Matchings in Graphs

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1 Matching using Linear Programming

We look at the linear programming method for the maximum matching and perfect matching problems. Given a graph $G = (V, E)$, an integer linear program (ILP) for the maximum matching problem can be written by defining a variable x_e for each edge $e \in E$ and a constraint for each vertex $u \in V$ as follows:

$$\begin{aligned} \text{Maximize} \quad & \sum_{e \in E} x_e \text{ subject to} \\ & \forall u \in V \sum_{e \sim u} x_e \leq 1 \\ & \forall e \in E x_e \in \{0, 1\} \end{aligned}$$

Here $e \sim u$ denotes e is incident on u. It can be seen that the optimum solution to this ILP is indeed a maximum matching in G. Therefore an algorithm to solve ILP can be used to get a maximum matching in G. However, ILP is known to be NP-complete and hence there is no polynomial-time algorithm known for it.

LP relaxation One way to deal with this is to relax the integrality constraints and allow $x_e \in [0, 1]$ to get a linear program, which can be solved in polynomial-time. However, this gives rise to fractional matchings. Characteristic vectors of matchings in G can be seen as points in $\mathbb{R}^m$ where $m = |E|$. The convex hull of all the matchings forms a polytope called *the matching polytope* $\mathbb{M}$. However, the LP relaxation may give matchings that are outside $\mathbb{M}$. Figure 1 shows some examples.

It can be seen that, in examples (1) and (2) in Figure 1, the matching polytope contains all the fractional matchings which form the feasible region of the relaxed LP. However, in Example (3), the maximum value of the relaxed LP is attained at the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, which lies outside $\mathbb{M}$. We will see next that the matching polytope always contains all the fractional matchings if and only if the graph is bipartite.

Definition 1 (Matching polytope) *For a given graph G, the* matching polytope $\mathbb{M}$ *is the convex hull of all the matchings in G. Thus*

$$\mathbb{M} = \{\sum_i \lambda_i M_i \,|\, \forall i\, M_i \text{ is a matching and } \sum_i \lambda_i = 1\}$$

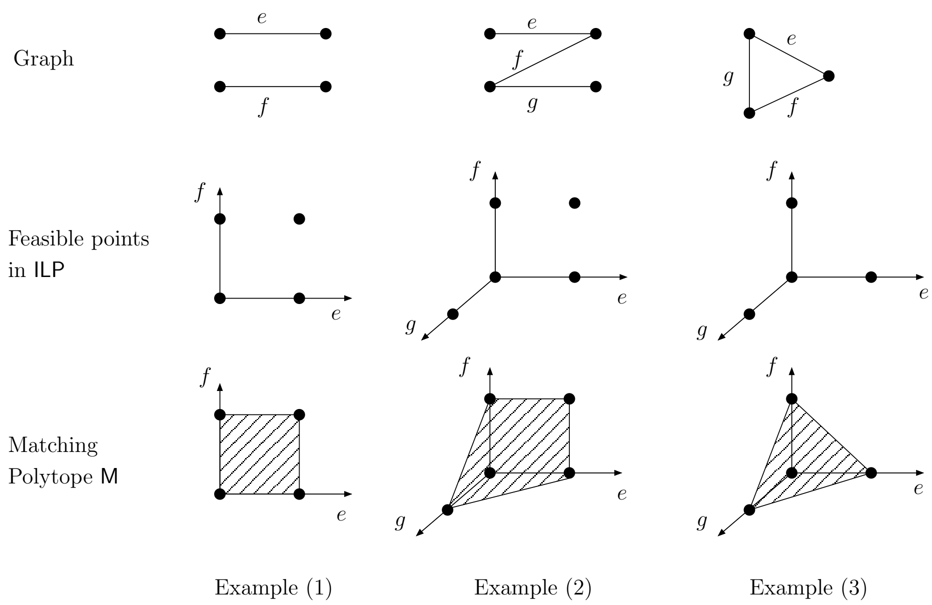

Figure 1: Some examples of LP relaxation

Definition 2 (Fractional matching polytope) *The* fractional matching polytope FM *is defined as the feasible region for the* LP

$$\forall u \in V \sum_{e \sim u} x_e \leq 1, \ \forall e \in E \, x_e \in [0,1]$$

For a graph G, M and FM may not be the same. But they are the same for bipartite graphs.

Claim 3 *For bipartite graphs, the* LP *relaxation gives a matching as an optimal solution.*

We define the *perfect matchings polytope* PM and the *fractional perfect matchings polytope* FPM.

Definition 4 (Perfect Matching Polytope) *For a given graph G, the* perfect matching polytope PM *is the convex hull of all the perfect matchings in G. Thus*

$$FM = \{\sum_i \lambda_i M_i \,|\, \forall i \, M_i \text{ is a perfect matching in } G \text{ and } \sum_i \lambda_i = 1\}$$

Definition 5 (Fractional Perfect Matching Polytope) *The* fractional perfect matching polytope FPM *is defined as the feasible region for the* LP

$$\forall u \in V \sum_{e \sim u} x_e = 1, \ \forall e \in E \, x_e \in [0,1]$$

Claim 6 *1.* M$\subseteq$ FM*,* PM$\subseteq$FPM

2. *If G is bipartite then* M=FM *and* PM=FPM

3. *If G is non-bipartite,* M$\neq$FM

Proof:

Proof of 1: It can be seen that any matching in G satisfies the constraints for FM. Thus the extreme points of M are all contained in FM and hence M$\subseteq$FM. Similarly, PM$\subseteq$FPM.

Proof of 2: Note that all the vertices of M are in $\{0,1\}^m$, where $m = |E|$. Further, any vertex in FM$\cap\{0,1\}^m$ is also in M. Thus, it suffices to prove that all the vertices of FM are integral. Assume that there is a non-integral vertex x of FM. We will show that x can be written as a convex combination of two points in FM, which contradicts the above assumption.

Define $G_x = (V, E_x)$ where $E_x = \{e | x_e \notin \{0,1\}\}$. Suppose G_x has a cycle C. Let a and b be the minimum and maximum values of x_e for $e \in C$ respectively. We refer to x_e as the weight on e. Define $\epsilon = \min 1 - b, a$. Define two matchings x^+ and x^- obtained from x by adding and subtracting ϵ from the weights on alternate edges of C. Then $x = \frac{1}{2}(x^+ + x^-)$. (See *e.g.* Figure 2.)

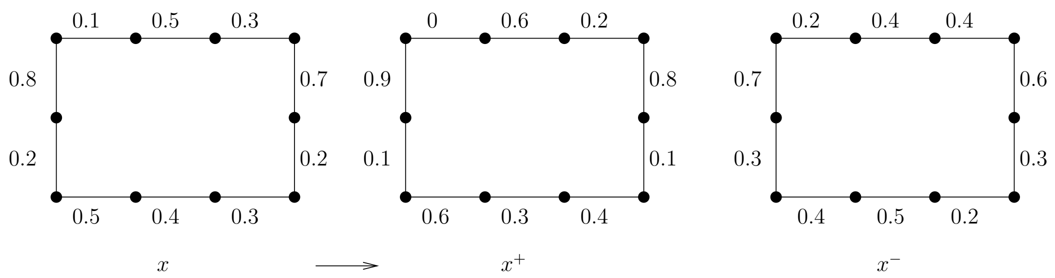

Figure 2: Expressing a cycle in x as a convex combination of x^+ and x^-

If G_x has no cycle, pick any maximal path in G_x. The end-points of the path have weights strictly less than 1 on the edges incident on them. Define a, b, ϵ and construct x^+ and x^- as before, which again contradicts the extremality of x.

Similar argument works for PM=FPM, except that the case G_x being acyclic does not arise.

Proof of 3: Let G be non-bipartite. Take an odd cycle C in G. Consider a fractional matching x that has weights $\frac{1}{2}$ on each of the edges in C. It can be seen that this matching is in FM but it can not be written as a convex combination of any two or more matchings. ■

Claim 7 *All the vertices of* FM *are half-integral.*

We will prove the following stronger version:

Theorem 8 *For any graph G, $x \in$*FPM *if and only if x satisfies the following conditions:*

1. $\forall e \in E\, x_e \in \{0, 1, \frac{1}{2}\}$

2. Let $G_0 = (V, E_0)$ *where* $E_0 = \{e \in E | x_e = \frac{1}{2}\}$*. Then* G_0 *is a collection of disjoint odd cycles.*

We first recall the following properties of a convex polytope $\mathbb{P}$:

1. A point x is a vertex of $\mathbb{P}$ if and only if x can not be written as a convex combination of any $y, z \in \mathbb{P}$.

2. There is a hyperplane H such that $\{x\} = H \cap \mathbb{P}$.

3. If $\mathbb{P}$ is given as an intersection of half-spaces, then x is a unique solution to a set of m linearly independent constraints met as equalities.

Proof of Theorem 8: First we prove that, if x satisfies the two conditions, then x is a vertex. Thus assume that $x \in \{0, 1, \frac{1}{2}\}^m$. Define

$$\begin{aligned} E_0 &= \{e \in E | x_e = \frac{1}{2}\} \\ E_1 &= \{e \in E | x_e = 1\} \\ E_2 &= \{e \in E | x_e = 0\} \end{aligned}$$

To show that x is a vertex, define a hyperplane $H = \{y | \sum_e w_e y_e = a\}$, $w_e = -1$ if $x_e = 0$ and $w_e = 0$ if $x_e > 0$. As $w^T x = 0$, $a = 0$. Suppose $\{y\} = H \cap \mathsf{FPM}$. Then $y|_{E_2} = 0$. Thus $y_e = 0$ whenever $x_e = 0$. From the constraints of FPM, it can be seen that $y|_{E_1} = 1$ and $y|_{E_0} = \frac{1}{2}$, since we have only odd cycles in the graph restricted to E_0. Thus y can not be different from x and thus x is a vertex with $\{x\} = H \cap \mathsf{FPM}$.

Now we prove the other direction. Let x be a vertex of FPM. We will prove that x satisfies the two conditions. Let H be the hyperplane such that $\{x\} = H \cap \mathsf{FPM}$. Further, let $H = \{z | w^T z = a\}$. Interpret the adjacency matrix of G as bipartite adjacency matrix. Equivalently, define a bipartite graph $G' = (V', V'', E')$ such that $V' = V'' = V$. Thus each $u \in V$ has a copy $u' \in V'$ and $u'' \in V''$. Further, for each $E' = \{(u', v''), (v', u'') | (u, v) \in E\}$. Thus $|E'| = 2m$. Each matching x in G also has a corresponding matching y in G' such that $(u, v) \in x \Rightarrow (u', v''), (v', u'') \in y$. As G' is bipartite, $y \in \mathsf{FPM}(G')$ and $y \in \mathsf{PM}(G')$.

Define hyperplane $H' = \{z | w'^T z = 2a\}$ where w' is obtained by concatenating the vector w with itself. Clearly, $x \in H \Leftrightarrow y \in H'$. Now we show that $\{y\} = H' \cap \mathsf{FPM}(G') \Leftrightarrow \{x\} = H \cap \mathsf{FPM}(G)$.

Let $\{z\} = H' \cap \mathsf{FPM}(G')$. Therefore $w'^T z = 2a$. Define a new vector x' such that $\forall e \in E, x'_e = \frac{z_{e_1} + z_{e_2}}{2}$ where e_1 and e_2 are copies of $e \in E$. Clearly, $w^T x' = a$ and $x' \in \mathsf{FPM}(G)$. Therefore $x' \in H \cap \mathsf{FPM}$and thus $x' = x$. Therefore $z = y$ and hence y is integral. This shows that x is half-integral, which proves the first condition of the theorem.

For the second condition, let x have an even cycle with weights $\frac{1}{2}$ on all its edges. Then x can be written as a convex combination of two matchings which contain alternate edges from the cycle. Thus presence of an even cycle in G_0 implies that x is not a vertex. Therefore G_0 contains disjoint odd cycles. ■

As we have seen, for non-bipartite graphs, $\mathsf{FPM} \not\subseteq \mathsf{PM}$. This happens precisely because an odd subset of vertices can have a fractional perfect matching but not an integral one. Hence we need to introduce more constraints for the FPM polytope, which essentially require that at least one vertex of each odd subset be matched outside the subset. With this additional constraint, we define a new perfect matching polytope

$$\mathbb{P}(G) = \{x \in \mathbb{R}^m | x_e \geq 0,\, \forall e \in E \sum_{e \sim u} x_e = 1,\, \forall S \subseteq V,\, |S| \text{ odd },\, \sum_{e \in E(S,\bar{S})} x_e \geq 1\}$$

Similarly define a new fractional matching polytope

$$\mathbb{M}P(G) \;=\; \{x \in \mathbb{R}^m | x_e \geq 0,\, \forall e \in E \sum_{e \sim u} x_e \leq 1,\, \forall S \subseteq V,\, |S| \text{ odd },\, \sum_{e \in E(S)} x_e \leq \frac{|S|-1}{2}\}$$

Theorem 9 (Edmonds Theorem) $\mathsf{PM}(G) = \mathsf{P}(G)$

Before proving Edmonds theorem, we prove that Edmonds theorem implies the following claim:

Claim 10 $\mathsf{MP}(G) = \mathsf{M}(G)$

Proof: Construct a new graph $H = (V', E')$ as follows: Take two disjoint copies of G, say G_1, G_2 and for each $u \in V$, add edges (u_1, u_2) such that u_1 and u_2 are the copies of u in G_1 and G_2 respectively. For $x \in \mathsf{MP}(G)$, we construct $y \in \mathsf{P}(H)$ such that both the copies of the edge $e \in E$ have weight x_e in y. Moreover, for each vertex $u \in V$ that has a deficit in x, we add the edge (u_1, u_2) to y with a weight equal to the deficit of u in x.

We show that y satisfies the constraints of $\mathsf{P}(H)$. It is easy to see that y satisfies the first two constraints of $\mathsf{P}(H)$. To see that y satisfies the third constraint as well, consider an odd cardinality subset of vertices in H. Let $S = X_1 \cup Y_2$ where $X_1 \subseteq V(G_1)$ and $Y_2 \subseteq V(G_2)$. Let $\sum_e \in E'(S, \bar{S}) = \delta(S)$. Therefore

$$\delta(S) \geq \delta(X_1 \setminus Y_1) + \delta(X_2 \setminus Y_2)$$

where Y_1 and X_2 are copies of Y_2 in G_1 and of X_1 in G_2 respectively.

Without loss of generality, we assume that X and Y are disjoint in G, and also $Y \neq \emptyset$ and that $|X|$ is odd. Since $|X|$ is odd, there is a deficit of at least $|S| - \frac{|S|-1}{2}$, which is added to the edges going out of X. Thus y satisfies the third constraint too and hence $y \in \mathsf{P}(H)$. $\therefore x \in \mathsf{MP}(G) \Rightarrow y \in \mathsf{P}(H)$. But $\mathsf{P}(H) = \mathsf{PM}(H)$ by Edmonds theorem. Therefore we can write y as a convex combination of perfect matchings in H, from which, a convex combination of matchings for x can be computed. ■

Now we can restrict ourselves to perfect matchings and prove Edmonds theorem:
Proof of Theorem 9: We prove the theorem by contradiction. Let G be a graph such that $\exists x \in \mathsf{P}(G) \setminus \mathsf{PM}(G)$. Take the smallest such G *i.e.* a graph with minimum number of vertices and breaking ties by picking a graph with minimum number of edges, which satisfies this condition. Consider x as defined above for G. By minimality of G, we have the following:

1. $\forall e \in E$, $0 < x_e < 1$. Otherwise we can discard edges with $x_e = 0$, contradicting the minimality of G.

2. There are no pendant vertices in G.

3. $\exists v \in V$ such that degree$(v) > 2$. Otherwise G is a collection of disjoint cycles. The cycles can not be odd as G satisfies the constraints of $\mathsf{P}(G)$. Then G is bipartite and hence can not be a counter example by Claim 6. Thus $m > n$.

Without loss of generality, assume x to be a vertex of $\mathsf{P}(G)$. Therefore it is the unique solution of m linearly independent constraints satisfied as equalities. By 1 above, the constraints can not be of the form $x_e \leq 0$. There are only n constraints of the form $\sum_{e\sim u} x_e = 1$. Therefore at least one constraint should be of the form $\sum_{e\in E(S,\bar{S})} x_e \geq 1$. Let S be a subset where such a constraint is satisfied by x with equality. Further, $|S|, |\bar{S}| > 1$, otherwise the constraint will be of the form $\sum_{e\sim u} x_e = 1$.

Construct G_1 and G_2 as follows: In G_1, $\bar{S}$ is contracted to a single vertex $\bar{s}$, S remains the same as in G, and multiple edges obtained are replaced by single edges. Similarly, define G_2 by contracting S to s and leaving $\bar{S}$ as in G. Let x_1 be the restriction of x to G_1 such that the edges incident on $\bar{s}$ get a weight equal to the sum of the weights of the corresponding edges in x. Similarly construct x_2 as restriction of x to G_2. Clearly $x_1 \in \mathsf{P}(G_1)$. But G_1 is smaller than G and hence $\mathsf{P}(G_1)$ $=\mathsf{PM}(G_1)$. Similarly, $\mathsf{P}(G_2)$ $=\mathsf{PM}(G_2)$. Therefore x_1 and x_2 can be written as convex combinations of perfect matchings in G_1 and G_2 respectively:

$$
\begin{aligned}
x_1 &= \sum_i \alpha_i L_i \; L_i: \text{ perfect matching in } G_1, \sum_i \alpha_i = 1, \forall i\, \alpha_i \geq 0 \\
x_2 &= \sum_j \beta_j N_j \; N_j: \text{ perfect matching in } G_2, \sum_j \beta_j = 1, \forall j\, \beta_j \geq 0
\end{aligned}
$$

Consider perfect matchings L in G_1 and N in G_2 from above convex combinations. Let u_L (respectively v_N) be the vertex in G_1 (G_2) which is matched to $\bar{s}$ (s) in L (N). If $(u_L, v_N) \in E$, then $L \setminus \{(u_L, \bar{s})\} \cup N \setminus \{(s, v_N)\} \cup \{(u_L, v_N)\}$ is a perfect matching in G. Construct such perfect matchings M_{ij} for all the pairs (L_i, N_j) wherever $(u_{L_i}, v_{N_j}) \in E$. We claim that x can be written as a convex combination of M_{ij}s.

Claim 11 *$\exists \gamma_{ij}$ such that $x = \sum_{i,j} \gamma_{ij}\chi_{M_{ij}}$. In fact $\gamma_{ij} = \frac{\alpha_i\beta_j}{x_{u_i v_j}}$ such that $\forall e \in E(S)\, x_e = \sum_{i:e\in L_i} \sum_{j:v_j\sim u_i} \gamma_{ij}$, and $\sum_{i,j} \gamma_{ij} = 1$.*

The claim contradicts the assumption that x is a vertex of $\mathsf{P}(G)$. ■