SUCCINCT HITTING SETS
AND
BARRIERS TO PROVING ALGEBRAIC CIRCUITS LOWER BOUNDS

Ben Lee Volk

Joint with

Michael A. Forbes
Amir Shpilka

ALTERNATE TITLE:

HOW NOT TO PROVE ALGEBRAIC CIRCUITS LOWER BOUNDS

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WHY IS IT HARD TO PROVE CIRCUIT LOWER BOUNDS?

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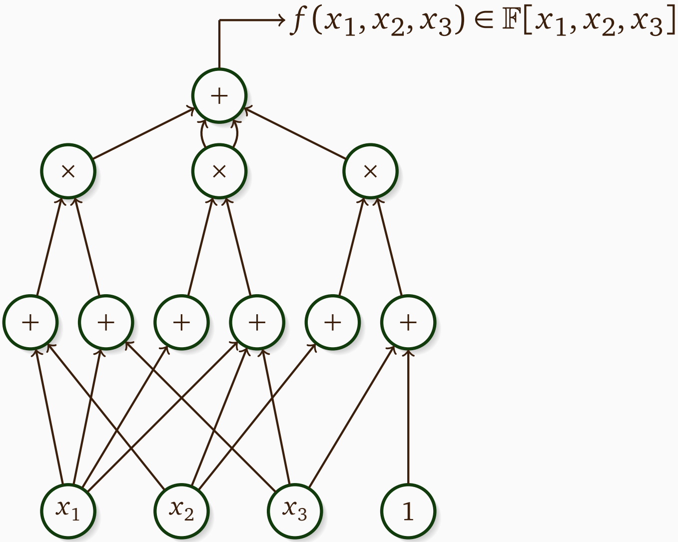
[Razborov-Rudich]: Most known lower bounds are natural and if there's a pseudorandom function in $\mathscr{C}$ then no natural lower bound against $\mathscr{C}$.

ALGEBRAIC CIRCUITS



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(also asked by **[Aaronson-Drucker]** and **[Grochow]**)

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Thus, the property $\{g : \det(M'_g) \neq 0\}$ is useful, constructive (determinant is efficiently computable) and large.

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Toy example: $\mathscr{C}$ is the class of perfect squares among polynomials of the form $ax^2 + bx + c$, and $D(a, b, c) = b^2 - 4ac$.

[Grochow]: almost all known lower bounds can be cast in this form.

GEOMETRIC COMPLEXITY THEORY

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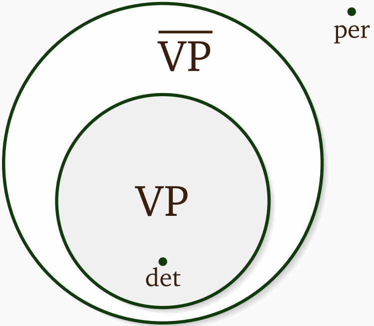

Geometric Complexity Theory

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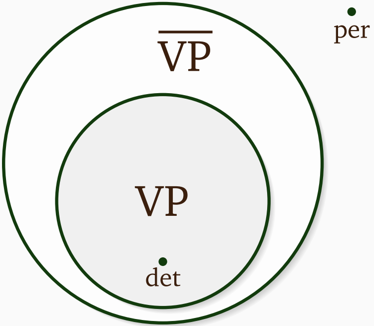

$\overline{\mathrm{VP}}$ is a zero set of a set of polynomials $\mathscr{T}$. What is the complexity of $\mathscr{T}$?

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In other words: if $\{\mathrm{coeff}(f) : f \in \mathscr{C}\}$ is a hitting set for a class $\mathscr{D}$, then no natural proof for $\mathscr{C}$ with the distinguisher coming from $\mathscr{D}$.

SUCCINCT HITTING SETS

Def: Let $\mathscr{C} \subseteq \mathbb{F}[x_1, \ldots, x_n]$ be a class of degree d polynomials, and $\mathscr{D} \subseteq \mathbb{F}[X_1, \ldots, X_N]$ for $N = \binom{n+d}{d}$. $\mathscr{C}$ is a **succinct hitting set** for $\mathscr{D}$ if $\mathscr{H} := \{\mathrm{coeff}(f) : f \in \mathscr{C}\}$ is a hitting set for $\mathscr{D}$.

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Question: are $\mathrm{poly}(n)$ size and $\mathrm{poly}(n)$ degree circuits a hitting sets for $\mathrm{poly}(N)$ size and $\mathrm{poly}(N)$ degree circuits?

("does VP hit VP?")

Algebraic Natural Proofs Barrier

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(note: $\mathscr{H}$ may a-priori be infinite but we'll soon see that this actually implies there exists some small $\mathscr{H}'$)

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$\{G(\mathbf{x}, \alpha) : \alpha \in \mathbb{F}^\ell\}$ is a $\mathscr{C}$-succinct hitting set against $\mathscr{D}$.

So succinct generator $\implies$ succinct hitting set (and even a "uniform" one).

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We can, but

1. unnecessary for succinct hitting sets which imply barriers
2. $G(\mathbf{x},\alpha)$ might be in even smaller class (e.g., if $\mathbf{y}$ has high deg)

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In particular, if $\mathscr{H} := \{\mathrm{coeff}(f) : f \in \mathrm{VP}(n)\}$ is an (infinite) hitting set for $\mathrm{VP}(N)$, there is a $N^{\mathrm{poly}\log(N)}$ size hitting set ($\mathscr{H}$ is in the image of a universal circuit).

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Challange: establish this under some crypto hardness assumption.

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- sparse polynomials,

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Thm: coefficient vectors of $\mathrm{poly}(\log s, n)$ size multilinear formulas are hitting sets for $N = 2^n$ variate size s:

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- circuits composed with sparse polynomials of transcendence degree $O(1)$.

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We can’t make all known hitting sets succinct but we have some excuses (more on that later).

HOW TO CONSTRUCT A SUCCINCT HITTING SET

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A: Yes. Each $\mathbf{v}$ is a coefficient vector of a $\mathrm{poly}(n)$ $\Sigma\Pi$ circuit in $x_1, \ldots, x_n$ (only $\mathrm{poly}(n)$ monomials with non-zero coefficient). $\square$

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Cor: $\mathrm{poly}(\log s, n)$-$\Sigma\Pi\Sigma$ succinct hitting set for sparse polynomials.

MORE SUCCINCT HITTING SETS

In the paper: many other succinct generators, most of them follow from various combinations of basic constructs such as **Shpilka-Volkovich generator** and Gabizon-Raz's **rank condenser**, which we make succinct.

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Builds on a lot of previous work: **[Dvir-Shpilka, Karnin-Shpilka, Kayal-Saraf, Saxena-Seshadhri, Shpilka-Volkovich, Forbes-Shpilka, Forbes-Shpilka-Saptharishi, Beecken-Mittmann-Saxena, Agrawal-Saha-Saxena-Saptharishi,...]**.

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Cor: Super-polynomial lower bounds on defining equations of $\overline{\mathrm{VP}}$ in many models.

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Challenge: make it succinct, i.e., find a small circuit in $\{x_1, \ldots, x_n, y\}$ such that the coefficient of $\mathbf{x}_S$ is $y^{k^{\mathrm{bin}(S)} \mod p}$, $\mathrm{bin}(S) =$ integer whose binary expansion is the characteristic vector of S.

USEFULNESS OF KS

The generator $X_i \mapsto y_1^{k^i \mod p_1} \cdots y_m^{k^i \mod p_m}$ for $m = O(\log n)$ hits roABPs (in any order) and read-once determinants

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Our constructions are all small multilinear formulas, so we might need new ideas.

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THANK YOU