Non-commutative computations: lower bounds and polynomial identity testing

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Joint work with G. Lagarde, S. Perifel

IMSc Workshop on Arithmetic Complexity

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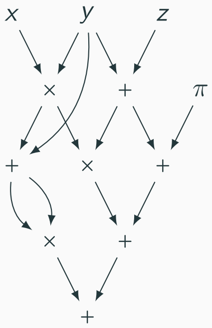
Introduction

Arithmetic circuits



Non-commutative circuits

- $\mathbb{F}$ commutative field.
- **Non-commutative** : $xy \neq yx \rightarrow$ distinguish left and right arguments in a computation gate.
- Various motivations

Some results

- ☹ No better lower bound for NC circuits than for commutative circuits

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  But for ABPs (Algebraic Branching Programs) :
- ☺ (Nisan 1991) Exact characterization of complexity
- ☺ (Nisan 1991) Exponential lower bounds for the *permanent*
- ☺ (Arvind, Joglekar, Srinivasan 2009) Deterministic poly-time PIT

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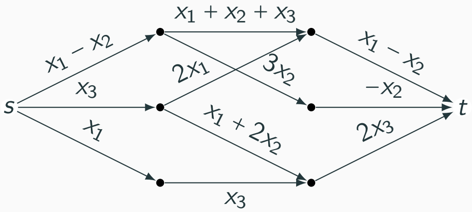
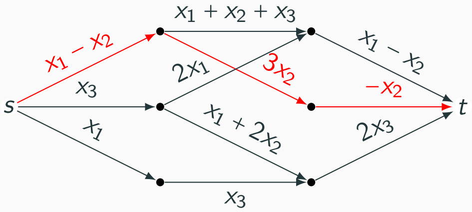
  Also (Limaye,Malod,Srinivasan 2016) Exponentiel lower bounds for *skew* circuits

Nisan's results

ABP (Branching programs)



ABP (Branching programs)

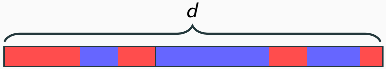


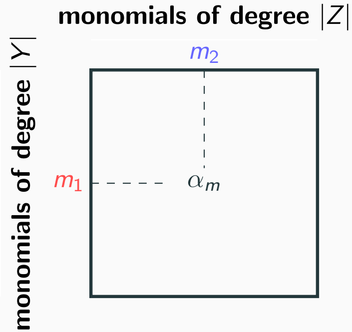
$(x_1 - x_2)(3x_2)(-x_2)$

- **DAG** : source s, sink t, edges with linear forms
- **Weight of a path** : product of edge weights
- **Computed polynomial** : sum of path weights from s to t.

Coefficient matrices

- $\Pi = ({\color{red}Y}, {\color{blue}Z})$ partition of $[d]$





- $f = \sum_m \alpha_m . m$, homogeneous, degree d, n variables

- Define matrix $M^{\Pi}(f)$

$$m \ t.q \begin{cases} m_{|Y} = m_1 \\ m_{|Z} = m_2 \end{cases}$$

Coefficient matrices

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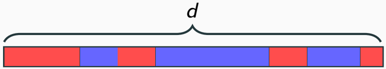

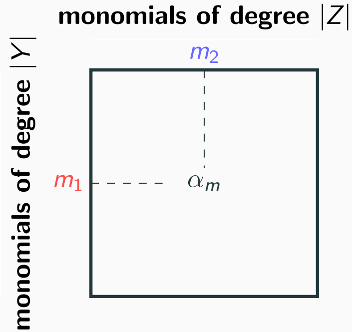

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- Define matrix $M^{\Pi}(f)$

- Complexity measure : $\text{rank}(M^{\Pi}(f))$.

Exercise: the palindrome polynomial

$$w = (w_1, \ldots, w_{d/2}) \in [n]^{d/2} \longrightarrow w^R = (w_{d/2}, \ldots, w_1)$$

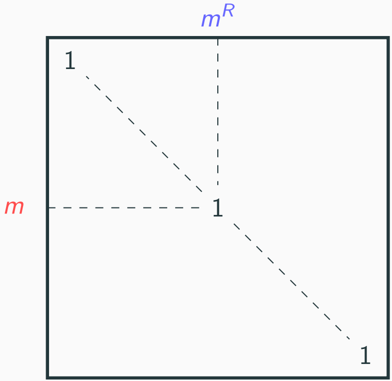
$$\tilde{x}_w = x_{w_1} x_{w_2} \ldots x_{w_{d/2}}$$

$$\mathrm{Pal}_d\, X = \sum_{w \in [n]^{d/2}} \tilde{x}_w \cdot \tilde{x}_{w^R}$$

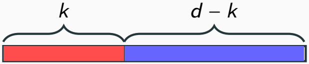
$$\mathrm{Pal}_{d+1}\, X = \sum_{i=1}^{n} x_i \cdot \mathrm{Pal}_d\, X \cdot x_i$$

What is the matrix if we cut in the middle?

Exercise: the palindrome polynomial



- $\Pi_i = (\{1, 2, \ldots, k\}, \{k+1, k+2, \ldots, d\})$



Theorem (Nisan, 1991)

For any homogeneous polynomial f of degree d, the size of a smallest homogeneous algebraic branching program for f is equal to

$$\sum_{k=0}^{d} \operatorname{rank}(M_k(f))$$

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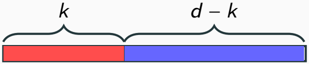


Theorem (Nisan, 1991)

For any homogeneous polynomial f of degree d, the size of a smallest homogeneous algebraic branching program for f is equal to

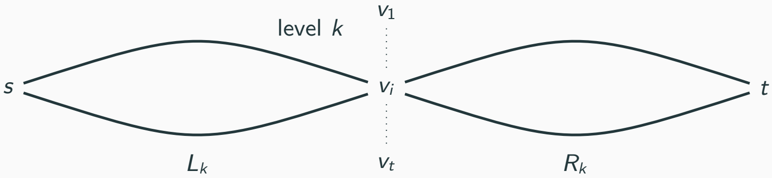
$$\sum_{k=0}^{d} \mathrm{rank}(M_k(f))$$

Corollary

Any homogeneous ABP computing the palindrome of degree d over n variables has size $\geq n^{d/2}$

Any homogeneous ABP computing the permanent has size $\geq 2^n$

Proof (lower bound)



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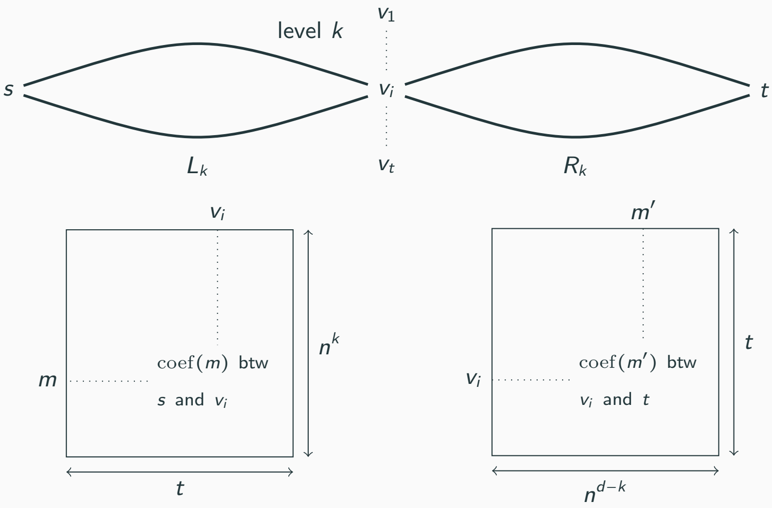

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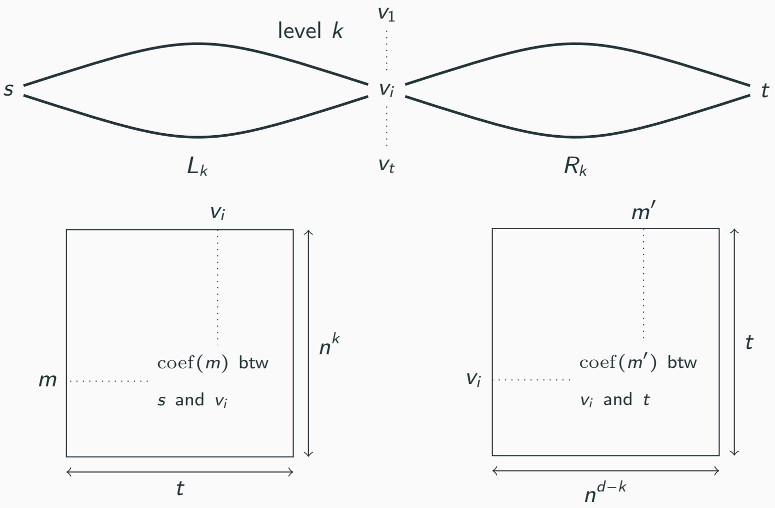
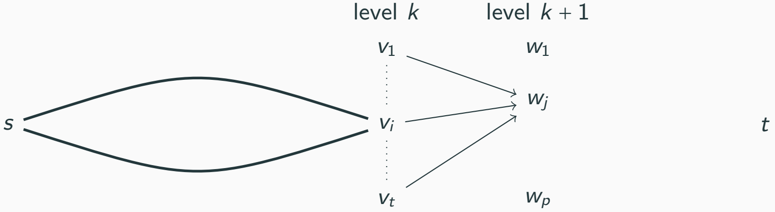


$$M_k(f) = L_k R_k \quad \text{and} \quad \text{rank}(M_k(f)) \leq t$$

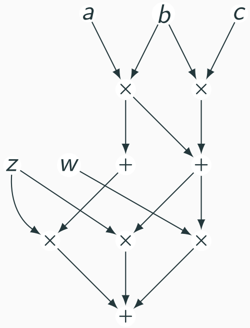
Proof (upper bound)



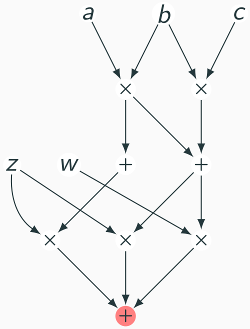
- suppose $\mathrm{rank}(L_k) < t$, then there is a column i which is a linear combination of the others
- the polynomial computed between s and v_i is a linear combination of the polynomials computed by the other vertices $v_j's$
- we could delete v_i and update the weights from level k to level $k+1$.
- so $\mathrm{rank}(L_k) = \mathrm{rank}(R_k) = t = \mathrm{rank}(M_k(f))$.

Unambiguous circuits

Parse trees



Parse trees



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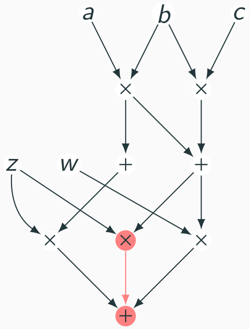

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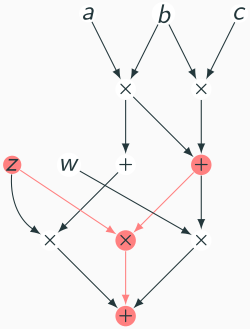

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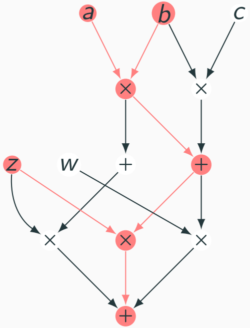


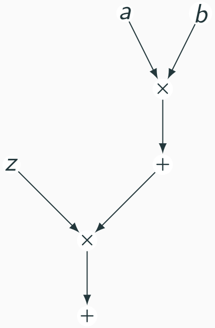


Figure 1: $val(T) = zab$

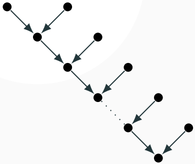
- Each parse tree computes a monomial.

Lemma

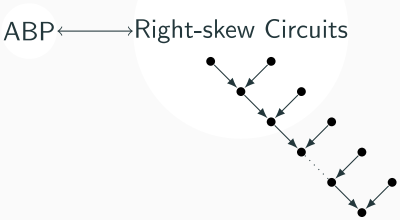
$f = \sum_{T} val(T)$

Parse trees of ABPs

ABP $\longleftrightarrow$ Right-skew Circuits



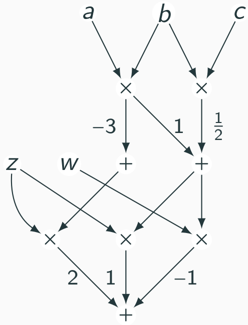
Parse trees of ABPs

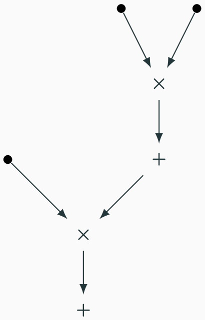


Définition

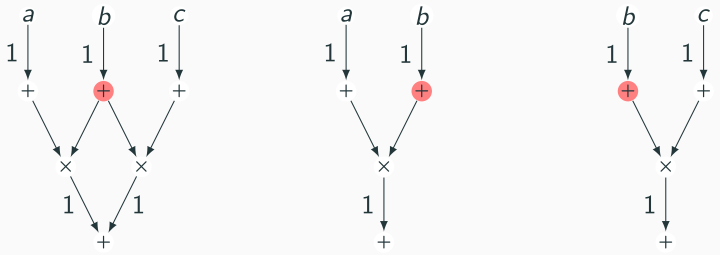
A circuit is unambiguous if all its parse trees are isomorphic.

Unambiguous circuits

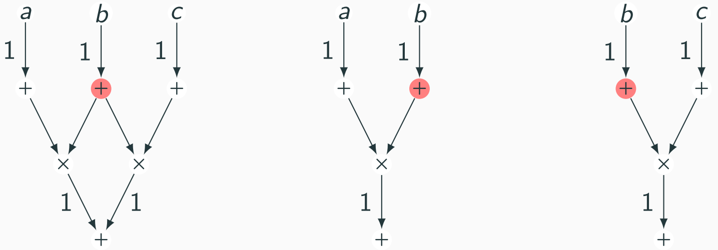




Canonical circuits

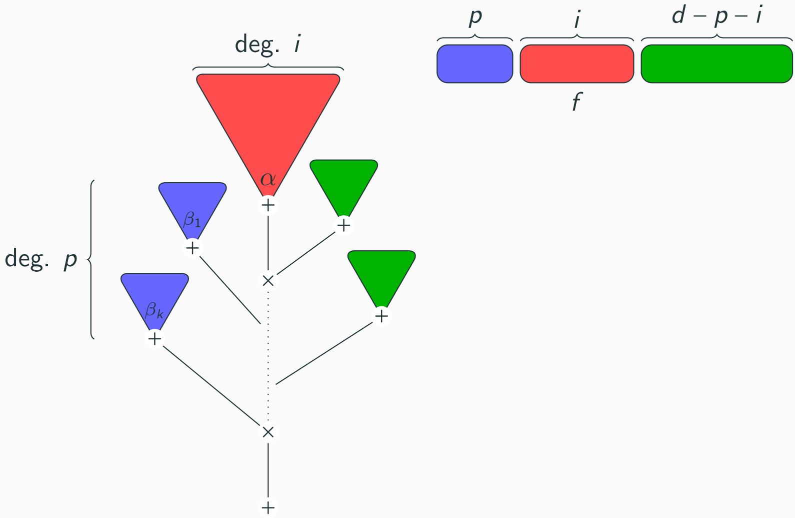


Canonical circuits



Any unambiguous circuit can be rendered canonical at a polynomial cost.

Type of a gate



Results

Theorem

Let P be a homogeneous polynomial of degree d and $\mathcal{T}$ a shape with d leaves. Then the minimal number of addition gates needed to compute P by a canonical unambiguous circuit with shape $\mathcal{T}$ is exactly equal to

$$\sum_{(i,p)\in S} \operatorname{rank}\left(M^{(i,p)}(P)\right),$$

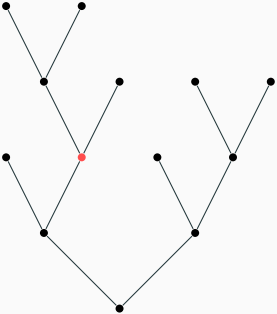
where S is the set of all existing types of +-gates in the shape $\mathcal{T}$.

Corollary

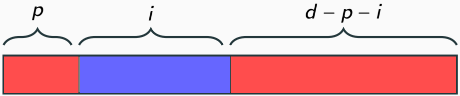
Any UC computing the permanent has size $2^{\Omega(n)}$

Proof (lower bound)

Parse tree shape



$\Pi_{(p,i)} =$



- $rg(M^{\Pi_{(p,i)}}(f)) \leq$ number of gates of type (p, i)

Proof (upper bound)

Proof (upper bound)

Clearly it works

Other results

PIT

Hadamard product (Arvind, Joglekar, Srinivasan)

Given two polynomials $P = \sum_{\vec{x}} a_{\vec{x}} \vec{x}$ and $Q = \sum_{\vec{x}} b_{\vec{x}} \vec{x}$, the *Hadamard product* of P and Q, written $P \odot Q$, equals $\sum_{\vec{x}} a_{\vec{x}} b_{\vec{x}} \vec{x}$.

Hadamard product of two unambiguous circuits

Let $\mathcal{C}$ and $\mathcal{D}$ be two unambiguous circuits in canonical form, of the same shape, and of size s and s', that compute two polynomials P and Q. Then $P \odot Q$ is computed by an unambiguous circuit of size at most ss'; moreover, this circuit can be constructed in polynomial time.

Theorem

There is a deterministic polynomial-time algorithm for PIT for polynomials computed by non-commutative unambiguous circuits over $\mathbb{R}$ (or $\mathbb{C}$).

Relationship to other classes

$ABP \subsetneq UC$

There are polynomials computed by polynomial-size UC that need exponential-size ABPs.

UC and skew are incomparable

There are polynomials computed by polynomial-size UC that need exponential-size skew circuits.
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The future

- Lower bounds for circuits with “similar” shapes.
- Lower bounds for circuits with not too many shapes.
- Poly-time PIT for a sum of UC circuits.

The future (much later?)

Lower bounds for general non-commutative circuits?

The future (much later?)

Lower bounds for general non-commutative circuits?

Theorem (Limaye, Malod, Srinivasan 2016)

There exists a polynomial computed by a small non-commutative circuit which is full rank for any partition.

Thank you!