 2014

Talk based on:

[arXiv:1312.0622] (w/ R. Volpato)

[arXiv:1302.5425] (w/ M. Gaberdiel, & R. Volpato)

[arXiv:1211.7074] (w/ M. Gaberdiel, H. Ronellenfitsch, R. Volpato)

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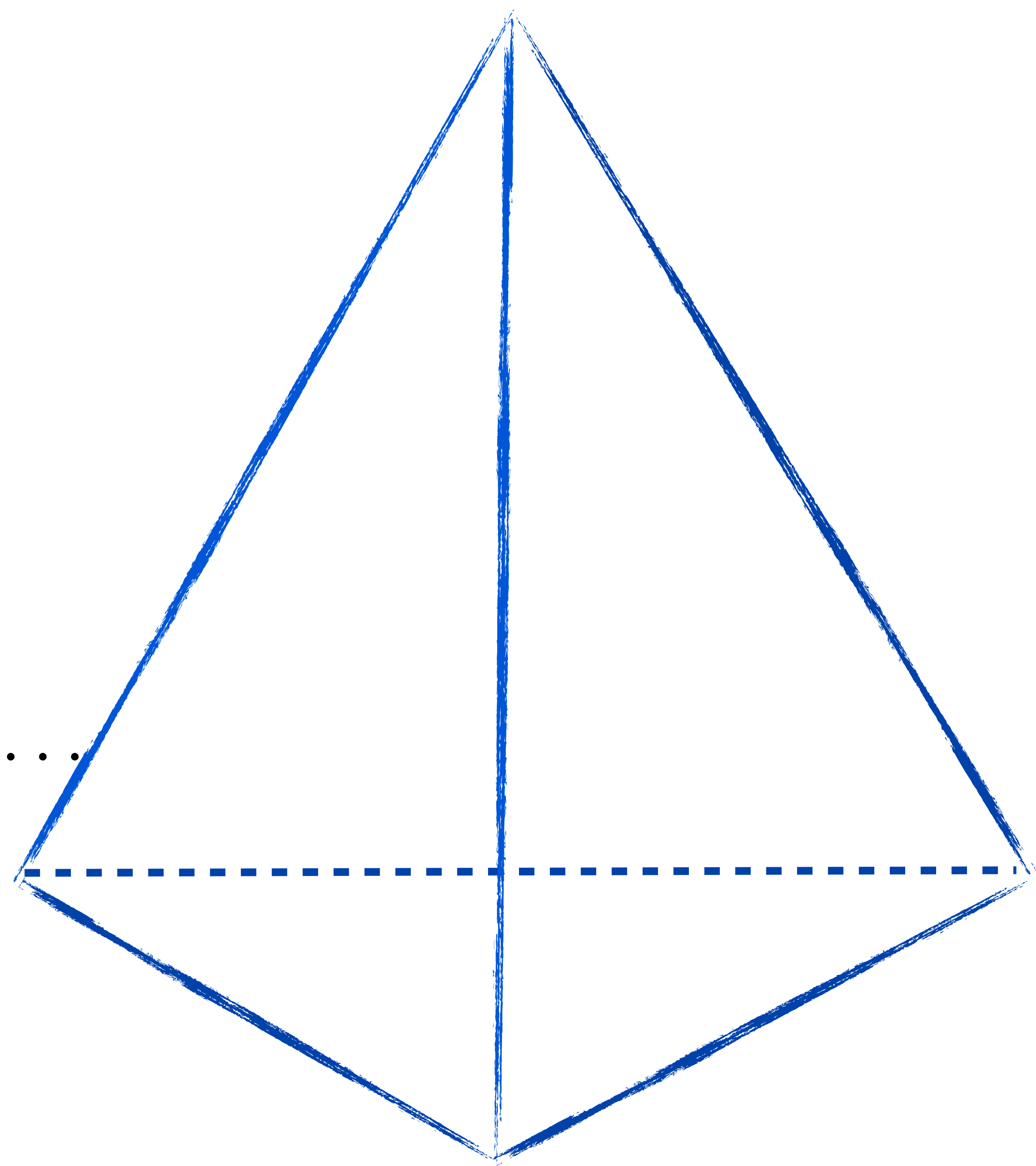
The most famous example is **Monstrous Moonshine**.

Monstrous Moonshine

monster group

$\mathbb{M}$



$J(\tau) = q^{-1} + 196884q + \cdots$

modular function

bosonic string theory on

$(T^{24}/\Lambda_{\text{Leech}})/\mathbb{Z}_2)$

(holomorphic VOA $V^\natural$)

monster Lie algebra $\mathfrak{m}$

(Figure stolen from Jeff's talk!)

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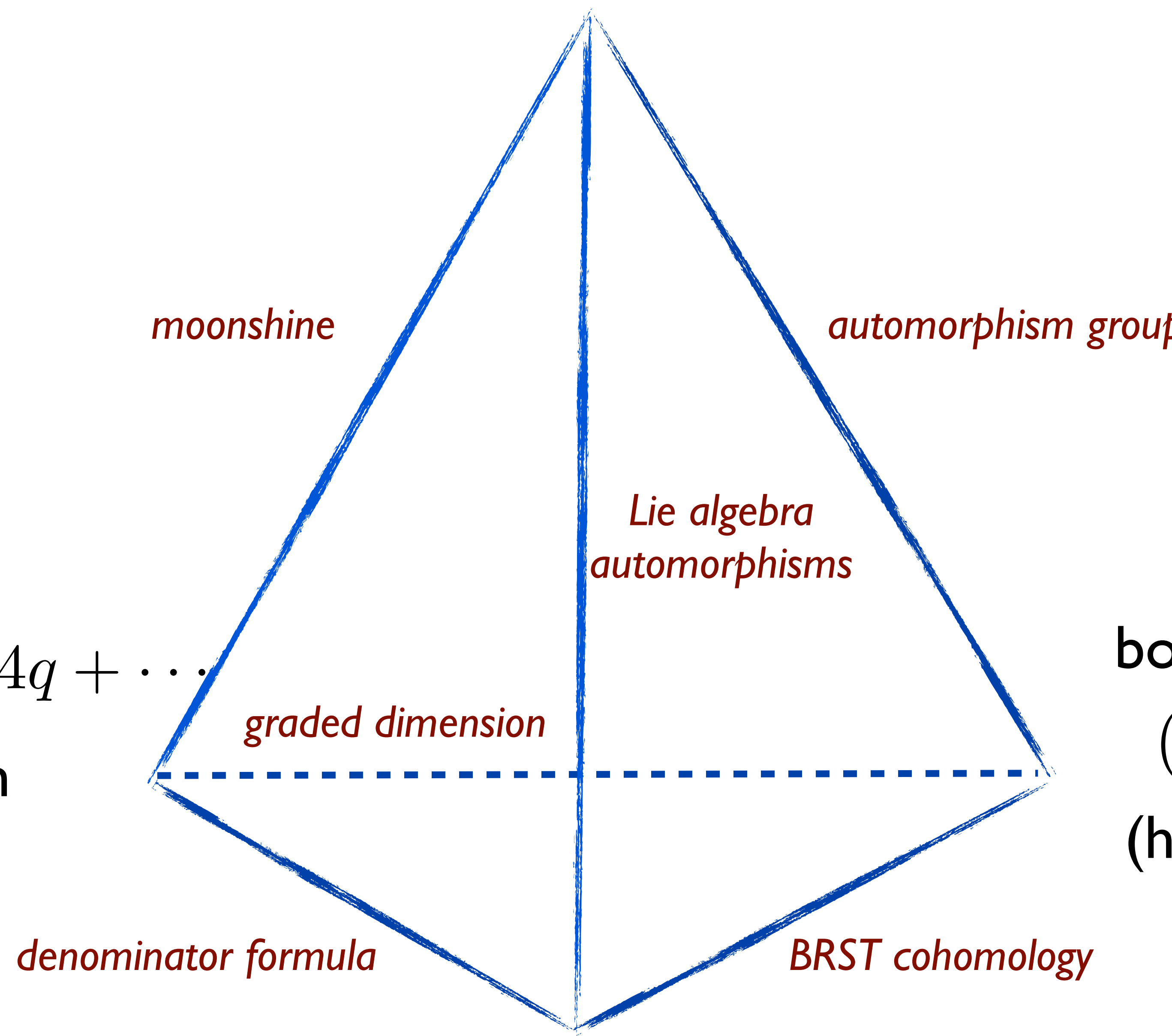

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In 2010, Eguchi, Ooguri, Tachikawa conjectured that there is Moonshine in the elliptic genus of K3 connected to the finite sporadic group $M_{24} \subset S_{24}$

EOT observation: Fourier coefficients of K3-elliptic genus are (sums of) dimensions of irreps of M_{24}

A completely new moonshine phenomenon to explore!

Monstrous Moonshine	Mathieu Moonshine
monster group $\mathbb{M}$	Mathieu group M_{24}
bosonic CFT	superconformal field theory
Virasoro algebra	$\mathcal{N}=(4,4)$ superconformal algebra
J-function	elliptic genus of K3
McKay-Thompson series	twining genera
monster module $V^\natural$	?
monster Lie algebra $\mathfrak{m}$	?

[Eguchi, Ooguri, Tachikawa][Cheng][Gaberdiel, Hohenegger, Volpato]
[Eguchi, Hikami][Taormina, Wendland][Gannon]

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What does M_{24} act on?

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We have considered a "two-step generalization" of Mathieu moonshine that sheds light on this question.

Mathieu moonshine $\longrightarrow$ **generalized Mathieu moonshine** $\longrightarrow$ **second-quantized Mathieu moonshine**

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Mathieu moonshine → **generalized Mathieu moonshine** → **second-quantized Mathieu moonshine**

Roberto's talk

This talk

Outline

1. Recap: Generalized Mathieu moonshine

2. Second quantization & black hole counting

3. Second quantization of generalized Mathieu moonshine

4. Connection with umbral moonshine

5. Summary and outlook

1. Generalized Mathieu moonshine

Generalized Mathieu Moonshine

[Gaberdiel, D.P., Ronellenfitsch, Volpato]

Introduce a family of functions, the ***twisted twining genera:***

$$\phi_{g,h} \; : \; \mathbb{H} \times \mathbb{C} \to \mathbb{C}$$

for each commuting pair

$$g, h \in M_{24}$$

such that for $g = e$ we recover the twining genera $\phi_{e,h} = \phi_h$

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This is the analogue of Norton's **generalized monstrous moonshine**

$$Z_{g,h} \; : \; \mathbb{H} \to \mathbb{C} \qquad g, h \in \mathbb{M}$$

$$Z_{e,h}(\tau) = T_h(\tau)$$

McKay-Thompson series

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Partially explained by orbifolds of the FLM monster VOA $V^\natural$.

[Dixon, Ginsparg, Harvey]
[Tuite]

Proven in special cases but the full conjecture still open.

[Dong, Li, Mason][Höhn][Tuite]
[Carnahan]

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Can we also interpret generalized Mathieu moonshine in terms of orbifolds?

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→ multiplier phases of characters $Z_{g,h}(\tau)$ determined by 2-cocycle

$$c_g \in H^2(C_G(g), U(1))$$

obtained from a class $[\alpha] \in H^3(G, U(1))$ via

$$c_h(g_1, g_2) = \frac{\alpha(h, g_1, g_2)\alpha(g_1, g_2, (g_1 g_2)^{-1} h (g_1 g_2))}{\alpha(g_1, h, h^{-1} g_2 h)}$$

In particular, for the S- and T-transformations we have

[Bantay][Coste, Gannon, Ruelle]

$$Z_{g,h}(\tau+1) = c_g(g,h) Z_{g,gh}(\tau)$$

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Moreover, under conjugation of g, h one has the general relation

$$Z_{g,h}(\tau) = \frac{c_g(h,k)}{c_g(k, k^{-1}hk)} Z_{k^{-1}gk, k^{-1}hk}(\tau) \qquad \forall k \in G$$

Cohomological Obstructions from $H^3(G)$

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So $Z_{g,h} = 0$ unless the 2-cocycle c_g is *regular*:

$$c_g(h,k) = c_g(k,h)$$

When this is not satisfied we have **obstructions**! [Gannon]

Conjecture (generalized Mathieu moonshine) [GHPV]:

For each $g \in M_{24}$ *there exists a graded unitary representation* $\mathcal{H}_g$ *of* $\mathcal{N} = 4$

with central charge $c = 6$ *carrying a projective representation*

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$R_{g,r}$ representation of a central extension of $C_{M_{24}}(g)$

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- The phases $\xi_{g,h}$, $\chi_{g,h}$ and the central extension of $C_{M_{24}}(g)$ are determined by the same class $[\alpha] \in H^3(M_{24}, U(1))$

Example: $8A$ -**twist** and $2B$ -**twine**:

$$\phi_{8A,2B}(\tau,z)=\frac{\eta\left(\frac{\tau}{2}\right)^6}{\eta(\tau)^6}\frac{\vartheta_1(\tau,z)^2}{\vartheta_4(\tau,0)^2}$$

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$\phi_{8A,2B}(\tau,z)$ is a Jacobi form of weight 0 index 1 for the group

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Multiplier given by:

$$\phi_{8A,2B}(\tau+4,z)=\frac{\prod_{i=0}^{3}c_g(g,g^ih)}{c_{g^4h}(g,g^{-1})c_{g^{-1}}(g^4h,g^4h)}\frac{c_{g^{-1}}(g^4h,k)}{c_{g^{-1}}(k,h)}\phi_{8A,2B}(\tau)=-\phi_{8A,2B}(\tau)$$

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using our result for $c_{g_1}(g_2,g_3)$ in terms of $\alpha \in H^3(M_{24},U(1))$

Theorem [GHPV]:

- *For each commuting pair* $g, h \in M_{24}$ *there exists functions* $\phi_{g,h}(\tau, z)$ *satisfying all the expected modular properties with respect to subgroups* $\Gamma_{g,h} \subset SL(2,\mathbb{Z})$
- *There is a unique class* $[\alpha] \in H^3(M_{24}, U(1))$ *which determines all the modular phases.*
- *Many of the* $\phi_{g,h}$ *vanish due to cohomological obstructions controlled by* $H^3(M_{24}, U(1))$

(in deriving these results we use the fact that $H^3(M_{24}, U(1)) \cong \mathbb{Z}_{12}$ [Ellis, Dutour-Sikiric] *)*

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This was verified for the first 500 coefficients.

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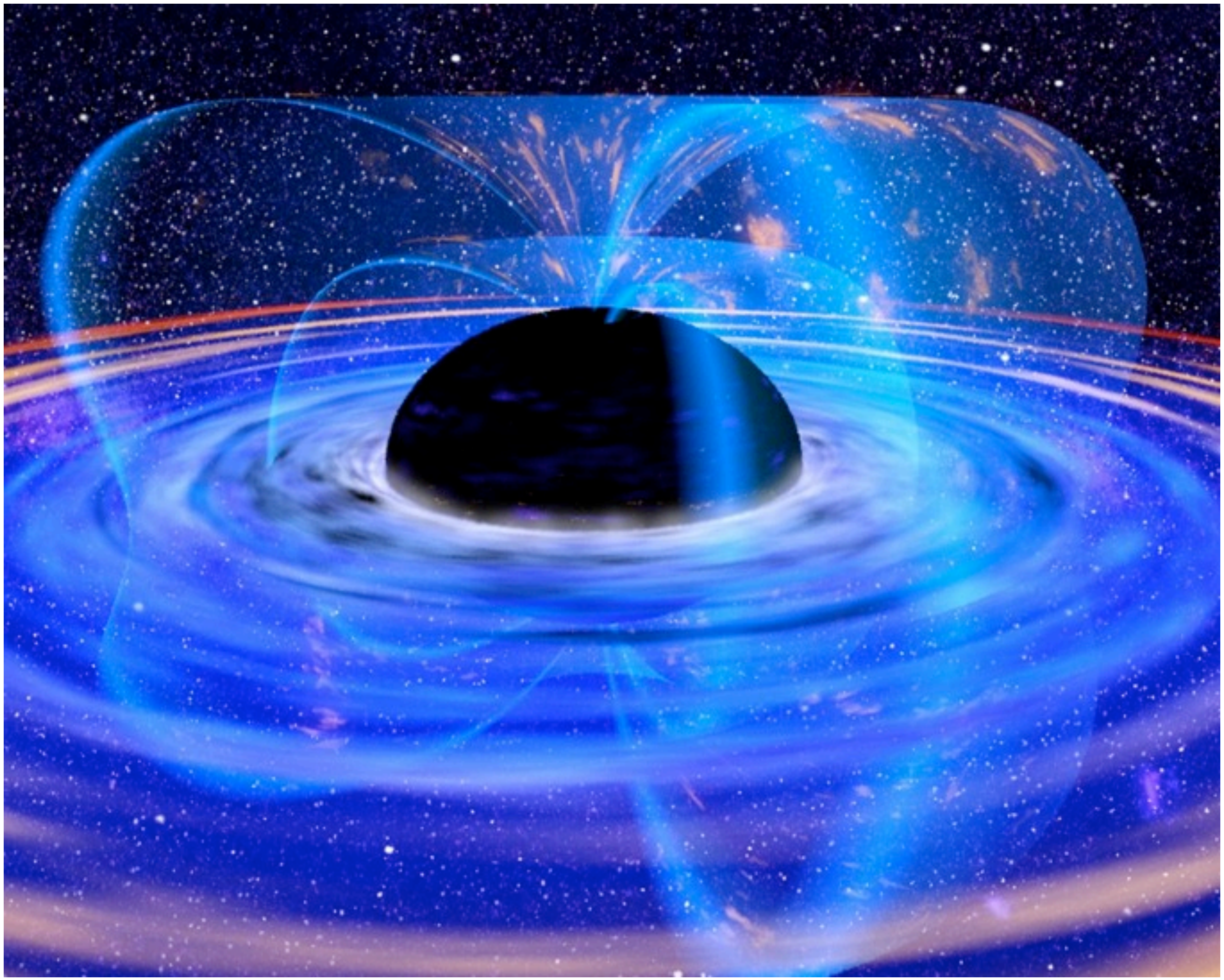
This is very strong evidence that generalized Mathieu Moonshine holds!

But what is the physical interpretation?

2. Second quantization & black hole counting



Second quantized elliptic genus

Let X be a Calabi-Yau manifold and $\chi(X;\tau,z)$ its elliptic genus.

$\longrightarrow$ $\chi(X;\tau,z)$ is a weak Jacobi form of weight zero and index $(\dim_{\mathbb{C}} X)/2$ [Gritsenko]

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This is the generating function of elliptic genera of symmetric products of X

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DMVV proved the following remarkable formula:

$$\Psi_X(\sigma,\tau,z) = \exp\left[\sum_{L=1}^{\infty} p^L T_L \chi(X;\tau,z)\right] = \prod_{\substack{n>0,m\geq 0\\ \ell\in\mathbb{Z}}} (1-p^n q^m y^\ell)^{-c_X(mn,\ell)}$$

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Hecke operator

$$T_L : J_{0,m} \to J_{0,mL}$$

Fourier coefficients of

$$\chi(X;\tau,z) = \sum_{k\geq 0,\ell\in\mathbb{Z}} c_X(k,\ell) q^k y^\ell$$

Second quantized elliptic genus

Gritsenko later showed that

$$\Phi_X(\sigma,\tau,z) := \frac{A_X(\sigma,\tau,z)}{\Psi_X(\sigma,\tau,z)}$$

is a Siegel modular form of weight $c_X(0,0)/2$

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is a Siegel modular form of weight $c_X(0,0)/2$

A_X is called the "Hodge anomaly"; only depends on the Hodge numbers of X

This is an example of a (multiplicative) **Borcherds lift:**

$$\Phi \ : \ \boxed{\begin{array}{c}\text{Jacobi}\\ SL(2,\mathbb{Z})\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{automorphic}\\ SO(3,2;\mathbb{Z})\end{array}}$$

Second quantized elliptic genus

For X a K3-manifold we have that

$$\Phi_X = \Phi_{10} = pqy \prod_{m,n,\ell)>0} (1 - p^m q^n y^\ell)^{c(mn,\ell)}$$

Igusa cusp form of weight 10 for

$$Sp(4;\mathbb{Z})$$

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This is a multiplicative Borcherds lift of the **K3 elliptic genus**

$$\chi(K3;\tau,z) = 2\phi_{0,1}(\tau,z) = \sum_{n\geq 0,\ell\in\mathbb{Z}} c(n,\ell) q^n y^\ell$$

The *inverse is the partition function* of 1/4 BPS dyons in Het/T^6 or $\mathrm{IIA}/(K3\times T^2)$

[Dijkgraaf, Verlinde, Verlinde][Shih, Strominger, Yin]

Counting dyons in $\mathcal{N}=4$ string theory

Large **moduli space** of such theories:

$$\mathcal{M} = O(6,22;\mathbb{Z})\backslash O(6,22;\mathbb{R})/(O(6)\times O(22))$$

The discrete duality group preserved the lattice of electric-magnetic charges:

$$(P,Q) \in \Gamma^{6,22} \oplus \Gamma^{6,22}$$

The full **non-perturbative duality group** is

$$SL(2,\mathbb{Z}) \times O(6,22;\mathbb{Z})$$

(P,Q) transform as a **doublet** under $SL(2,\mathbb{Z})$

Hilbert space of states decomposes as

$$\mathcal{H} = \bigotimes_{(P,Q)\in\Gamma^{6,22}\oplus\Gamma^{6,22}} \mathcal{H}_{Q,P}$$

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We are interested in **BPS-states:**

$\longrightarrow$ **1/2 BPS:** Purely electric $(0,Q)$ or magnetic $(P,0)$

$\longrightarrow$ **1/4 BPS (generic):** Dyonic (Q,P)

1/2 BPS-states are counted by [Dabholkar, Harvey]

$$\frac{1}{\eta(\tau)^{24}} = \sum_{n\in\mathbb{Z}} d(n) q^n$$

→ $d(n) =$ number of 1/2 BPS-states with charge Q such that $n = Q^2/2$

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In general, **1/4 BPS states** are counted by the 6th helicity supertrace [Kiritsis]

$$B_6(P,Q) := \frac{1}{6!}\mathrm{Tr}_{\mathcal{H}_{P,Q}}\left((-1)^J (2J)^6\right) \qquad J \text{ = helicity}$$

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J = helicity

- invariant under $SL(2,\mathbb{Z}) \times SO(6,22;\mathbb{Z})$ $\longrightarrow$ can only depend on the combinations $P^2, Q^2, Q \cdot P$
- locally constant on $\mathcal{M}$

Generating function: $(q := e^{2\pi i\tau}, y := e^{2\pi i z}, p := e^{2\pi i\sigma})$

$$\frac{1}{\Phi_{10}(\sigma,\tau,z)} = \sum_{m,n,\ell} d(m,n,\ell)p^m q^n y^\ell$$

with the identification

$$\boxed{B_6(P,Q) = d\left(\frac{Q^2}{2}, \frac{P^2}{2}, P\cdot Q\right)}$$

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Φ_{10} has a double pole at $z=0$. In the limit, we have a factorization

$$\underbrace{\lim_{z\to 0}\frac{\Phi_{10}(\sigma,\tau,z)}{(2\pi i z)^2}}_{\text{1/4-BPS}} = \underbrace{\eta(\sigma)^{24}}_{\text{1/2-BPS}}\,\underbrace{\eta(\tau)^{24}}_{\text{1/2-BPS}}$$

“wall-crossing formula”

3. Second quantization of generalized Mathieu moonshine

Second quantized twisted twining genera

Inspired by the aforementioned results we seek a similar **spacetime interpretation** for the twisted twining genera $\phi_{g,h}(\tau, z)$ of generalized Mathieu moonshine.

This generalizes earlier results by Cheng and Govindarajan.

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We define the ***second quantized twisted twining genus*** as:

$$\Psi_{g,h}(\sigma, \tau, z) := \exp\left[\sum_{L=1}^{\infty} p^L \mathcal{T}_L^{\alpha} \phi_{g,h}(\tau, z)\right]$$

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Note that this depends on the choice of 3-cocycle $\alpha \in H^3(M_{24}, U(1))$ but different representatives in each class $[\alpha]$ simply amounts to a shift of σ

Twisted equivariant Hecke operators

Geometric interpretation following Ganter. Let

$$\mathcal{M}_{M_{24}} = \mathcal{P} \times (\mathbb{H}_+ \times \mathbb{C}) \,/\, M_{24} \times (SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2)$$

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Explicitly one can represent this action by

$$\mathcal{T}_L^{\alpha}\phi_{g,h}(\tau,z) := \frac{1}{L}\sum_{\substack{a,d>0\\ ad=L}}\sum_{b=0}^{d-1}\chi_{g,h}\left(\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\right)\phi_{g^d,g^{-b},h^a}\left(\frac{a\tau+b}{d},az\right)$$

This is a generalization of similar Hecke operators used in generalized monstrous moonshine by Ganter & Carnahan. (see also [Tuite][Govindarajan])

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multiplier phase determined by

$[\alpha] \in H^3(M_{24}, U(1))$

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Example: for $g, h \in 2B$ we have

$$\mathcal{T}_2^{\alpha}\phi_{g,h}(\tau, z) = \frac{1}{2}\Big[- \phi_{g,e}(2\tau, 2z) + \phi_{e,h}(\tfrac{\tau}{2}, z) - \phi_{e,gh}(\tfrac{\tau+1}{2}, z)\Big]$$

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signs come from the multiplier system $\chi_{g,h}$

On the other hand, for $g, h \in 2B$ we in fact have

$$\phi_{g,h}(\tau, z) = 0$$

by **cohomological obstructions** from $H^3(M_{24}, U(1))$

Since

$$\mathcal{T}_L^\alpha \; : \; \mathcal{L}^\alpha_{g,h} \longrightarrow (\mathcal{L}^\alpha_{g,h})^{\otimes L}$$

This implies that for L sufficiently large $\mathcal{T}_L^\alpha \phi_{g,h}$ has **trivial multiplier phase**

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This implies that for L sufficiently large $\mathcal{T}_L^{\alpha}\phi_{g,h}$ has **trivial multiplier phase**

Even if $\phi_{g,h}$ vanishes by cohomological obstructions, all the second quantized twisted twining genera $\Psi_{g,h}$ are unobstructed!

$$\Psi_{g,h}(\sigma,\tau,z) := \exp\left[\sum_{L=1}^{\infty} p^L \mathcal{T}_L^{\alpha}\phi_{g,h}(\tau,z)\right]$$

Theorem (D.P., Volpato):

The second quantized twisted twining genera satisfy the following properties

- *Infinite product formula*

$$\frac{1}{\Psi_{g,h}(\sigma,\tau,z)} = \prod_{d=1}^{\infty}\prod_{m=0}^{\infty}\prod_{\ell\in\mathbb{Z}}\prod_{t=0}^{M-1}(1-e^{\frac{2\pi i t}{M}}q^{\frac{m}{N\lambda}}y^{\ell}p^{d})^{\hat{c}_{g,h}(d,m,\ell,t)}$$

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$$M=\mathcal{O}(h) \qquad N=\mathcal{O}(g)$$

λ *length of the shortest cycle of* g *in its 24-dim permutation reps*

$$\hat{c}_{g,h}(d,m,\ell,t) := \sum_{k=0}^{M-1}\sum_{b=0}^{\lambda N-1}\frac{e^{-\frac{2\pi i t k}{M}}}{M}\frac{e^{\frac{2\pi i b m}{\lambda N}}}{\lambda N}\chi_{g,h}\left(\begin{smallmatrix}k & b\\ 0 & d\end{smallmatrix}\right)c_{g^d,g^{-b}h^k}(\frac{md}{N\lambda},\ell)$$

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- *The ratio*

$$\Phi_{g,h}(\sigma,\tau,z):=\frac{A_{g,h}(\sigma,\tau,z)}{\Psi_{g,h}(\sigma,\tau,z)}$$

is a Siegel modular form for a subgroup $\Gamma^{(2)}_{g,h}\subset Sp(4;\mathbb{R})$

For $g=e$ *this was conjectured by Cheng and partially proven by Raum.*

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$$A_{g,h} = -p\frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6}\eta_{g,h}(\tau)$$

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- *"Wall-crossing formula"*

$$\lim_{z\to 0}\frac{\Phi_{g,h}(\sigma,\tau,z)}{(2\pi i z)^2} = \eta_{g,h}(\tau)\eta_{g,h}(N\lambda\sigma)$$

Automorphy of $\Phi_{g,h}$ follow from

- **"Electric-magnetic duality"**

$$\Phi_{g,h}(\sigma,\tau,z) = \Phi_{g,h'}(\frac{\tau}{N\lambda}, N\lambda\sigma, z)$$

where h' is not necessarily in the same conjugacy class $[h]$

This generalizes the electric-magnetic duality in Φ_{10} [Dijkgraaf, Verlinde, Verlinde]

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This generalizes the electric-magnetic duality in Φ_{10} [Dijkgraaf, Verlinde, Verlinde]

- Using results of Gritsenko-Nikulin, one also has invariance under (an extension of) the **para-modular group**

$$\Gamma_t(N) = \{ \begin{pmatrix} * & t* & * & * \\ * & * & * & t^{-1}* \\ N* & Nt* & * & * \\ Nt* & Nt* & t* & * \end{pmatrix} \in Sp(4,\mathbb{Q}), \ * \in \mathbb{Z}\}$$

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We can therefore view this our construction as a **twisted equivariant** generalization of a multiplicative Borcherds lift

$$\mathbf{Mult}_G[\phi_{g,h}] := A_{g,h}(\sigma,\tau,z)\exp\left[-\sum_{L=1}^{\infty} p^L \mathcal{T}_L^{\alpha}\phi_{g,h}(\tau,z)\right]$$

This resolves a puzzle about the connection with Mason's old version of generalized M_{24} -moonshine for eta-products

(For $g = e$ this was observed previously by Cheng and Govindarajan.)

twisted twining genera
(weak Jacobi forms)

$$\phi_{g,h}(\tau, z)$$

twisted equivariant multiplicative lift

("second quantization")

second-quantized twisted twining genera
(Siegel modular forms)

$$\Phi_{g,h}(\sigma, \tau, z)$$

$$z \to 0$$

("wall-crossing")

generalized eta-products
(modular forms)

$$\eta_{g,h}(\tau)\eta_{g,h'}(N\lambda\sigma)$$

Physical interpretation: CHL-models

Can we interpret the second quantized twisted twining genera as counting spacetime BPS-states?

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Suppose (g, h) are commuting symmetries of the internal superconformal CFT of type $\mathrm{II}/(K3\times T^2)$ or Het/T^6

- Consider the orbifold of this theory by g $\longrightarrow$

new $\mathcal{N}=4$ theory

"CHL-model"

[Chaudhuri, Hockney, Lykken]

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"CHL-model"

[Chaudhuri, Hockney, Lykken]

In this orbifold theory we have "twisted" dyon states counted by the twisted BPS-index

$$B_{6;g,h}(P,Q) := \frac{1}{6!}\mathrm{Tr}_{\mathcal{H}^g_{Q,P}}(h(-1)^{2J}(2J)^6)$$ [Sen]

Computed for some pairs of symmetries [Dabholkar, Gaiotto][Dabholkar, Nampuri][Jatkar, Sen][David][Dabholkar, Cheng][Govindarajan][Sen]...

Expanding the second quantized twisted twining genera

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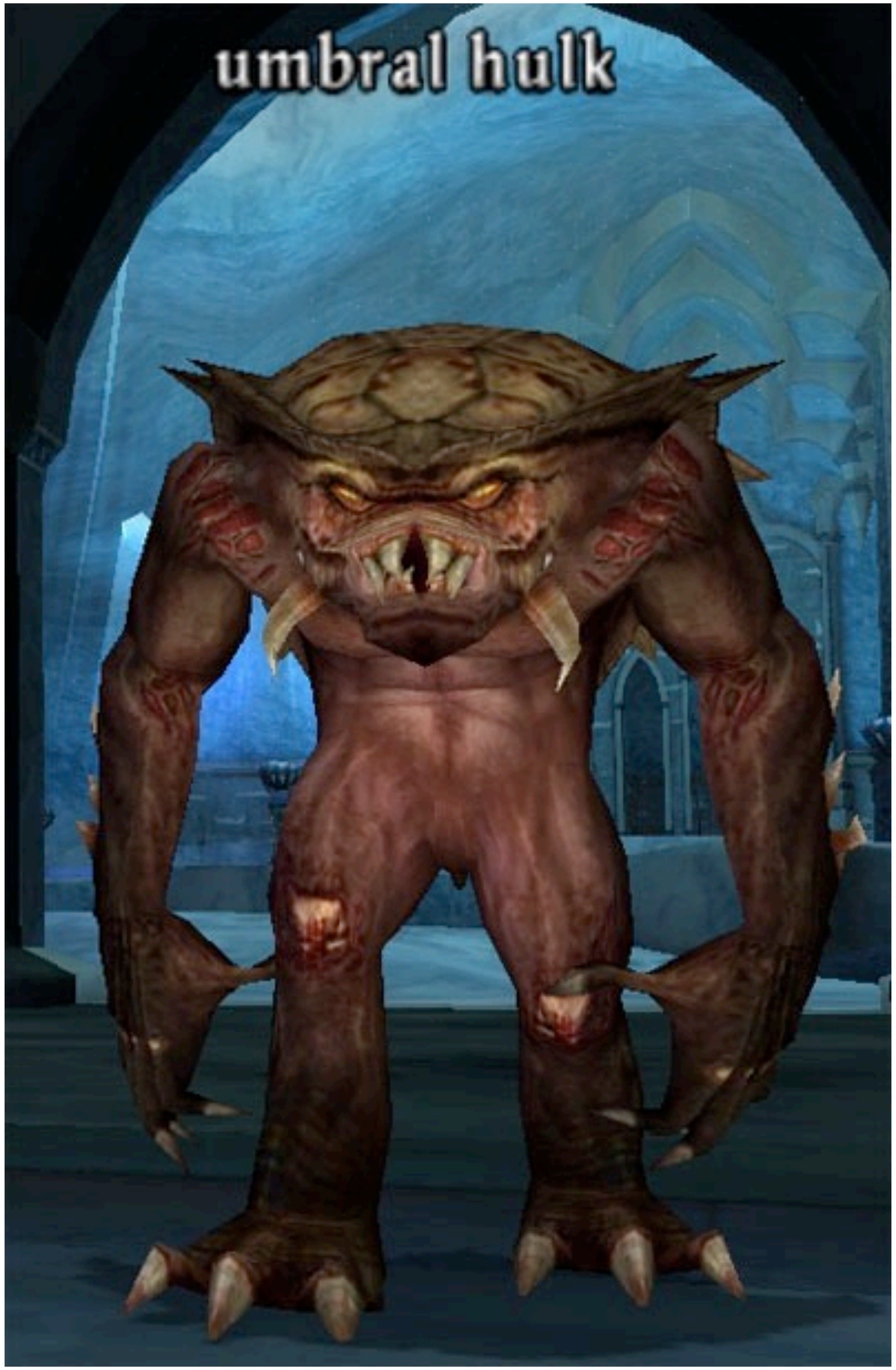
$$B_{6;g,h}(P,Q) = d_{g,h}\left(\frac{Q^2}{2}, \frac{P^2}{2}, Q \cdot P\right)$$

Coincides with Fourier coefficients of

$$\Phi_{g,h}$$

for some pairs (g,h) *!*

Could it be that all of the $\Phi_{g,h}$ **have interpretations as partition functions for BPS-dyons?**



4. Connection with umbral moonshine

Umbral moonshine

Cheng, Duncan, Harvey proposed a generalization of Mathieu moonshine involving 23 examples **labelled by ADE-type root systems**.

Here we focus on the 6 cases corresponding to *pure A-type root systems*.

$$(G^{(\ell)}, Z^{(\ell)}) \qquad\qquad \ell \in \{2, 3, 4, 5, 7, 13\}$$

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finite group   Jacobi form

$$(G^{(2)}, Z^{(2)}) = \big(M_{24}, \chi(K3;\tau,z)\big)$$

Mathieu moonshine corresponds to $\ell = 2$

We shall now see that there appears to be a relation between umbral moonshine and generalized Mathieu moonshine.

Let us consider the case when $g, h \in 2A$ in M_{24}

$\rightarrow \quad \phi_{g,h} = 0$ but $\mathcal{T}_2^{\alpha} \phi_{g,h} \in J_{0,2}^{weak}$

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$\longrightarrow$ $\phi_{g,h} = 0$ but $\mathcal{T}_2^{\alpha}\phi_{g,h} \in J_{0,2}^{weak}$

In fact, this is nothing but the **umbral Jacobi form** for $\ell = 3$

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In fact, this is nothing but the **umbral Jacobi form** for $\ell = 3$

$$\mathcal{T}_2^{\alpha} \phi_{g,h} = Z^{(3)}(\tau, z)$$

The same holds for a few other conjugacy classes in M_{24} that we checked

$(3A, 3A)$ $\qquad \mathcal{T}_3^{\alpha} \phi_{g,h} = Z^{(4)}(\tau, z)$

$(4B, 4B)$ $\qquad \mathcal{T}_4^{\alpha} \phi_{g,h} = Z^{(5)}(\tau, z)$

Starting from the umbral Jacobi forms Cheng-Duncan-Harvey constructed a class of Siegel modular forms using a standard **Borcherds lift:**

$$\Phi^{(\ell)} = \mathbf{Mult}[Z^{(\ell)}] = p^{A(\ell)} q^{B(\ell)} y^{C(\ell)} \prod_{(m,n,r)>0} (1 - p^m q^n y^r)^{c^{(\ell)}(mn,r)}$$

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We observe that these Siegel modular forms coincide with some of the second quantized twisted twining genera in generalized Mathieu moonshine:

$$(2A, 2A) \; : \; \Phi_{g,h} = (\Delta_2)^2 = \Phi^{(3)}$$

$$(3A, 3A) \; : \; \Phi_{g,h} = (\Delta_1)^2 = \Phi^{(4)}$$

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Note that this is non-trivial since the LHS is constructed using an **equivariant lift** while the RHS is constructed using a **standard Borcherds lift**:

$$\mathbf{Mult}_G[\phi_{g,h}] = \mathbf{Mult}[Z^{(\ell)}]$$

These Siegel modular forms also appear in CHL-models. [Sen][Govindarajan]

In fact, following an observation by Govindarajan, for these cases one can also show that the same functions can be obtained using an **additive lift** from the "Hodge anomaly" $A_{g,h}(\sigma,\tau,z)$

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A modular coincidence or an indication of some deeper relation?

5. Summary and outlook

Summary

- We have established that generalised Mathieu moonshine holds by computing all twisted twining genera $\phi_{g,h}$.

- Twisted twining genera can be expanded in projective characters of $C_{M_{24}}(g)$.

- A key role is played by the third cohomology group $H^3(M_{24}, U(1))$.

- All the second quantized twisted twining genera found and verified to be Siegel modular forms

- Some of these correspond to partition functions of twisted dyons in CHL-models

- Intriguing connection with umbral moonshine

Outlook

- Can one construct a generalised Kac-Moody algebra for each conjugacy class $[g] \in M_{24}$? (c.f. [Borcherds][Carnahan])
- Twisted equivariant additive lifts: $\mathbf{Add}_G[A_{g,h}]$? (see also [Eguchi, Hikami])
- Relation with BPS-algebras à la Harvey Moore...?
- Generalised Umbral Moonshine...? [Cheng, Duncan, Harvey]
- Recent interesting results indicate that there is are N=2 and N=1 versions of Mathieu Moonshine in heterotic string theory.
  [Cheng, Dong, Duncan, Harvey, Kachru, Wrase][Harrison, Kachru, Paquette][Wrase]
- Does M_{24} play a role in mirror symmetry?
- Can one construct an action of M_{24} on the (cohomology) of the chiral de Rham complex of K3? See Katrin's talk!

What does M_{24} act on?

Our results strongly suggests that there is something like a **holomorphic vertex operator algebra** underlying Mathieu Moonshine

...but which one remains a mystery...