Aspects of Umbral Moonshine

Jeff Harvey

Mock Modular forms and Physics
IMSc
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based on work with
Miranda Cheng, John Duncan and Sameer Murthy
and work in progress with Sameer and Caner Nazaroglu.

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Amuse-bouche

Ramanujan's third order mock theta function:

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2 \cdots (1+q^n)^2} = 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 - 5q^6 + \cdots$$

$[g]$	FS	1A	2A	4A	2B
$[g^2]$		1A	1A	2A	1A
$[g^3]$		1A	2A	4A	2B
$[g^5]$		1A	2A	4A	2B
$[g^{11}]$		1A	2A	4A	2B
χ_1	+	1	1	1	1
χ_2	+	11	11	−1	3
χ_3	+	11	11	−1	3
χ_4	∘	16	16	4	0
χ_5	∘	16	16	4	0
χ_6	+	45	45	5	−3
χ_7	+	54	54	6	6
χ_8	+	55	55	−5	7
χ_9	+	55	55	−5	−1
χ_{10}	+	55	55	−5	−1
χ_{11}	+	66	66	6	2
χ_{12}	+	99	99	−1	3
χ_{13}	+	120	120	0	−8
χ_{14}	+	144	144	4	0
χ_{15}	+	176	176	−4	0
χ_{16}	∘	10	−10	0	−2

The first component of a two-component mock modular form appearing in works of Eguchi-Hikami and Dabholkar-Murthy-Zagier:

$$H^{(3,1)}(q) = 2q^{-1/12}(-1 + 16q + 55q^2 + 144q^3 + 330q^4 + 704q^5 + 1397q^6 + \cdots)$$

replace

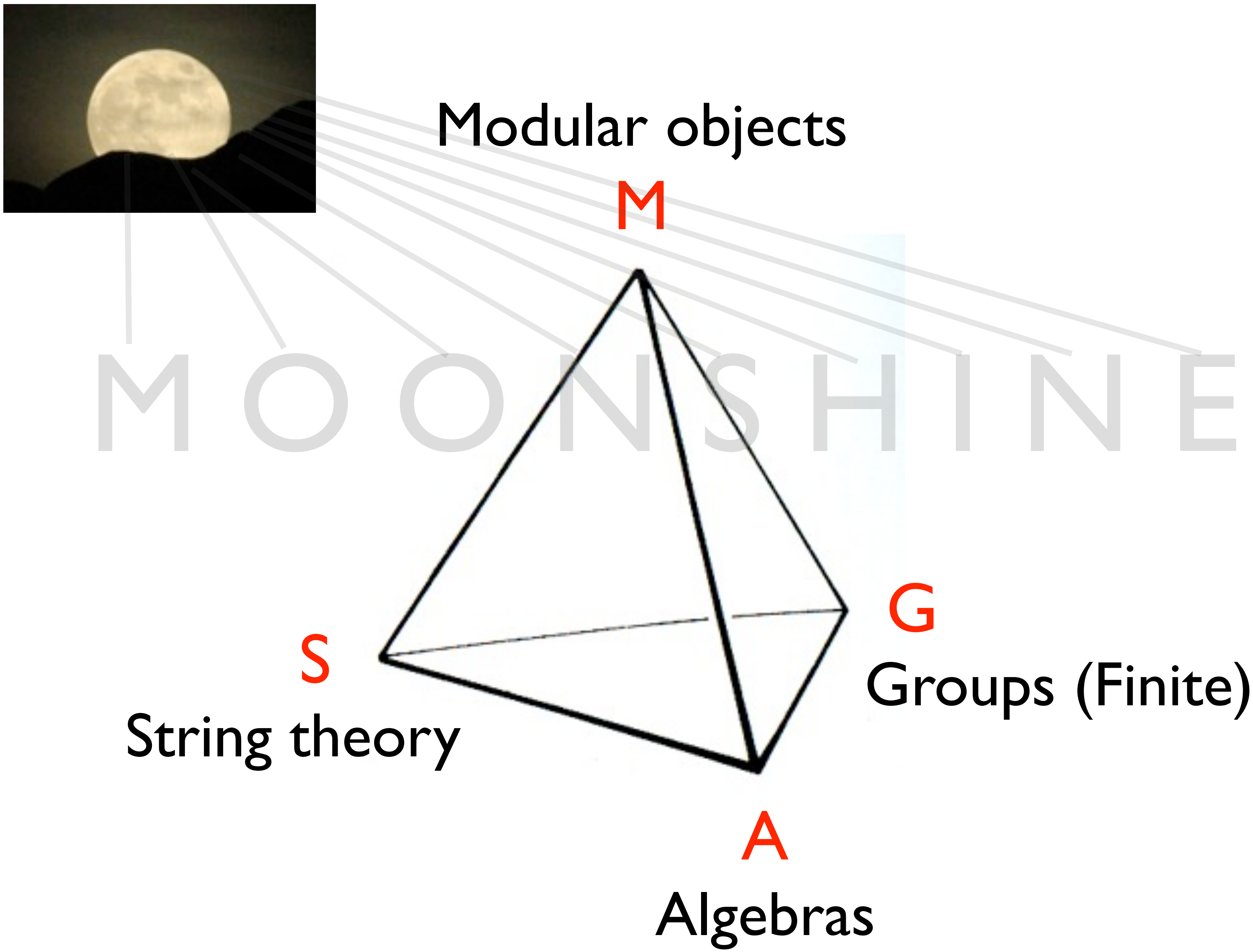
$$-Tr_{2\chi_1} 1A = -2 \to -Tr_{2\chi_1} 2B = -2$$

$$Tr_{\chi_4+\chi_5} 1A = 2 \times 16 \to Tr_{\chi_4+\chi_5} 2B = 0$$

$$Tr_{2\chi_9} 1A = 2 \times 55 \to Tr_{2\chi_9} 2B = -2$$

etc. using the character table of 2.M12 to find

$$H^{(3,1)}_{2B}(q) = -2q^{-1/12} f(q^2)$$


Modular objects
M
MOONSHINE
G
Groups (Finite)
S
String theory
A
Algebras

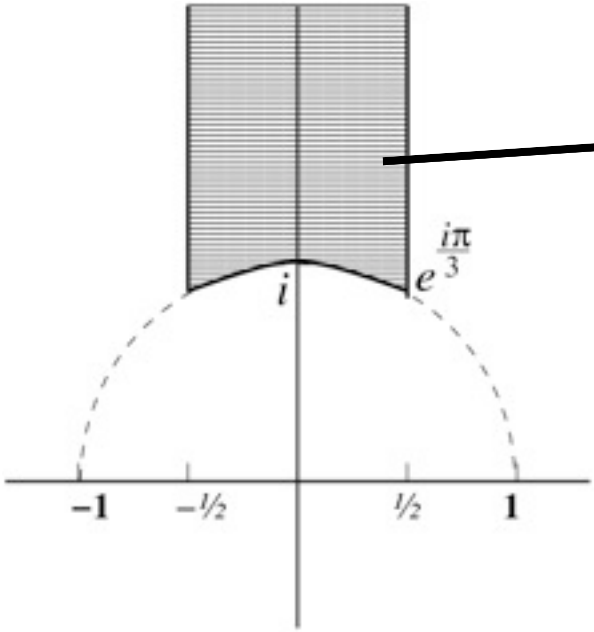
The original example of this structure is Monstrous Moonshine

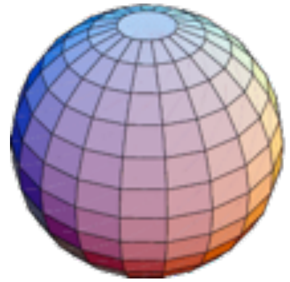
G The Monster sporadic group

M The modular function $j(\tau)$ and other genus zero hauptmoduln appearing as McKay-Thompson series.

A Vertex Operator Algebra (OPE of chiral Vertex operators)

S Bosonic or Heterotic String on an asymmetric orbifold background $(\mathbb{R}^{24}/\Lambda_L)/(\mathbb{Z}/2)$





$$J(\tau) = q^{-1} + 196884q + \cdots = \sum_{m=-1}^{\infty} a(m)q^m$$

196883+1

dimensions of first two irreps of the Monster simple group

$$SL(2,Z)\backslash\mathcal{H} \cup \{i\infty\}$$

In the asymmetric orbifold CFT construction

$$J(\tau) = \mathrm{Tr}\ q^{L_0 - c/24}$$

One of the most remarkable, and still poorly understood parts of Monstrous Moonshine is that the McKay-Thompson series

$$T_g(\tau) = \mathrm{Tr}\ g\ q^{L_0 - c/24}, \qquad g \in M$$

are all hauptmoduls, that is analogs of the modular function $J(\tau)$ in that they map genus zero quotients of the UHP by subgroups of $SL(2,R)$ into the Riemann sphere.

The development of the Monster CFT by Frenkel-Lepowsky-Meurman and the proof of the above genus-zero property by Borcherds introduced a great deal of new technology into mathematics and string theory: orbifolds, vertex operator algebras...

OUTLINE

- A new kind of moonshine relating K3 and the Mathieu group M24 (Eguchi, Ooguri, Tachikawa 2010)

- A mathematical extension called Umbral Moonshine (M.Cheng, J.Duncan, JH)

- Trying to relate Mathieu and Umbral Moonshine to Black Holes and BPS state counting via Fivebranes (S.Murthy and JH)

K3 elliptic genus and M24 moonshine

A K3 surface X defines a (4,4) SCFT with c=6 and from that a modular object known as the elliptic genus:

$$Z^X_{ell}(\tau, z) = \mathrm{Tr}_{R\times R}(-1)^{J_0-\bar{J}_0} q^{L_0-1/4} \bar{q}^{\bar{L}_0-1/4} y^{J_0}$$

$$q = e^{2\pi i\tau}, y = e^{2\pi i z}; \qquad \tau \in \mathfrak{H}, z \in \mathbb{C}$$

$Z^X_{ell}(\tau, z)$ is a (weak) Jacobi form of weight 0, index 1

The elliptic genus is independent of moduli and can be computed directly, say in an orbifold limit. It can also be determined by an indirect argument using the fact that there is a unique such Jacobi form up to normalization.

A weak Jacobi form of (weight,index)=(k,m) is a holomorphic map $\phi : \mathfrak{H} \times \mathbb{C} \to \mathbb{C}$ obeying

modular: $$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} \phi(\tau,z)$$

elliptic: $$\phi(\tau, z+\lambda\tau+\mu) = q^{-m\lambda^2} y^{-2m\lambda} \phi(\tau,z)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}); \qquad \lambda, \mu \in \mathbb{Z}$$

weak: $$\phi(\tau,z) = \sum_{n,r} c(n,r) q^n y^r, \; c(n,r) = 0 \text{ unless } n \geq 0$$

It is known that the ring of weak Jacobi forms for k even has the form (Eichler-Zagier)

$$J_{k,m}^{\text{weak}} = \oplus_{j=0}^{m} M_{k+2j} \cdot \phi_{-2,1}^{j} \phi_{0,1}^{m-j}$$

Modular forms of wt. k+2j, generated by E_4, E_6

$$\frac{\theta_{11}(\tau,z)^2}{\eta(\tau)^6}$$

$$4\left[\left(\frac{\theta_{01}(\tau,z)}{\theta_{01}(\tau,0)}\right)^2 + \left(\frac{\theta_{10}(\tau,z)}{\theta_{10}(\tau,0)}\right)^2 + \left(\frac{\theta_{00}(\tau,z)}{\theta_{00}(\tau,0)}\right)^2\right]$$

So, for example

$$J^{\text{weak}}_{0,1} = \{a\phi_{0,1}\}$$
$$J^{\text{weak}}_{0,2} = \{a\phi^2_{0,1}, bE_4\phi^2_{-2,1}\}$$
$$J^{\text{weak}}_{0,3} = \{a\phi^3_{0,1}, bE_4\phi_{0,1}\phi^2_{-2,1}, cE_6\phi^3_{-2,1}\}$$

Since $J^{\text{weak}}_{0,1}$ is one-dimensional and

$$Z^X_{ell}(\tau, 0) = \text{Euler}(X) = 24$$

we have

$$\boxed{Z^X_{ell}(\tau, z) = 2\phi_{0,1}(\tau, z)}$$

K3 elliptic genus and M24 moonshine

Since we have an N=4 SCA we can decompose into characters of unitary reps:

$$Z_{ell}^{X} = \sum_{(h,\ell)} m_{h,\ell} \tilde{ch}_{h,\ell}$$

The unitary reps fall into two classes. For c=6 we have

"massless"

$h = 1/4, \ell = 0, 1/2 \quad \tilde{ch} =$ complicated Lerch sum

"massive"

$$h > 1/4, \ell = 1/2 \qquad \tilde{ch}_{h,1/2} = q^{h-3/8} \frac{\theta_{11}(\tau, z)^2}{\eta(\tau)^3}$$

At h=1/4: $\tilde{ch}_{1/4,1/2} + 2\tilde{ch}_{1/4,0} = q^{-1/8} \frac{\theta_{11}^2}{\eta^3}$

One finds the decomposition

$$Z_{ell}^{X} = 20\tilde{ch}_{1/4,0} - 2\tilde{ch}_{1/4,1/2} + \sum_{n\geq 1} 2c(n-1/8)\tilde{ch}_{n+1/4,1/2}$$

$$\begin{aligned} c(7/8) &= 45 \\ c(15/8) &= 231 \\ c(23/8) &= 770 \\ c(31/8) &= 2277 \end{aligned}$$

dimensions of M24 irreps

(Eguchi, Ooguri, Tachikawa)

$$c(n-1/8) = \sum_{i}^{26} r_n^i \,\mathrm{dim} R_i$$

positive integer multiplicity

irreps of M24

K3 elliptic genus and M24 moonshine

It is useful to repackage this using an identity between massless and massive reps as

$$Z^X_{ell} = 24\tilde{ch}_{1/4,0} + H^{(2)}(\tau)\frac{\theta_{11}^2}{\eta^3}$$

with

$$H^{(2)}(\tau) = -2q^{-1/8} + \sum_{n\geq 1} 2c(n-1/8)q^{n-1/8}$$

This gives us a q series with coefficients related to dimensions of M24 representations. This is reminiscent of monstrous moonshine for the modular function $J(\tau) = q^{-1} + \mathrm{const} + 196884q + \cdots$

$196883 + 1$

Sunday, April 20, 14

Monstrous Moonshine

$$J(\tau) = \sum_{n \geq -1} a(n) q^n$$

a(n)=dim of Monster reps,
J=modular function for $SL(2,\mathbb{Z})$

FLM:

$$V^\natural = \oplus_{m \geq -1} V_m, \quad a(m) = \mathrm{dim} V_m$$

Constructed as c=24 asymmetric orbifold CFT

K3/M24

$$H^{(2)}(\tau) = 2 \sum_{n \geq 0} c(n-1/8) q^{n-1/8}$$

c(n-1/8)=dim of M24 reps,
$H^{(2)}$ =mock modular form of weight 1/2 for $SL(2,\mathbb{Z})$

??

$$W^\natural = \oplus_{m \geq 0} W_{m-1/8}$$

??

In either case we can define McKay-Thompson series $J_g, H_g^{(2)}$ by

$$\mathrm{dim} V_m = \mathrm{tr} 1|_{V_m} \to \mathrm{tr} g|_{V_m}$$

Mock modular forms

Recall a modular form of weight k is a holomorphic function $f(\tau)$ obeying

$$f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$h(\tau)$ is a mock modular form of weight k if there is a pair $(h(\tau), g(\tau))$ where $g(\tau)$ is a holomorphic modular form of weight 2-k, known as the shadow of h, such that the non-holomorphic function

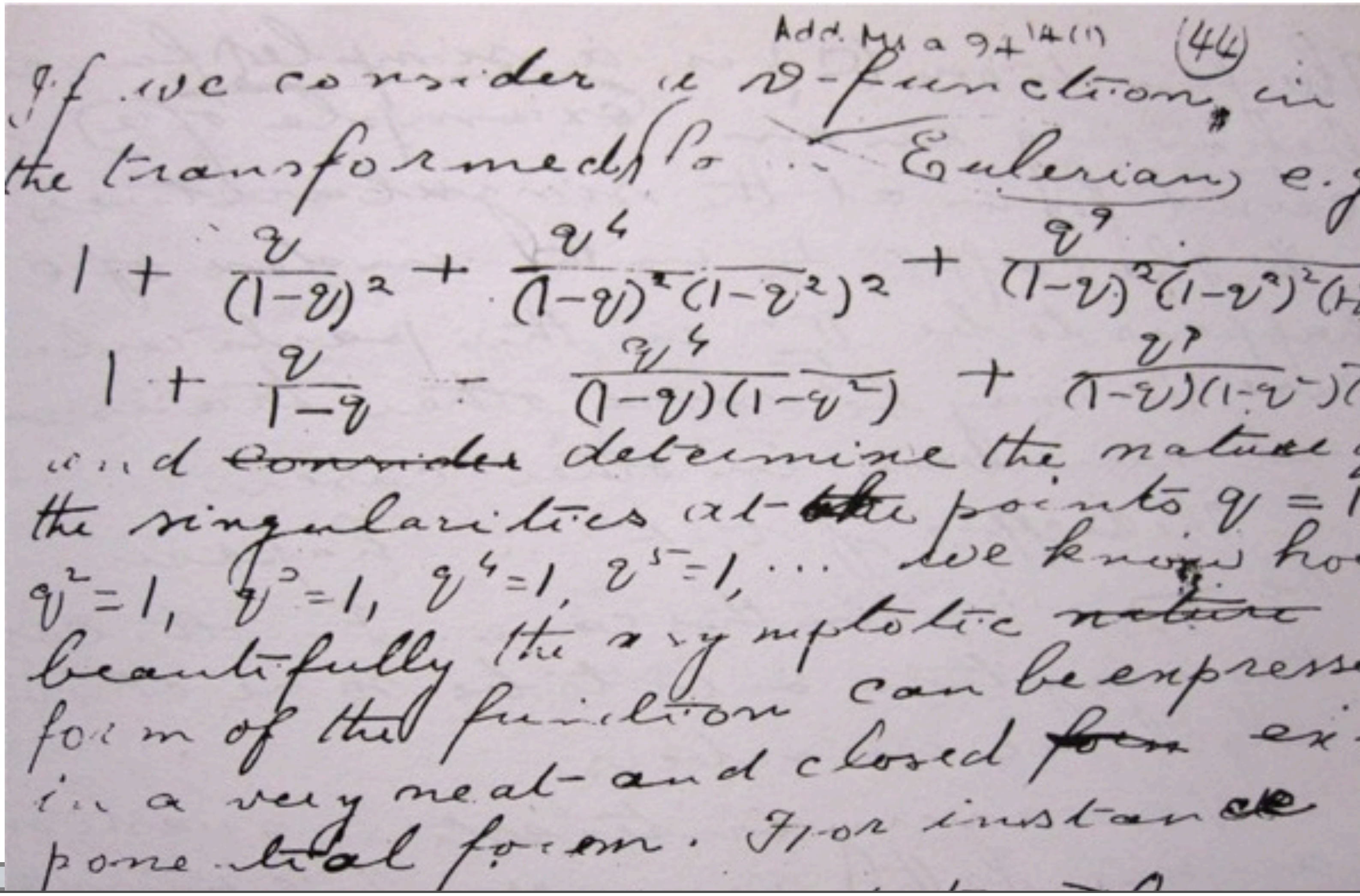
$$\hat{h}(\tau) = h(\tau) + \text{const} \int_{-\bar{\tau}}^{\infty} \overline{g(-\bar{z})} (z+\tau)^{-k} dz$$

is modular of weight k.

$(H^{(2)}(\tau), \eta(\tau)^3)$ is such a pair with k=1/2

Mock modular forms

Mock theta functions first appeared in 1920 in the last letter Ramanujan wrote to Hardy. He wrote them down as q expansions but did not explain how he had found them or how they were defined.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If we consider a ϑ-function in the transformed or Eulerian form e.g.

$$1 + \frac{q}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^2)^2} + \frac{q^9}{(1-q)^2(1-q^2)^2(1+\cdots}$$

$$1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(\cdots}$$

and ~~consider~~ determine the nature of the singularities at ~~the~~ points $q=1$, $q^2=1$, $q^3=1$, $q^4=1$, $q^5=1$, … we know how beautifully the asymptotic ~~nature~~ form of the function can be expressed in a very neat and closed ~~form~~ exponential form. For instance

Umbral Moonshine

In Mathieu moonshine we have a triple

$$H^{(2)}(\tau), 24\eta^3(\tau), M24$$

of a mock modular form, its shadow, and a finite group which seems to act on the mock modular form. Are there generalizations of this structure?

After much work, both theoretical and "experimental" the answer is "yes" and some remarkable new structure is revealed, but at the moment I cannot tell you the origin of this structure.

Umbral Moonshine

Let X be a root system with A,D,E components, rank 24 and with all components having equal Coxeter number.

$$A_1^{24},\ A_2^{12},\ A_3^8,\ A_4^6,\ A_5^4D_4,\ A_6^4,\ A_7^2D_5^2,\ A_8^3,\ A_9^2D_6,\ A_{11}D_7E_6,\ A_{12}^2,\ A_{15}D_9,\ A_{17}E_7,\ A_{24},$$

$$D_4^6,\ D_6^4,\ D_8^3,\ D_{10}E_7^2,\ D_{12}^2,\ D_{16}E_8,\ D_{24},$$

$$E_6^4,\ E_8^3.$$

For each of the 23 X above we (M. Cheng, J. Duncan and JH) claim that there exist 6 mathematical objects determined by X

$$H^X, S^X, G^X, \Gamma^X, T^X, L^X$$

The Niemeier lattice L^X

Recall that even, unimodular (self-dual) lattices occur only in dimensions that are a multiple of 8:

8 : E8
16: E8xE8, Spin(32)/Z2
24: 23 Niemeier lattices Γ^X and the Leech lattice with no roots.

It is a classical result that each Niemeier lattice is uniquely determined by its roots (points of length squared 2) which are in one to one correspondence with the X. One must "glue" in a set of weights of the associated Lie algebra to form the Niemeier lattice according to a "glue code"

The finite group G^X

For each Niemeier lattice we define a finite group

$$G^X = \mathrm{Aut}(L^X)/W^X$$

Weyl group of X

If we consider the bosonic string on R^{24}/L^X then the gauge group would be X and G^X would act as a finite global symmetry group.

$$G^{A_1^{24}} = M_{24}, \qquad G^{A_2^{12}} = 2.M_{12}, \qquad G^{E_8^3} = S_3$$

Umbral Moonshine

The genus zero subgroup and its hauptmodul Γ^X, T^X

One of the new and surprising results in our paper was the observation that each X leads directly to a genus zero subgroup of SL(2,R) and its hauptmodul.

The construction uses the eigenvalues of the Coxeter elements of the components of X. I will not go into the details, but as an example that something non-trivial is going on, the groups

$$\Gamma_0(n) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z), \quad c = 0 \ (\mathrm{mod}) n$$

are genus zero for n=2,3,..9,10,12,13,16,18,25 which are exactly the Coxeter numbers of X with A-type components

Since Hauptmoduls also appear in the M-T series of Monstrous Moonshine, this gives a way of associating conjugacy classes of the monster to examples of Umbral Moonshine:

X	A_1^{24}	A_2^{12}	A_3^{8}	A_4^{6}	$A_5^{4}D_4$	A_6^{4}	$A_7^{2}D_5^{2}$
ℓ	2	3	4	5	6	7	8
π^X	$\frac{2^{24}}{1^{24}}$	$\frac{3^{12}}{1^{12}}$	$\frac{4^{8}}{1^{8}}$	$\frac{5^{6}}{1^{6}}$	$\frac{2^{1}6^{5}}{1^{5}3^{1}}$	$\frac{7^{4}}{1^{4}}$	$\frac{2^{2}8^{4}}{1^{4}4^{2}}$
Γ^X	$2B$	$3B$	$4C$	$5B$	$6E$	$7B$	$8E$

X	A_8^{3}	$A_9^{2}D_6$	$A_{11}D_7E_6$	A_{12}^{2}	$A_{15}D_9$	$A_{17}E_7$	A_{24}
ℓ	9	10	12	13	16	18	25
π^X	$\frac{9^{3}}{1^{3}}$	$\frac{2^{1}10^{3}}{1^{3}5^{1}}$	$\frac{2^{2}3^{1}12^{3}}{1^{3}4^{1}6^{2}}$	$\frac{13^{2}}{1^{2}}$	$\frac{2^{1}16^{2}}{1^{2}8^{1}}$	$\frac{2^{1}3^{1}18^{2}}{1^{2}6^{1}9^{1}}$	$\frac{25^{1}}{1^{1}}$
Γ^X	$9B$	$10E$	$12I$	$13B$	$16B$	$18D$	$(25Z)$

X	D_4^{6}	D_6^{4}	D_8^{3}	$D_{10}E_7^{2}$	D_{12}^{2}	$D_{16}E_8$	D_{24}
ℓ	6+3	10+5	14+7	18+9	22+11	30+15	46+23
π^X	$\frac{2^{6}6^{6}}{1^{6}3^{6}}$	$\frac{2^{4}10^{4}}{1^{4}5^{4}}$	$\frac{2^{3}14^{3}}{1^{3}7^{3}}$	$\frac{2^{3}3^{2}18^{3}}{1^{3}6^{2}9^{3}}$	$\frac{2^{2}22^{2}}{1^{2}11^{2}}$	$\frac{2^{2}3^{1}5^{1}30^{2}}{1^{2}6^{1}10^{1}15^{2}}$	$\frac{2^{1}46^{1}}{1^{1}23^{1}}$
Γ^X	$6C$	$10B$	$14B$	$18C$	$22B$	$30G$	$46AB$

X	E_6^{4}	E_8^{3}
ℓ	12+4	30+6,10,15
π^X	$\frac{2^{4}3^{4}12^{4}}{1^{4}4^{4}6^{4}}$	$\frac{2^{3}3^{3}5^{3}30^{3}}{1^{3}6^{3}10^{3}15^{3}}$
Γ^X	$12B$	$30A$

The mock modular forms and shadows H^X, S^X

Mathematically, the EOT result arose from computing the decomposition of a weak Jacobi form of weight 0 and index 1 into characters of the N=4 SCA. This can be generalized to weight 0 and index m-1 Jacobi forms, but the space of such forms grows with m. Dabholkar, Murthy and Zagier identified a special set of Jacobi forms "of optimal growth" and many of their examples also exhibit moonshine with groups in our list of G^X

Details of m=3

For m=3 associated to $X = A_2^{12}$

$$\phi_{0,2} = (\phi_{0,1}^2 - E_4\phi_{-2,1}^2)/24$$

$$\hat{G}^{(3)} = 2.M_{12}$$

and a decomposition in N=4 characters which now has $\ell = 0, 1/2$ for massive characters and thus generates two mock modular forms

$$H^{(3,1)}(\tau) = 2q^{-1/12}(-1 + 16q + 55q^2 + 144q^3 + \cdots)$$

$$H^{(3,2)}(\tau) = 2q^{2/3}(10 + 44q + 110q^2 + 280q^3 + \cdots)$$

$10, 44, 110, 120, 160$ dimensions of faithful irreps

$16, 55, 144$ irreps with trivial $\mathbb{Z}/2\mathbb{Z}$ action

Details of m=3

As was true for M_{24}, $2.M_{12}$ has 26 conjugacy classes so there are 2x26 Thompson series $H_g^{(3,1)}, H_g^{(3,2)}$ labelled by the conjugacy classes. Most of these can be identified either with products of eta functions, or with the order 3 mock theta functions of Ramanujan!

e.g.

$$\left.\begin{aligned} H_{2B}^{(3,1)} &= -2q^{-1/12} f(q^2) \\ H_{2B}^{(3,2)} &= -4q^{2/3}\omega(-q) \end{aligned}\right\}$$

Results of Zwegers can be used to verify the existence of a two-dimensional rep $\alpha_2 : \Gamma_0(2) \to GL(2,\mathbb{C})$ such that this pair defines a vector-valued mock modular form for $\Gamma_0(2)$

Details of m=3

$$H^{(3,1)}_{4C}(\tau) = -2\frac{\eta(2\tau)^3}{\eta(4\tau)^2} \qquad H^{(3,1)}_{12A}(\tau) = -2\frac{\eta(\tau)\eta(3\tau)}{\eta(6\tau)}$$

$$H^{(3,1)}_{6C}(\tau) = -2q^{-1/12}\chi(q^2) \; H^{(3,2)}_{6C}(\tau) = 2q^{2/3}\rho(-q)$$

$$H^{(3,1)}_{4A}(\tau) = -2\frac{\eta(\tau)^4}{\eta(2\tau)\eta(8\tau)^2}$$

$$H^{(3,1)}_{20AB}(\tau) = -2\frac{\eta(2\tau)^2\eta(5\tau)}{\eta(\tau)\eta(10\tau)}$$

$$H^{(3,1)}_{4B}(\tau) = -2q^{-1/12}[\phi(-q^2) + 2\psi(-q^2)]$$

We find eta products rather than mock theta functions whenever the coefficient of the shadow vanishes which is determined by the character of the permutation representation.

The Shadows of these two examples are proportional to the weight 3/2 theta functions

$$S_{m,r}(\tau) = \frac{1}{2\pi i}\frac{\partial}{\partial z}\theta_{m,r}(\tau, z)|_{z=0}$$

$$\theta_{m,r}(\tau, z) = \sum_{\substack{k \in Z \\ k = r (mod) 2m}} q^{k^2/4m} y^k$$

with shadow $S_{2,1} \propto \eta^3$ for m=2 and a vector-valued shadow $(S_{3,1}, S_{3,2})$ at m=3.

Umbral Moonshine

In some cases one can modify these shadows by a sort of twisting or folding operation with

$$S_{m,r} \to \sum_{r'} \Omega_{r,r'} S_{m,r'}$$

The classification of such matrices that preserve the modular properties (and a positivity condition) turns out to be isomorphic to the Capelli-Itzykson-Zuber classification of modular invariant affine SU(2) partition functions. In particular there is an ADE classification. We use this to associate a shadow to each of the Niemeier lattices and in each case find a mock modular form with this shadow S^X exhibiting moonshine for the group G^X .

In our paper arXiv:1307.5793 we not only provide constructions of the mock modular forms and shadows for each X, we also give the q expansions of the McKay-Thompson series for each $g \in G^X$

There are many other fascinating structures involving relations between these series at different X, relations between the group structure and structure of the mock modular forms many of which are conjectural and for which we provide a great deal of evidence. But proofs and a deeper understanding are for the future.

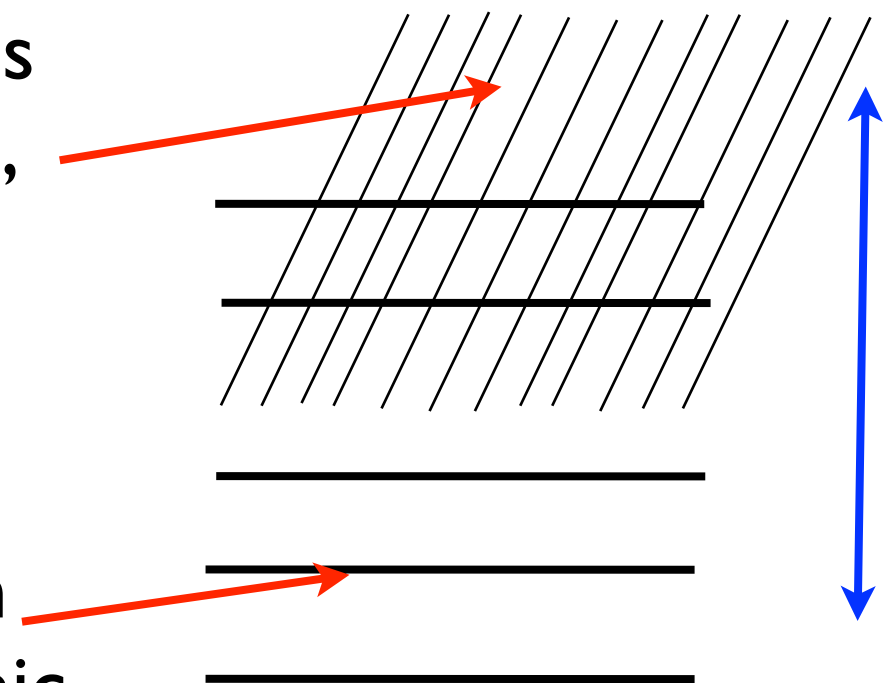
BPS state counting, 5-branes and Moonshine
(with S. Murthy)

Motivation

a) The results of EOT are world-sheet results, resulting from a decomposition of the elliptic genus into irreps of the world-sheet SCA. If we could find an analog involving counting of spacetime BPS state we could try to apply all the modern technology of dualities.

5-branes and Moonshine

b) Mock modularity occurs in Black Hole state counting and in the computation of the elliptic genus for non-compact sigma-models and gives a physical explanation for the tension between holomorphicity and modularity (Troost, Ashok, Eguchi, Sugawara)



5-branes and Moonshine

c) We have seen that that the ADE root lattices of a certain type classify Umbral Moonshine. Fivebranes have an ADE classification and are described by non-compact SCFT's.

With these motivations we consider the SCFT describing k NS-branes in type II in the "capped off" configuration where the near-horizon geometry they create is described by the "cigar" SCFT.

$$\left(SL(2,R)_k/U(1) \times SU(2)_k/U(1)\right)/Z_k$$

The second factor vanishes for k=2 and we consider this case in what follows.

We consider Type II string theory on

$$K3 \times S^1 \times R^{5,1}$$

and take the 5-branes to wrap $K3 \times S^1$

The resulting theory has 8 real spacetime supercharges, like N=2 in d=4, and BPS states are counted by the second BPS index

$$\chi_2(\tau) = \mathrm{Tr} J^2 (-1)^{F_s} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24}$$

using the technology developed by Ashok, Troost, Eguchi and Sugawara, and a fun integral first evaluated in unpublished work of Gaiotto-Zagier we find

5-branes and Moonshine

$$P(\tau,u) = \frac{\varphi_{0,1}(\tau,u)}{\varphi_{-2,1}(\tau,u)} = -\frac{3}{\pi^2}\wp(\tau,u)$$

$$H(\tau,u) \;=\; (2\tau_2)^{1/2} e^{-2\pi u_2^2/\tau_2} |\vartheta_1(\tau,u)|^2\,,$$

ı as:

$$\chi_2(\tau) \;=\; \int_{E(\tau)} P(\tau,u)\, H(\tau,u)\, \frac{du_1 du_2}{\tau_2}\,.$$

$$\begin{aligned}\chi_2(\tau) &= -\frac{1}{2}\eta^3(\tau)\Big(H^{(2)}(\tau) + 24\sum_{k=0}^{\infty}(-1)^k q^{-\frac{(2k+1)^2}{8}}\Big(1 - \mathrm{Erf}\Big[\frac{1+2k}{2}\sqrt{2\pi\tau_2}\Big]\Big)\Big) \\ &= -\frac{1}{2}\eta^3(\tau)\widehat{H}^{(2)}(\tau)\,.\end{aligned}$$

The M24 Moonshine mock modular form shows up in the holomorphic part of a spacetime BPS state counting problem!

There are 14 choices of X which are pure A, D E meaning they are powers of a single A, D, E component: $A_1^{24}, A_2^{12}, \cdots A_{24}, D_4^6, \cdots D_{24}, E_6^4, E_8^3$

For these one can consider m(X) fivebranes of type

$$A_1, A_2, \cdots A_{24}, D_4, \cdots D_{24}, E_6, E_8$$

defined by using A,D,E modular invariants in the SCFT description. For these we find that χ_2 is generalized to

$$\sum_r S_{m,r}(\tau)\hat{H}_r^X(\tau)$$

with the H_r^X the umbral mock modular forms for the corresponding root system X.

Conclusions

This all leads to many questions.

1. What is the generalization to fivebranes with mixed A,D,E structure?

2. Is it possible to understand explicitly the action of M24 and its generalizations? Is there an explicit construction of the modules implied by moonshine?

3. Can we learn something about physics? The appearance of large, sometimes sporadic groups with no obvious origin in supersymmetric compactifications of string theory certainly suggests we are missing something and could have interesting consequences.

Conclusions

4. Is there a way to construct the mock modular forms directly from the data of the Niemeier lattice?

5. Is there a physical explanation of why the Niemeier lattices classify examples of Umbral Moonshine?

6. Is there some algebraic structure like the VOA of Monstrous Moonshine connected to Umbral Moonshine?

Comment on Discriminants

The powers of q in the series expansion of $H^{(m,r)}$ are $q^{d(m,n,r)/4m}$ where $d(m,n,r) = 4mn - r^2$ is the discriminant of the binary quadratic form

$$\begin{pmatrix} 2m & r \\ r & 2n \end{pmatrix}$$

We conjecture that the representation $K^{(m,r)}_{d(m,n,r)/4m}$ of $\hat{G}^{(m)}$ associated to $q^{d(m,n,r)/4m}$ can be defined over no field smaller than $\mathbb{Q}[\sqrt{-d(m,n,r)}]$ whenever $\hat{G}^{(m)}$ has an element of order $d(m,n,r)$.

Comment on Discriminants

For example, the representations 45,231,770 of M24 are associated to terms with discriminant 7,15,23 and M24 has elements of this order and the characters of these representations involve $\sqrt{-d(m,n,r)}$

Table 4: Character table for $G^{(2)} = M_{24}$

	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	23A	23B
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	23	7	−1	5	−1	−1	3	−1	3	1	−1	2	2	1	−1	1	−1	−1	0	0	0	0	−1	−1	0	0
χ_3	45	−3	5	0	3	−3	1	1	0	0	−1	b_7	$\bar{b}_7$	−1	0	1	0	1	$-b_7$	$-\bar{b}_7$	0	0	b_7	$\bar{b}_7$	−1	−1
χ_4	45	−3	5	0	3	−3	1	1	0	0	−1	$\bar{b}_7$	b_7	−1	0	1	0	1	$-\bar{b}_7$	$-b_7$	0	0	$\bar{b}_7$	b_7	−1	−1
χ_5	231	7	−9	−3	0	−1	−1	3	1	1	0	0	0	−1	1	0	−1	0	0	0	b_{15}	$\bar{b}_{15}$	0	0	1	1
χ_6	231	7	−9	−3	0	−1	−1	3	1	1	0	0	0	−1	1	0	−1	0	0	0	$\bar{b}_{15}$	b_{15}	0	0	1	1
χ_7	252	28	12	9	0	4	4	0	2	1	0	0	0	0	2	−1	1	0	0	0	−1	−1	0	0	−1	−1
χ_8	253	13	−11	10	1	−3	1	1	3	−2	1	1	1	−1	−1	0	0	1	−1	−1	0	0	1	1	0	0
χ_9	483	35	3	6	0	3	3	3	−2	2	0	0	0	−1	−2	−1	0	0	0	0	1	1	0	0	0	0
χ_{10}	770	−14	10	5	−7	2	−2	−2	0	1	1	0	0	0	0	0	−1	1	0	0	0	0	0	0	b_{23}	$\bar{b}_{23}$
χ_{11}	770	−14	10	5	−7	2	−2	−2	0	1	1	0	0	0	0	0	−1	1	0	0	0	0	0	0	$\bar{b}_{23}$	b_{23}

$$b_d = (-1 + \sqrt{-d})/2$$