$AC^0[\oplus]$ FORMULAS VERSUS AC^0 CIRCUITS

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Boolean functions

$f : \{0,1\}^n \to \{0,1\}$ – Computational problem

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AND: $\wedge_m : \{0,1\}^m \rightarrow \{0,1\}$

$$\wedge_m(x_1, \ldots, x_m) = \min\{x_1, \ldots, x_m\}$$

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"Simple" operations

AND: $\wedge_m : \{0,1\}^m \to \{0,1\}$

$\wedge_m(x_1, \ldots, x_m) = \min\{x_1, \ldots, x_m\}$

OR: $\vee_m(x_1, \ldots, x_m) = \max\{x_1, \ldots, x_m\}$

NOT: $\neg x = 1 - x$

Boolean formulas

$$f(x_1, x_2, x_3, x_4, x_5) = \neg(x_1 \wedge x_2) \vee x_3 \vee (x_4 \wedge x_5)$$

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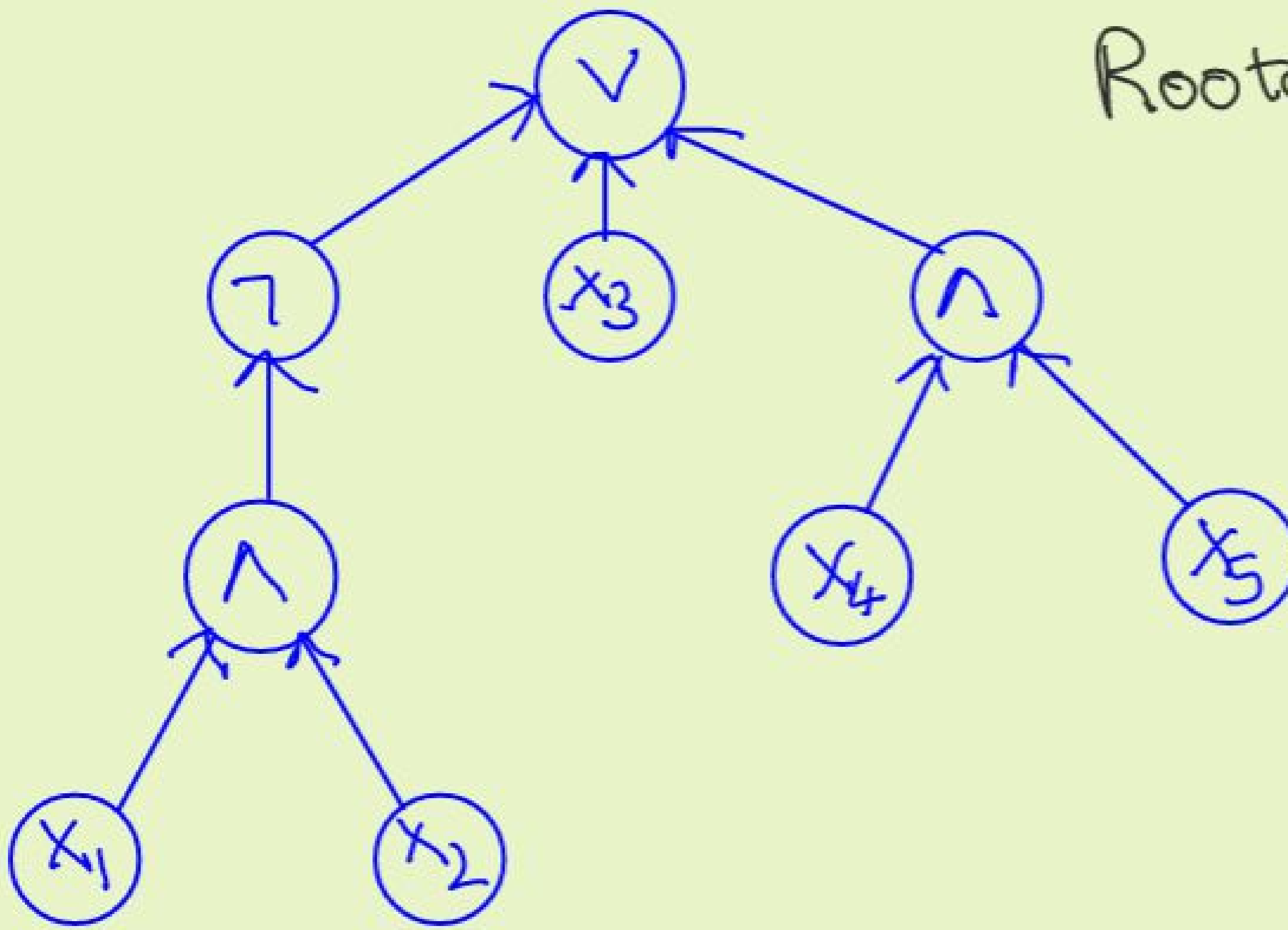

Rooted directed tree

Boolean formulas

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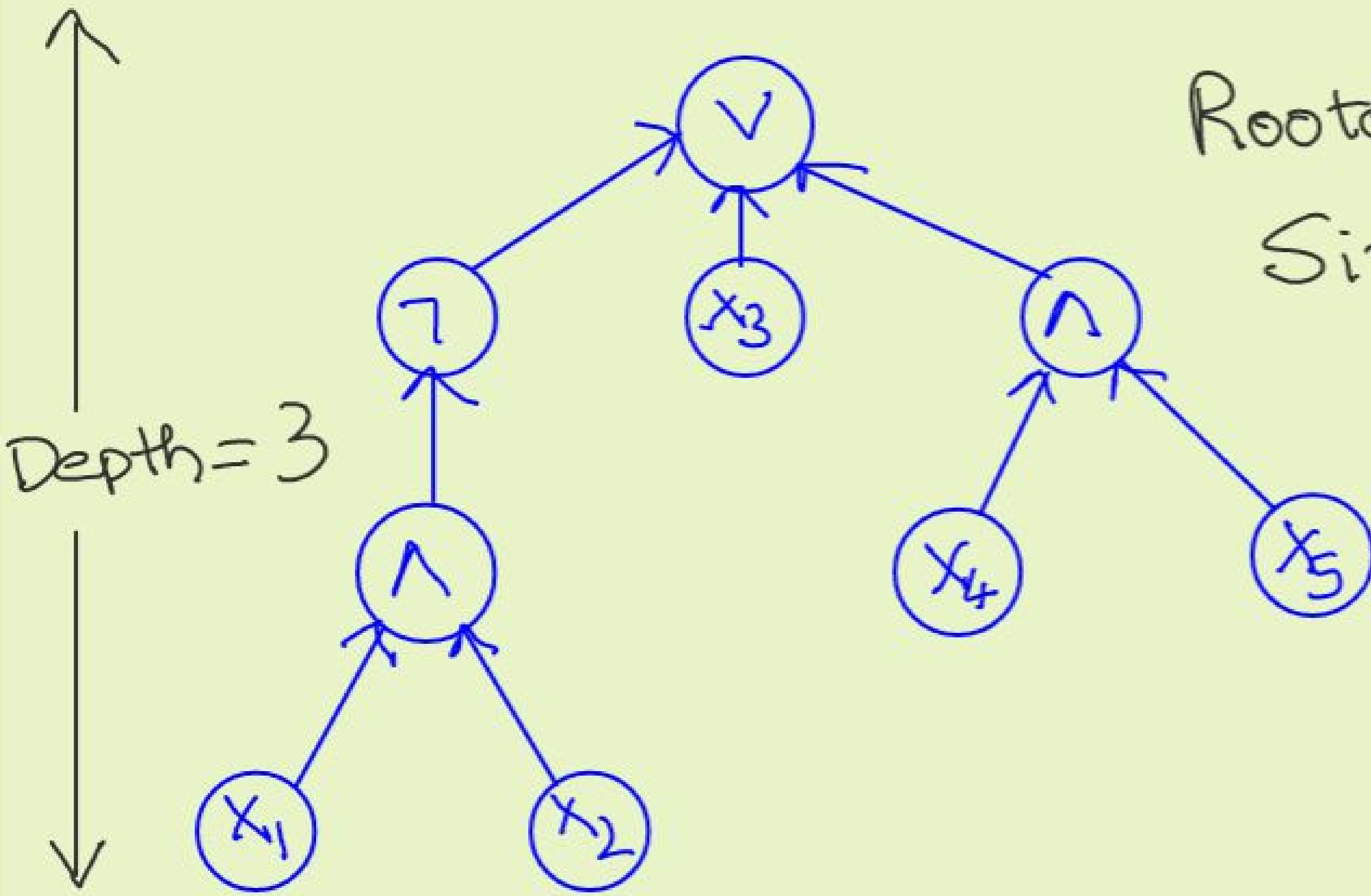
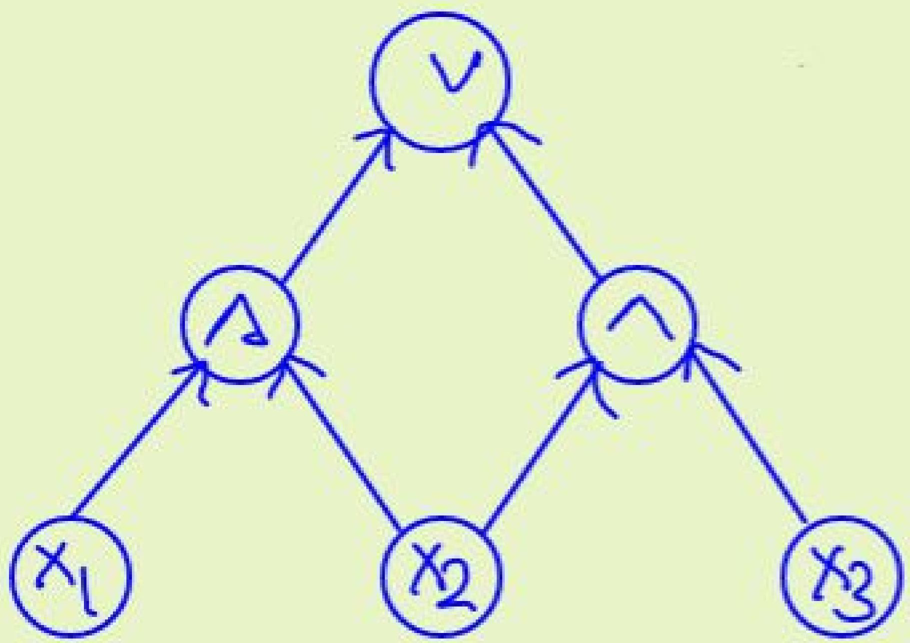


Rooted directed tree

Size = # of nodes
= 9

Boolean circuits

$f(x_1, x_2, x_3) =$



Boolean circuit

Boolean circuits

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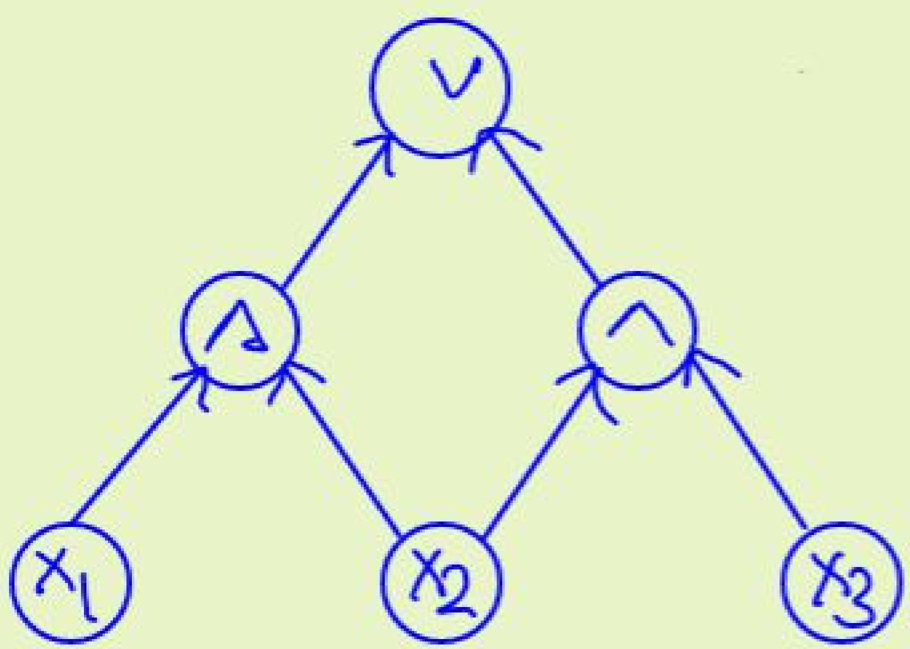


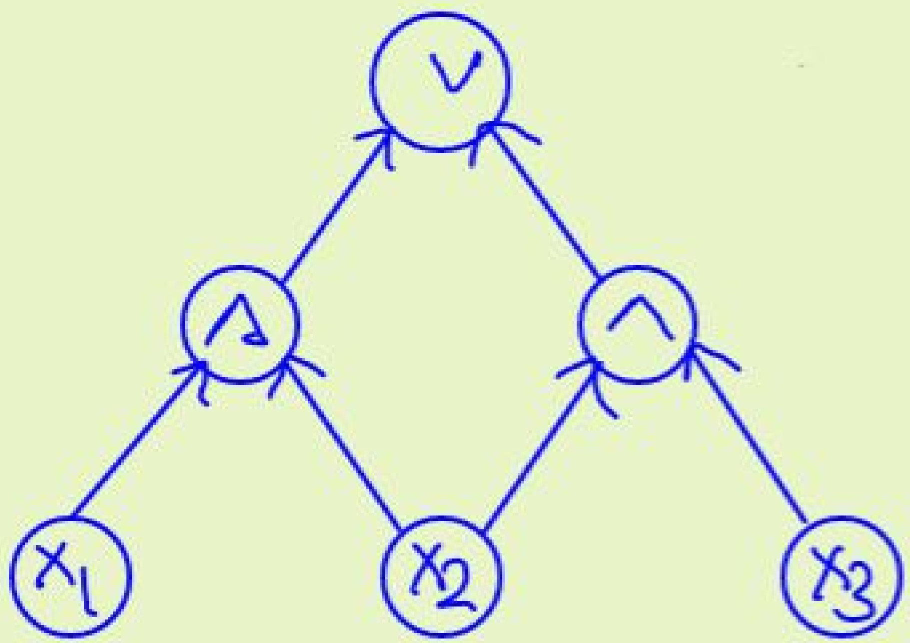
Boolean circuit

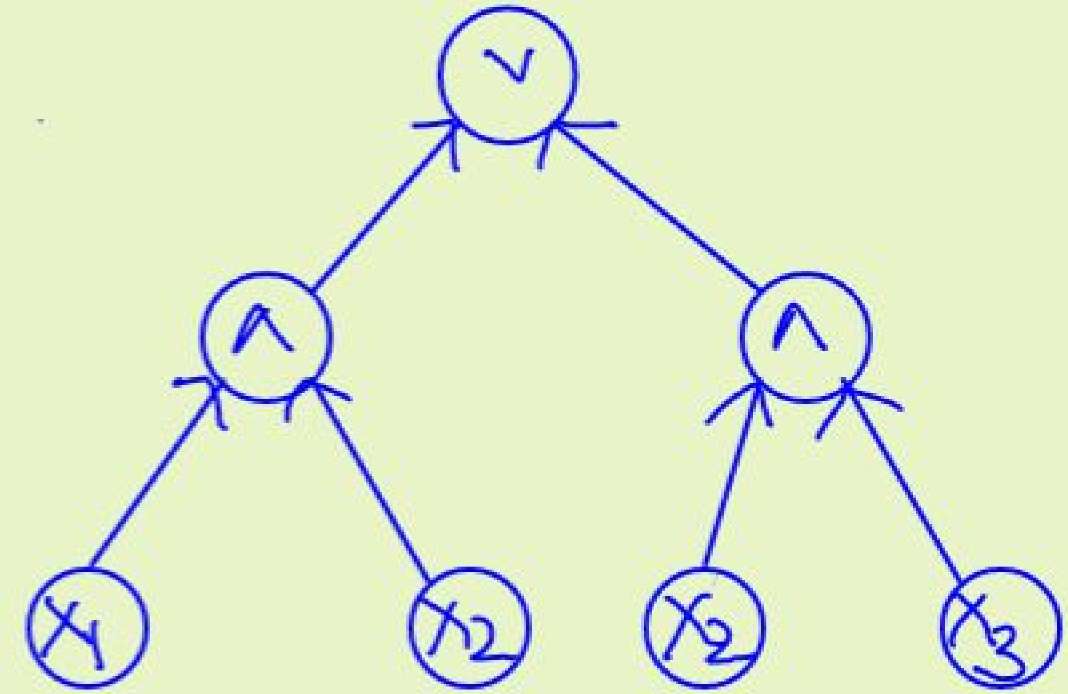
Size = # of nodes = 6

Depth = 2

Boolean circuits

$f(x_1, x_2, x_3) =$





Boolean circuit

Size = # of nodes = 6

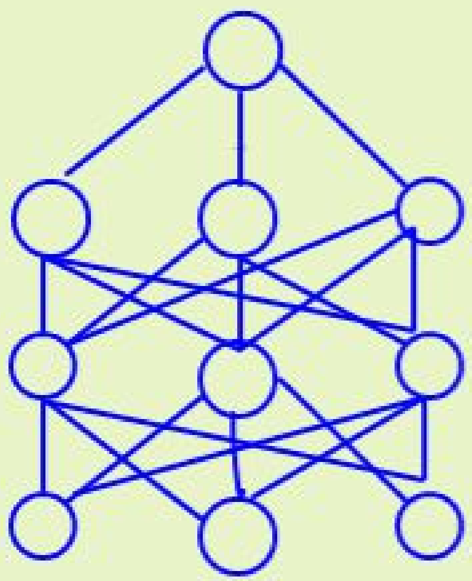
Depth = 2

Equivalent formula

Size = 7

Depth = 2

Converting circuits to formulas



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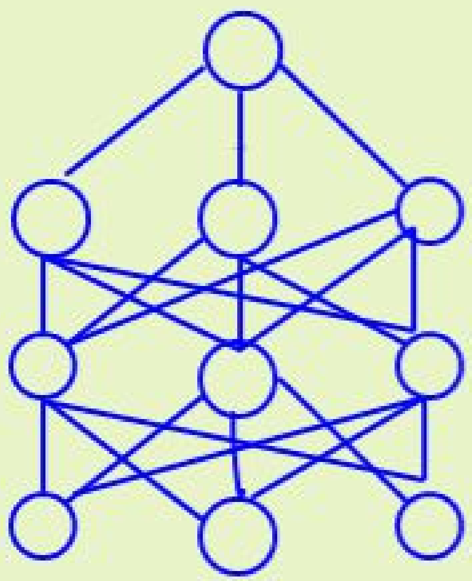

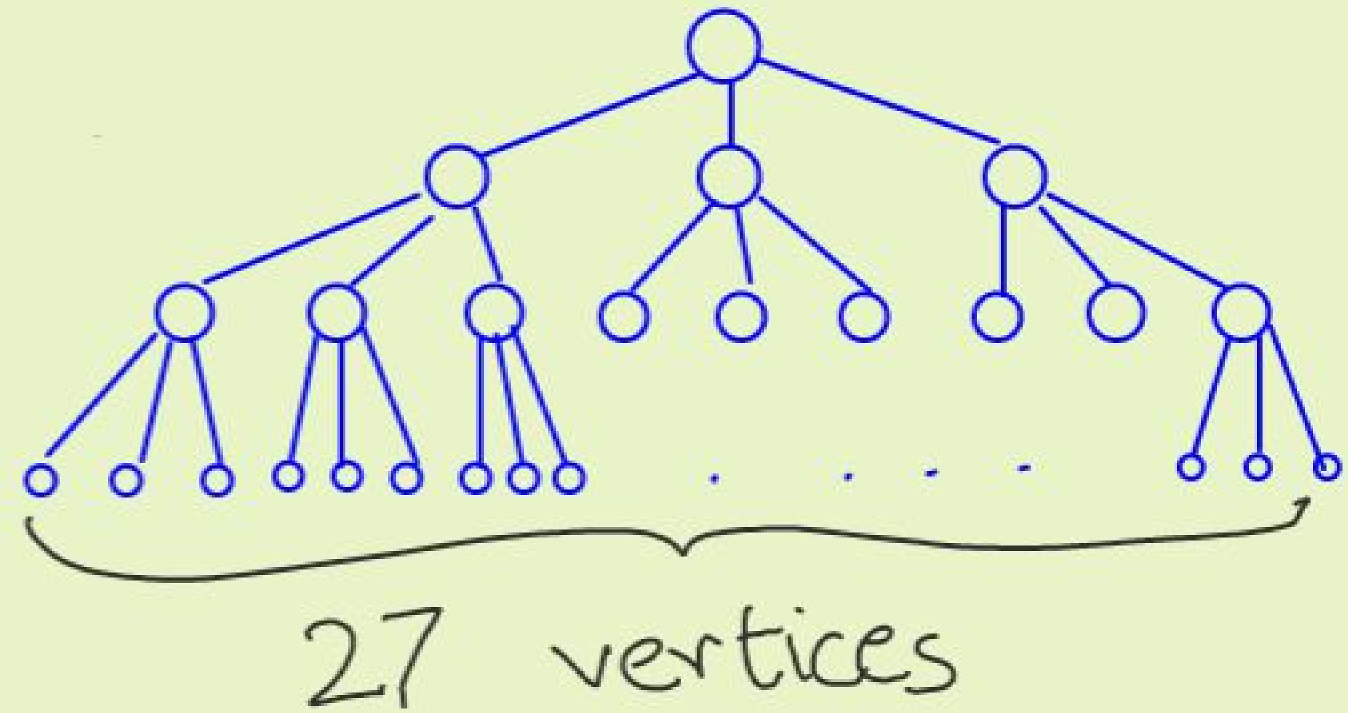

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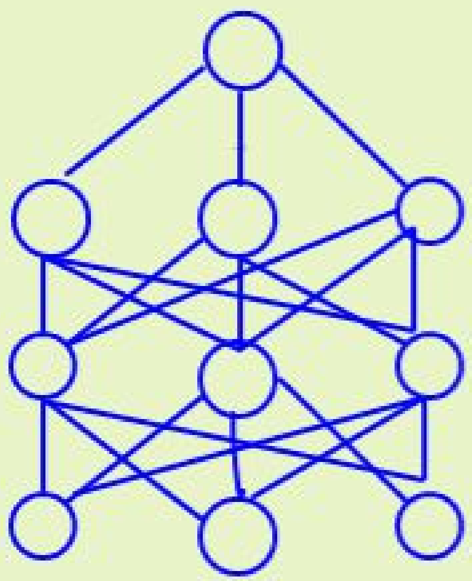
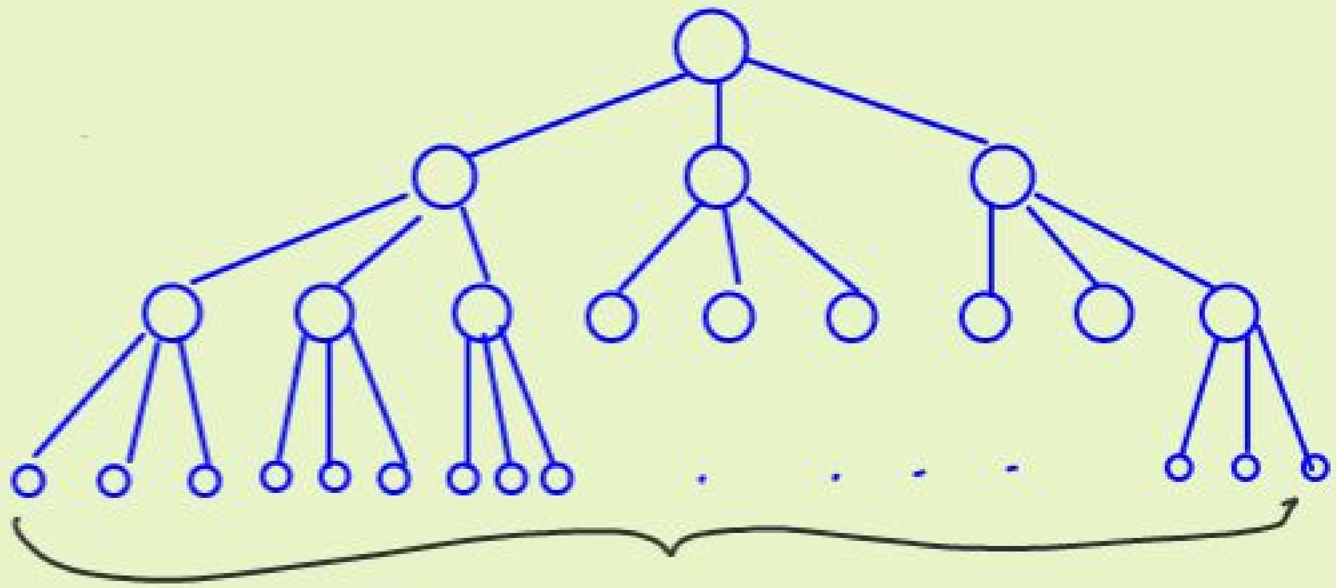


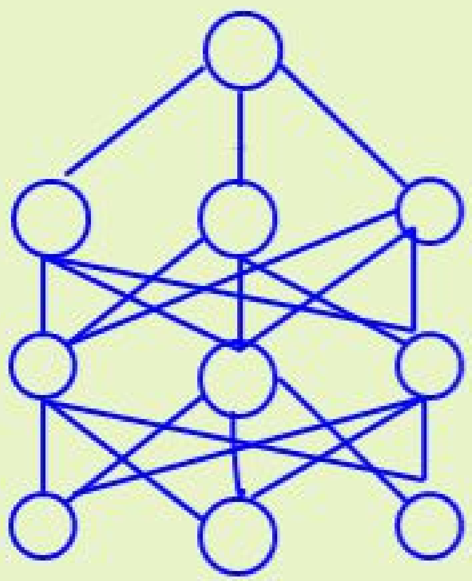


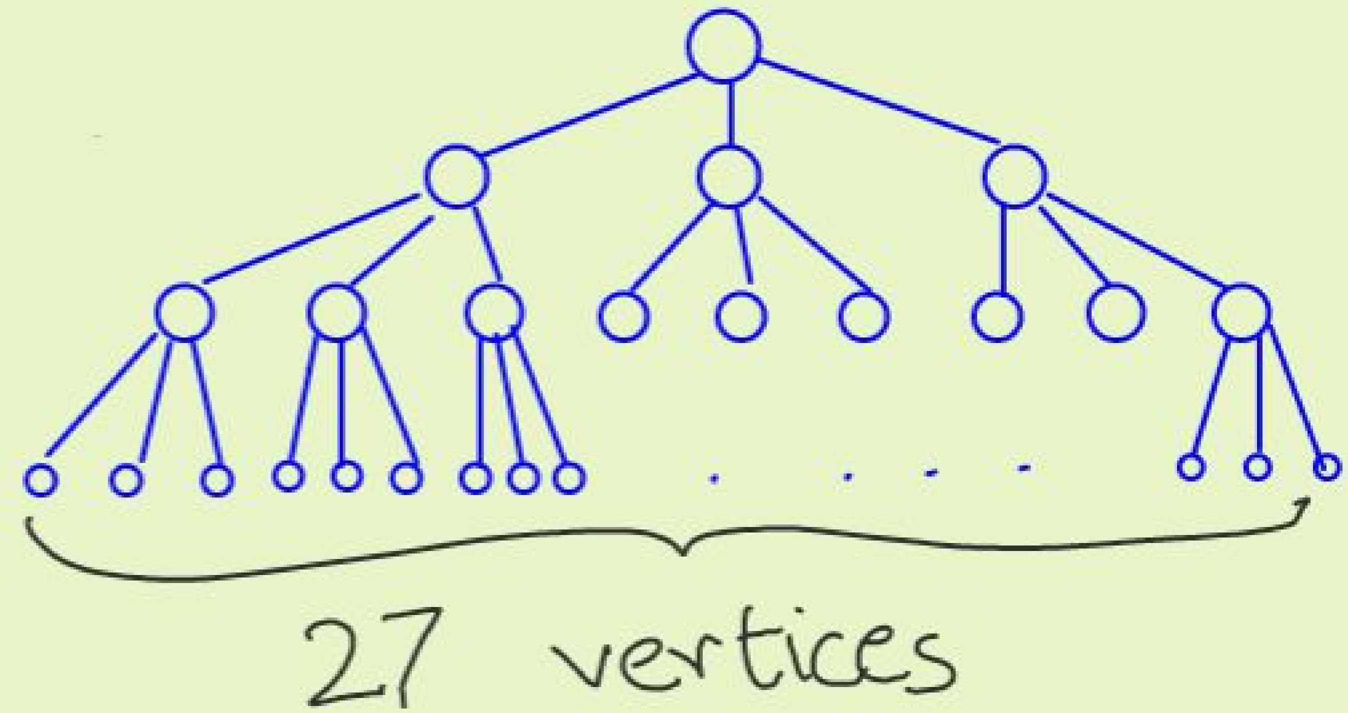
Circuit

size s $\longrightarrow$ Formula size s^d

depth d | depth d

Converting circuits to formulas





Circuit		Formula
size s	$\longrightarrow$	size s^d
depth d	Optimal?	depth d

Question

Can every circuit of size s be con-
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Expected answer: NO

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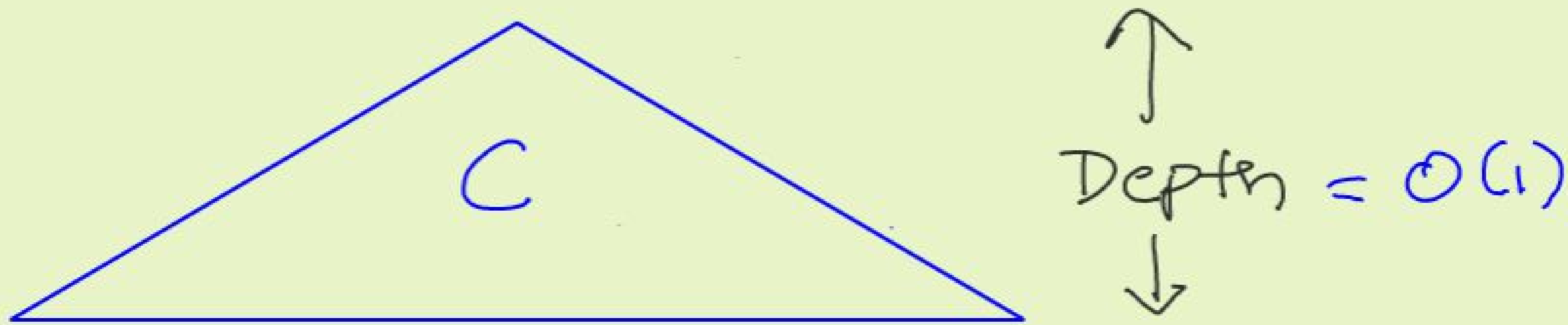
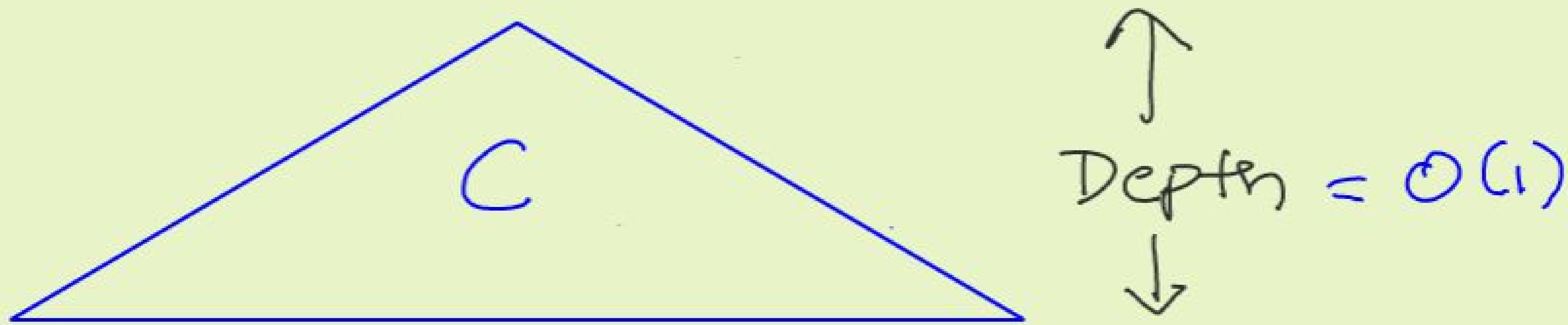
This work: $d = O(1)$

AC^0 ckts



Constant-depth $\{\wedge, \vee, \neg\}$ circuits

AC^0 ckts



Constant-depth $\{\wedge, \vee, \neg\}$ circuits

Eg: $d = 2$: DNF / CNF formulas.

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2. There are f such that any AC^0 circuit computing f has size $\geq 2^{n/2}$.

Background on AC^0

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$\vee(1, x_2, x_3) = 1$

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[Ajtai, Furst-Saxe-Sipser, ..., Håstad 80's]

Any AC^0 depth d ckt for PARITY has size $\geq 2^{\Omega(n^{1/d-1})}$.

Håstad's theorem

Any AC^0 depth d circuit for PARITY has size $2^{\Omega(n^{1/d-1})}$

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Folklore: PARITY has circuits of size $2^{O(n^{1/d-1})}$

Rossman's theorem

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Fact: PARITY has depth d circuits of size $2^{O(n^{1/d-1})}$

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Fact: PARITY has depth d circuits of size $2^{O(n^{1/d-1})}$

Corollary: For $d = O(1)$, trivial circuit to formula conversion is tight.

$AC^0[\oplus]$ - a stronger ckt class

Stronger circuit classes?

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Similar result for $AC^0[\oplus]$.

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<u>Corollary</u>: Trivial conversion of $AC^0[\oplus]$ circuits to formulas is tight.

Boolean functions & Polynomials

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3. $\deg(P) \leq n$.
4. P is unique.

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③ $f = \vee(x_1, \ldots, x_n)$

$= 1 + \prod_i (1 + x_i)$ $\qquad \deg(f) = n$

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Can be salvaged by

Polynomial Approximation

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Eg: $\wedge(x_1, \ldots, x_n)$ is $1/2^n$-approximated by the zero polynomial.

Polynomial approximation lemma

[Razborov 80_s]: f computed by an $AC^0[\oplus]$ circuit of size s and depth d.

f $1/10$-approximated by a poly. p of degree $O(\log s)^{d-1}$.

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$s = n^{O(1)} \Longrightarrow$ deg. $O(\log n)^{d-1} \ll n$

Approximation lower bounds

$$MAJ(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } |x| > n/2 \\ 0 & \text{otherwise} \end{cases}$$

$$|x| = \left| \{ i \mid x_i = 1 \} \right|$$

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[Razborov] Any P $1/4$-approximating MAJ must have deg $\Omega(\sqrt{n})$.

Lemma 1: f has $AC^0[\oplus]$ ckt. size s, depth d

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$$\Downarrow$$

Any $AC^0[\oplus]$ ckt of depth d for MAJ

has size $\geq 2^{\Omega(n^{1/2(d-1)})}$

Tightness of Razborov's theorem

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MAJ has AC^0 ckts of size

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$$2^{O(n^{1/d-1} (\log n)^{1-1/d-1})}$$

$d=3$: lower bound $2^{\Omega(n^{1/4})}$

upper bound $2^{O(\sqrt{n}\log n)}$

Variations in our proof

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$\Downarrow$ lemma 2 from Razborov

Any $AC^0[\oplus]$ formula of depth d for MAJ has size $2^{\Omega(d \cdot n^{1/2(d-1)})}$.

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Defn: f a δ-approx. Majority (δ-A.M.)

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f — $\leq 1/10$ — P Poly. approx. lemma

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MAJ P

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$$\begin{tikzcd}
 & f \arrow[dl, dash, "\leq 1/10"'] \arrow[dr, dash, "\leq 1/10"] & \\
MAJ \arrow[rr, dash, "\leq 1/5"'] & & P
\end{tikzcd}$$

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Also: Any $AC^0[\oplus]$ formula for f

has size $\geq 2^{\Omega(d n^{1/2(d-1)})}$.

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Lemma: There is a $1/10$-AM f that has an AC^0 circuit of depth d & size $2^{O(n^{1/2(d-1)})}$

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Builds on: Valiant; Amano; Hoory, Magen, Pitassi.

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$$\Downarrow$$

Main theorem.

Stronger approx. lemma f has $AC^0[\oplus]$ formula of size s and depth d.

$\exists P$ of deg $\Delta = O\left(\frac{\log s}{d}\right)^{d-1}$ that $1/10$-approx. f.

Probabilistic polynomials

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$\mathbb{P}$ of degree d if it takes values from degree $\leq d$ polynomials.

Probabilistic polynomials

$\mathbb{P}$ a Probabilistic polynomial (PP) if it is a random polynomial from $\mathbb{F}_2[x_1, \dots, x_n]$.

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$\mathbb{P}$ δ-computes $f : \mathbb{F}_2^n \to \mathbb{F}_2$ if

$$\forall x \in \mathbb{F}_2^n \qquad \Pr_{\mathbb{P}}[\mathbb{P}(x) \neq f(x)] \leq \delta.$$

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Check: For any $x_1, x_2 \in \mathbb{F}_2$

$$\Pr_{y_1, y_2}[\mathbb{P}(x_1, x_2) \neq \mathrm{AND}(x_1, x_2)] \leq 1/4$$

Probabilistic polynomials

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(Also true for OR_n)

Probabilistic polynomials

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Stronger approx. lemma f has $AC^0[\oplus]$ formula of size s and depth d.
$\exists \mathbb{P}$ of deg $\Delta = O\left(\frac{\log s}{d}\right)^{d-1}$ that $1/10$-computes f.

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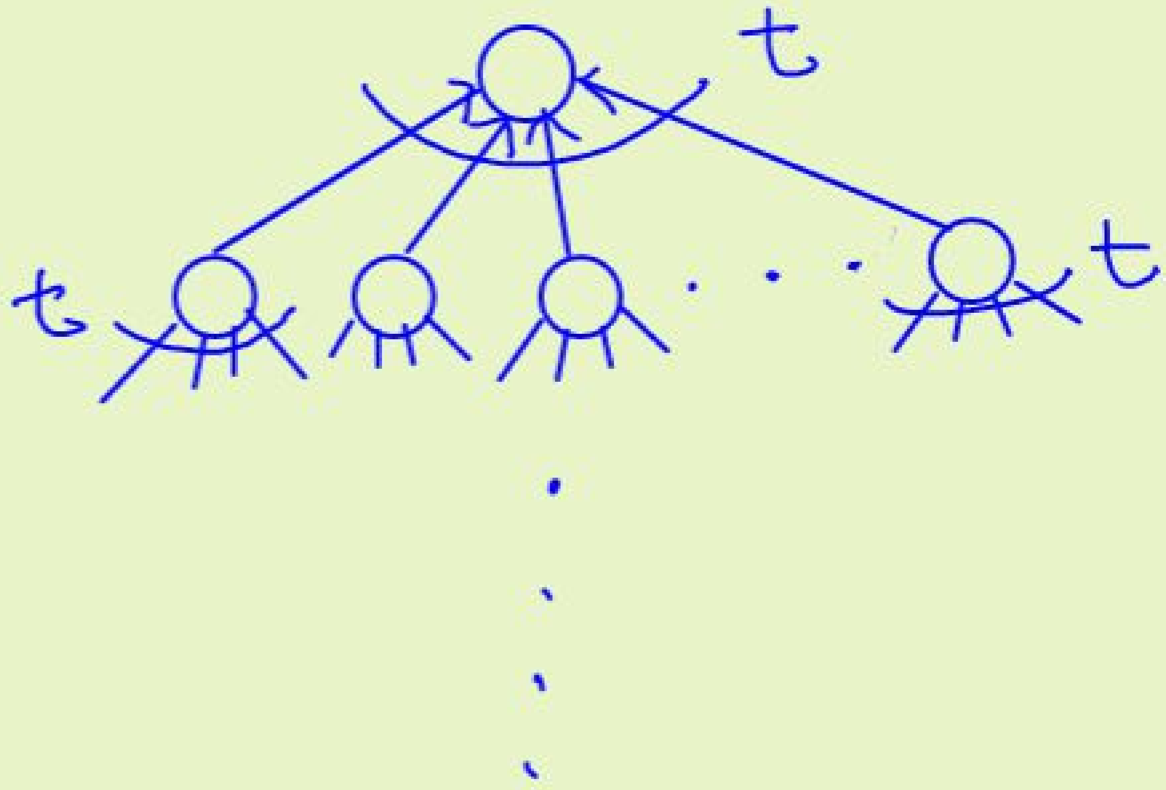
Simpler case:



t - Regular formula

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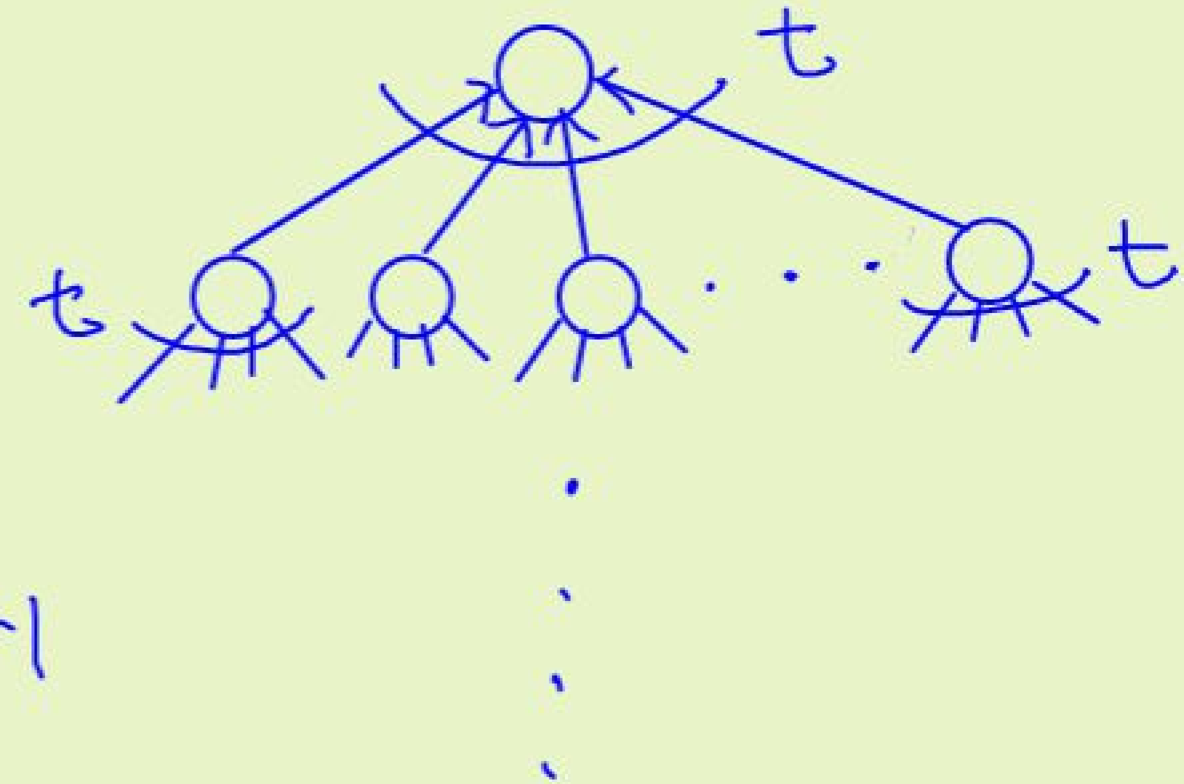

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<u>Pf</u>: Induction on d

$d=1$: Razborov's construction of probabilistic polys for OR_t / AND_t of deg $O(1)$.

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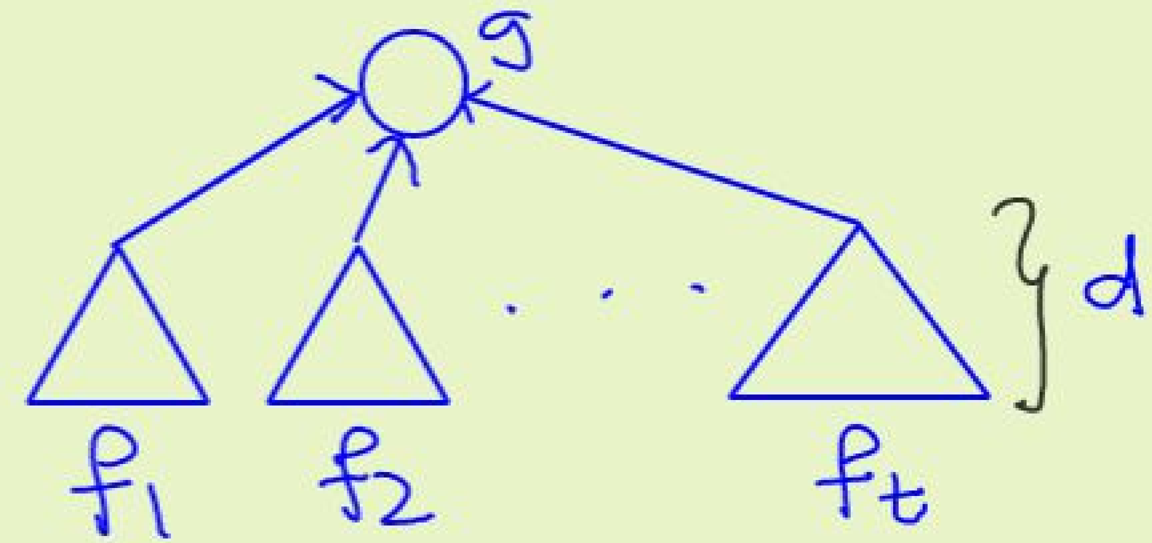
Pf: $d \to d+1$



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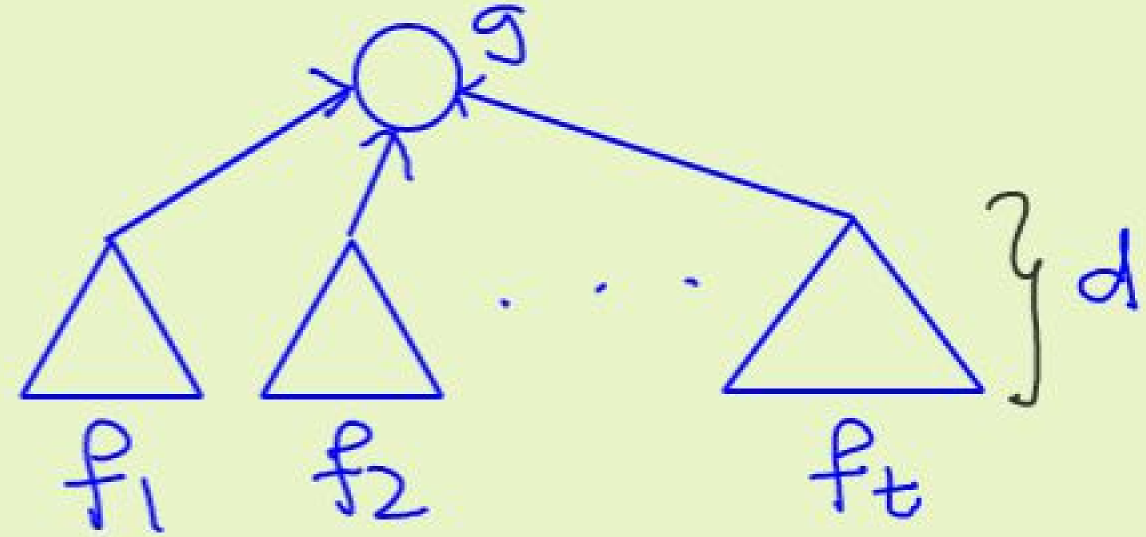

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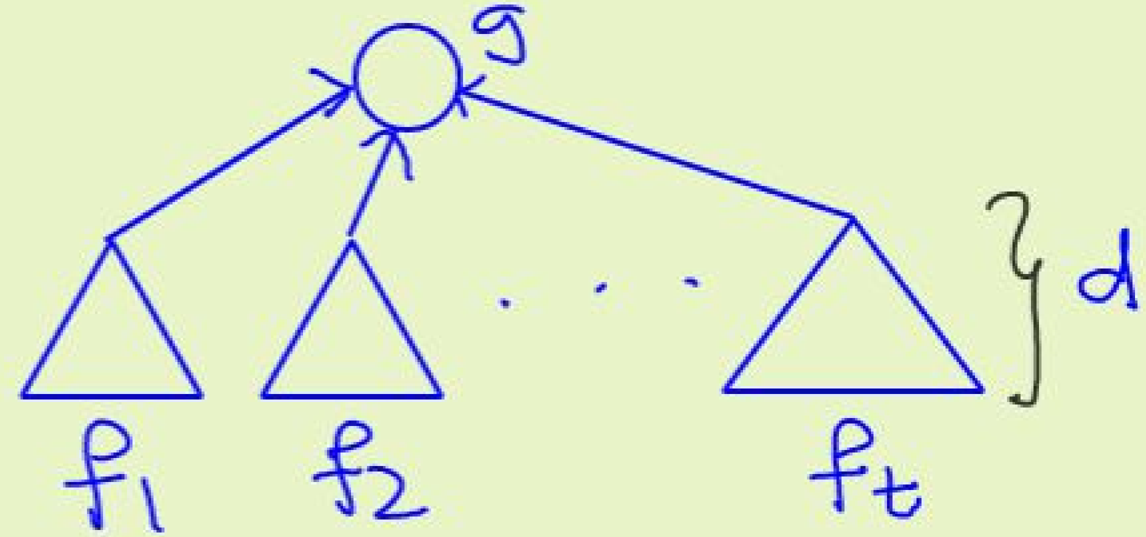

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$\Downarrow$ Error reduction

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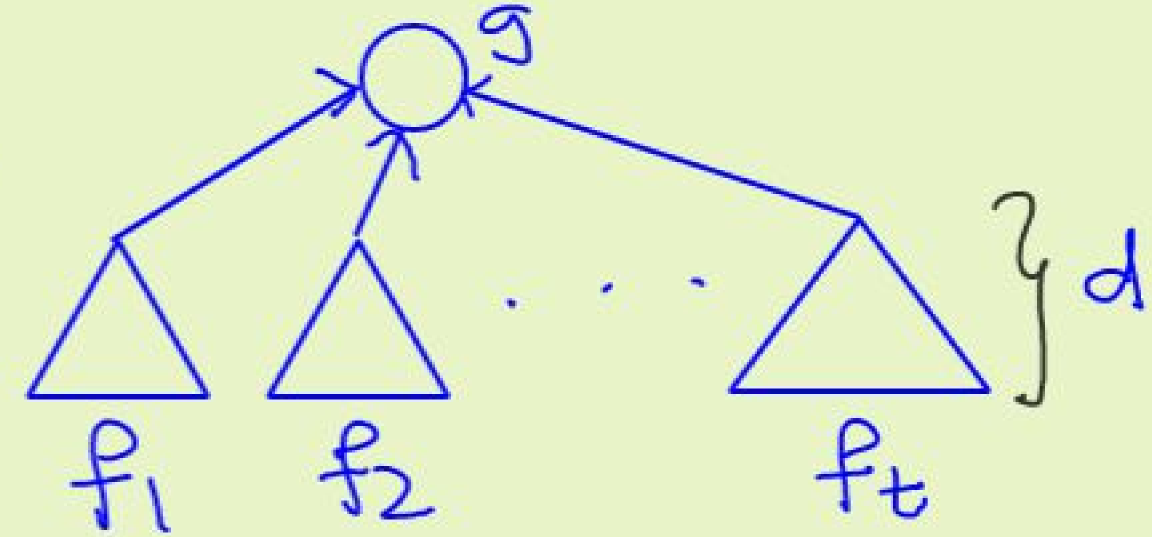

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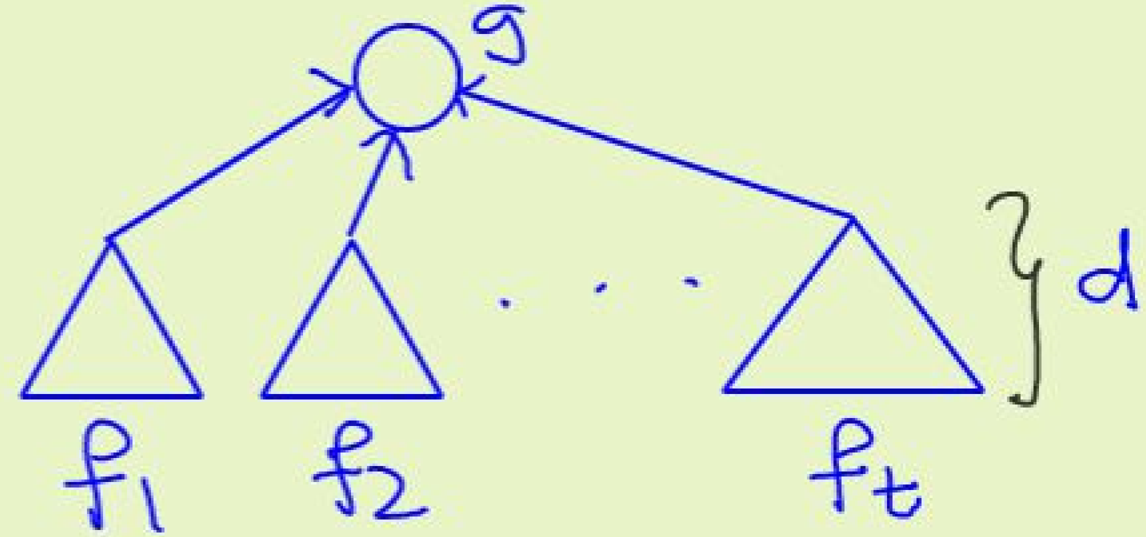

$\mathbb{P}_i'$ $1/20t$-computing f_i

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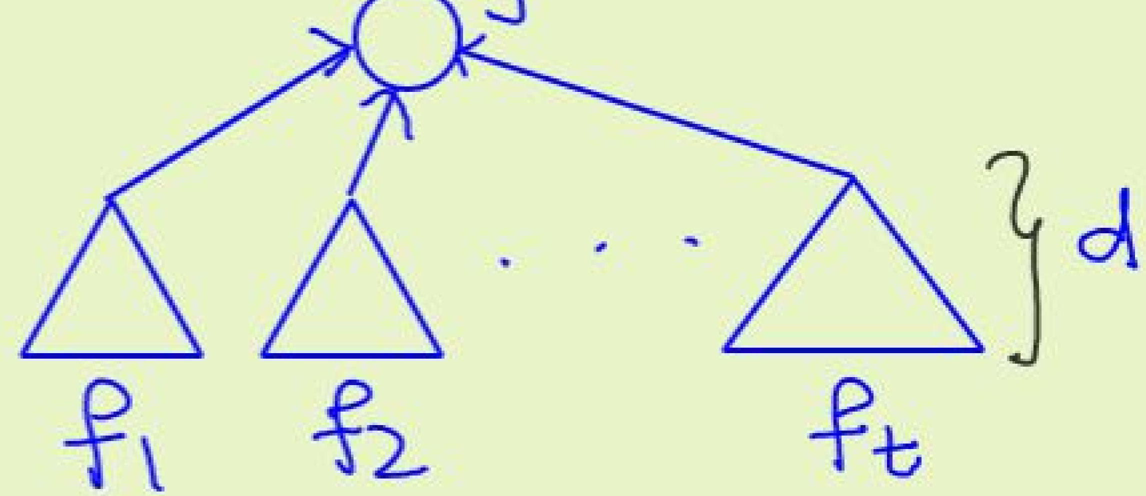

$\mathbb{P}'_i$ $\frac{1}{20t}$-computing f_i

of deg $O(\log t)^d$

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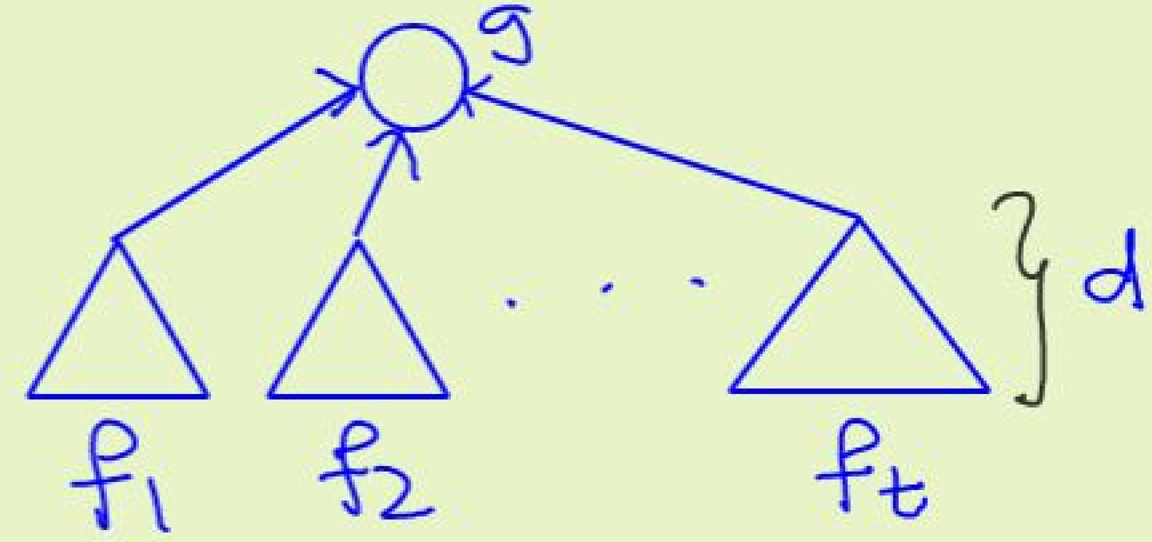
Irregular case



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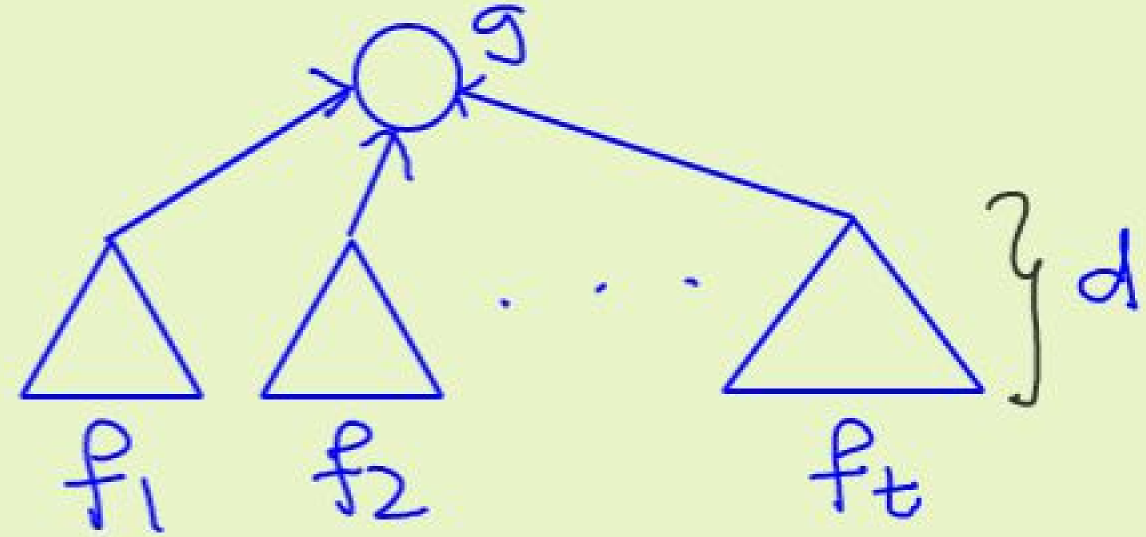
Irregular case

Π_i $\frac{1}{10}$ - computes f_i. of deg. Δ_i



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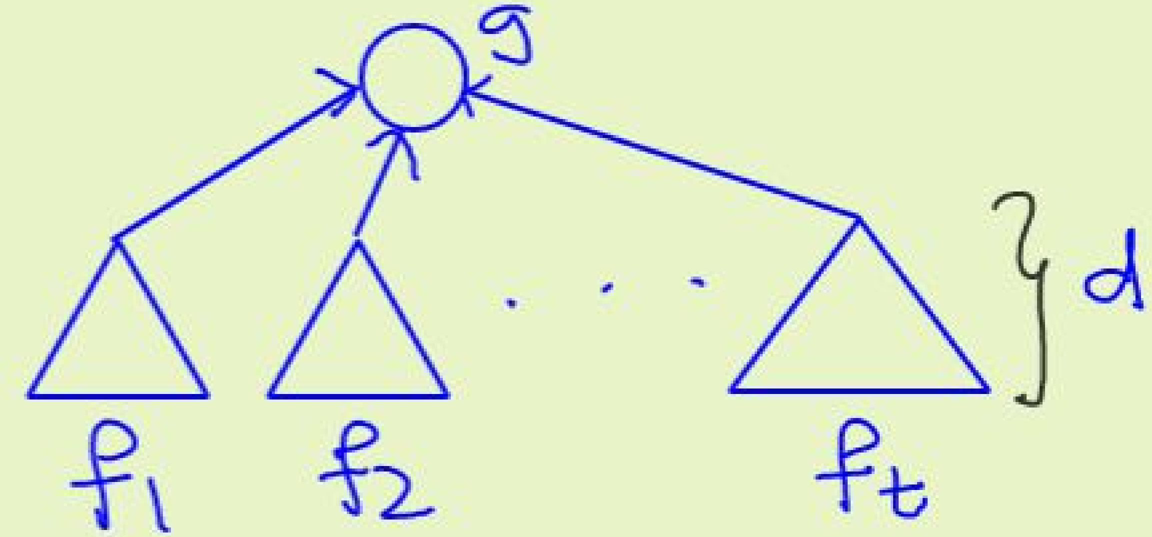

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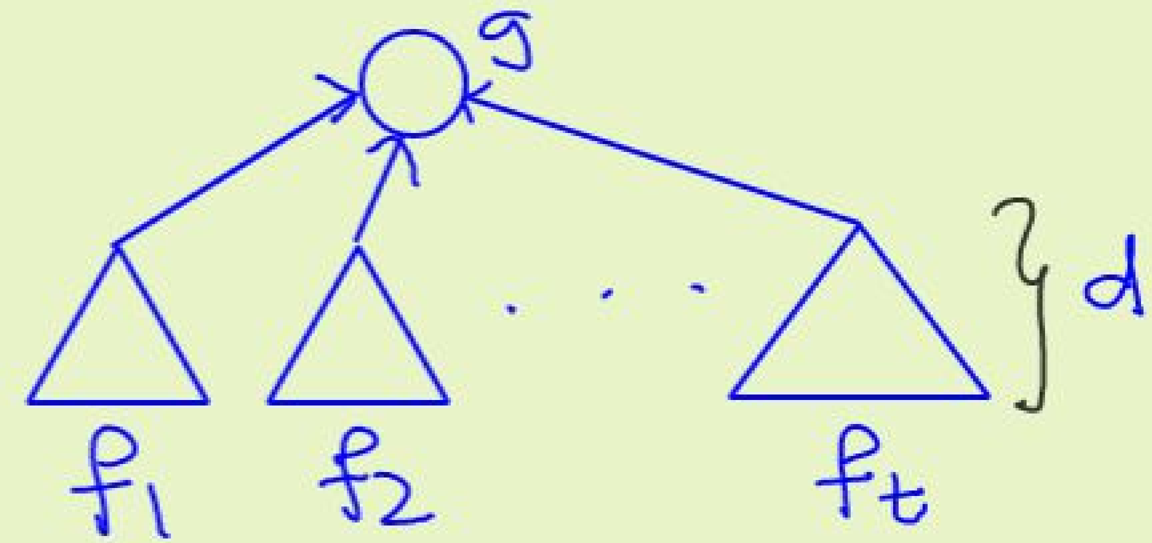

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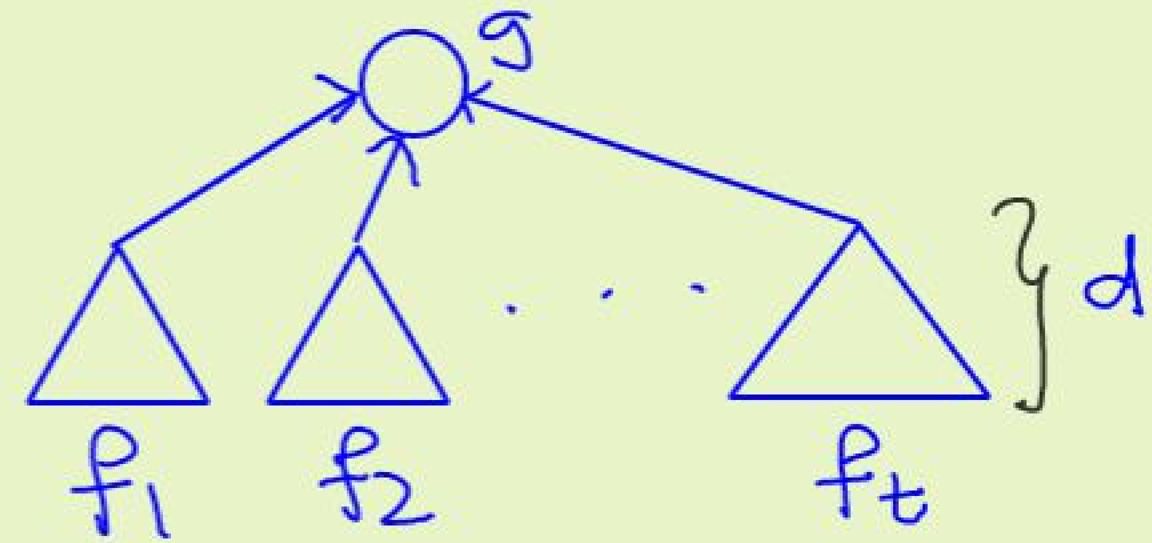

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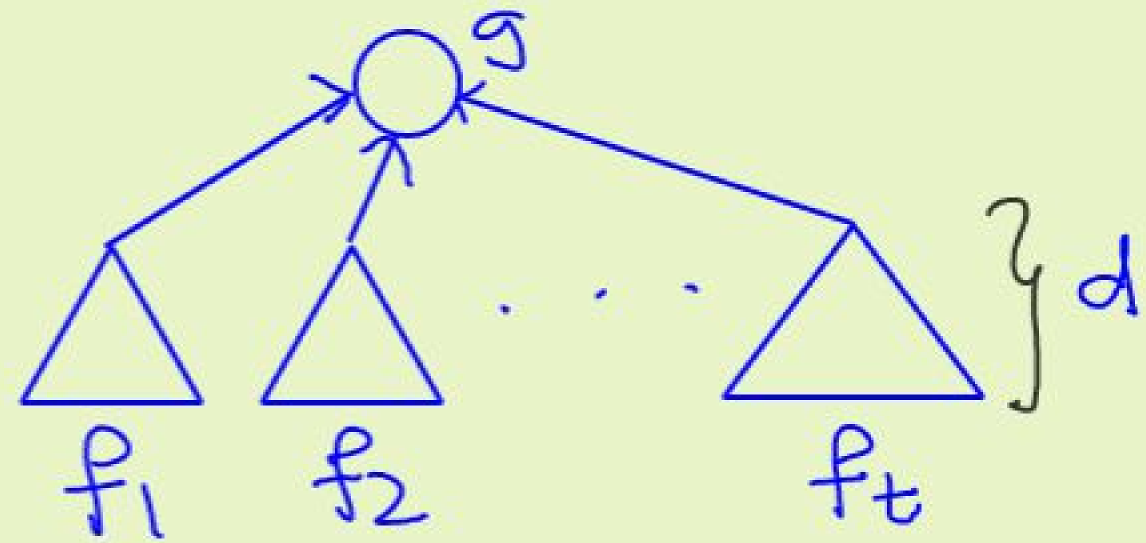

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$\deg(\mathbb{P}) = \deg(\tilde{\mathbb{P}}) \cdot \max_i \deg(\mathbb{P}_i')$

Questions

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① Optimal AC^0 circuit for MAJ

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Thanks!