Matchings in Graphs

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1 A General Algorithm for Matching

Let $G = (V, E)$ be a general undirected graph. We know from Berge's theorem that a matching M is maximum in G if and only if G has no augmenting paths with respect to M. This immediately gives us the following algorithm for maximum matching in undirected graphs.

```
M = ∅; n = |V|;
for v = 1 to n do
  Find an augmenting path with respect to M starting at v
  If success, we get a path π
  M ← M ⊕ π
end for
return M
```

Remark 1 *Although the above algorithm does a single pass on vertices, it does not miss maximum matching. The correctness of the algorithm follows from the fact that if there is no augmenting path from v with respect to M and π is some augmenting path with respect to M then there is no augmenting path from v with respect to $M \oplus \pi$.*

1.1 An algorithm for finding an augmenting path starting at a particular vertex

To find an augmenting path in G starting at v with respect to a matching M, let us construct a directed graph H_v from the original graph G. $V(H_v) = V(G)$ and $E(H_v) = \{(u, v) \mid \exists w \in V(G),\, (u, w) \notin M \,\text{and}\, (w, v) \in M\}$. Let us define,

Free $= \{v \mid v \text{ is not matched by } M\}$
Good $= \{u \mid \mathrm{N}(u) \cap \text{Free} \setminus \{v\} \neq \emptyset\}$

Consider the following algorithm to find an augmenting path from v.

```
if v is not free then
  return v
else if ∃u ∈ N(v) and u is free then
```

return (v, u)
else
Construct H_v
Construct Free
Construct Good
Search in H_v for a simple directed path, ρ from v to the set Good, let us call these paths pseudo augmenting paths.
return ρ if the original path corresponding to ρ in G is simple path
end if

Remark 2 *The correctness of the algorithm follows from the fact that there is an augmenting path π with respect to M starting at v iff there is a directed path ρ from v to set Good in H_v such that ρ corresponds to a simple path in the original graph.*

Remark 3 *The existence of an augmenting path in G with respect to M from v implies the existence of a pseudo augmenting path in H_v. But the converse is not true.*

Claim 4 *If the full path corresponding to a pseudo augmenting path is not augmenting then it contains an odd cycle.*

Proof: The full path corresponding to the pseudo augmenting path is not augmenting, which implies that full path is not simple. Hence it contains a cycle. If it is an even length cycle, then the pseudo augmenting path touch a vertex twice which contradicts that a pseudo augmenting path is a simple directed path. ■

2 Edmonds' Blossom Shrinking Algorithm

A blossom, B, with respect to M is an odd cycle with maximal number of matched edges i.e if it contains $2k+1$ vertices then k edges are matched. Shrinking a blossom B means shrinking all vertices of B in G to a single new vertex, say b, and making it adjacent to all vertices in $V \setminus B$ which are adjacent to any node of B.

Algorithm

1: $M = \emptyset$, $H = G$, $N = M$
2: EVEN $= \{v \,|\, v \text{ is free in } M\}$
ODD $= \emptyset$, $E_F = \emptyset$
$V_F =$ EVEN $\cup$ ODD
Construct forest $F = (V_F, E_F)$
3: Pick an edge (u, v) such that $u \in$ EVEN and $v \notin$ ODD
If no such edge exists, Output M
4: **if** $v \notin V_F$ **then**

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5:      Put v in ODD, M(v) in EVEN and, (u, v) and (v, M(v)) in E_F, where M(v) is a
        matched vertex to v.
6:  else if v ∈ EVEN & (u, v) connected in F then
7:      A blossom B has been found. Shrink B to b to get H \ B and N \ B
8:      Goto Step 3
9:  else if v ∈ EVEN & (u, v) not connected in F then
10:     We have an augmenting path η in H with respect to N. Retrieve an augmenting path
        ρ in G with respect to M.
11:     Augment M.
12:     Goto Step 1
13: end if
```

Claim 5 *When the algorithm terminates, N is maximum matching in H.*

Theorem 6 *Let N be a matching in H and B be a blossom. $N \setminus B$ is maximum in $H \setminus B$ if and only if N is maximum in H.*

Remark 7 *The proof of correctness of the algorithm follows from Claim 5 and Theorem 6.*

Proof of Theorem 6: Let us prove the contrapositive. Suppose that $N \setminus B$ has an augmenting path ρ. If ρ completely avoids the blossom node, b, in $H \setminus B$ then ρ itself is an augmenting path in H. So, let us assume that ρ touches b. let c be adjacent to b on ρ. Start from the one end on ρ and reaching b on path follow the path from b to c on blossom B and then follow the rest of the part of the path ρ. This gives an augmenting path in H with respect to M. Hence M is not maximum.
Assume Now that N is not maximum in H. Then there exists an augmenting path π in H. If π completely avoids B then π is also an augmenting path in $H \setminus B$. So let us suppose that π first intersects B at b from its one of the end points such that b is matched to a vertex not in blossom then shrinking B on π gives an augmenting path in $H \setminus B$. Hence $N \setminus B$ is not maximum in $H \setminus B$. ■

Proof of Claim 5: We know that $\forall\, U \subseteq V$, $\forall$ matching$M$,

$$\begin{aligned} |V| - 2 \times |M| &\geq \circ(G \setminus U) - |U| \\ \implies |M| &\leq \frac{|V| + |U| - \circ(G \setminus U)}{2} \end{aligned} \tag{1}$$

where $\circ(G \setminus U)$ is number of odd components of $(G \setminus U)$.
The set U is called a witness set iff the equality in the equation 1 holds and then M is a maximum matching. So to certify that a matching M is maximum in a graph G it is sufficient to show a witness set. Now, the proof follows from the following claim. ■

Claim 8 *The set ODD constructed by algorithm after termination is a witness set.*

Proof: When algorithm terminates the vertex set V is divided into three set EVEN, ODD and the Rest. The set EVEN is an independent set because if there is an edge between any two vertex in EVEN then either Step 6 or Step 9 of the algorithm is true which contradicts the fact that algorithm has terminated. The Step 5 of the algorithm ensures that the matched vertex of a vertex in ODD set is put in EVEN set and also the vertices in set EVEN are matched only to the vertices in set ODD. So, $\circ(H \setminus ODD) = |$ EVEN $| = |$ ODD $| +$ $|$ Free vertices $|$.

$$\begin{aligned} \implies |\text{ODD}| + |V| - \circ(H \setminus \text{ODD}) &= |V| - |\text{Free vertices}| \\ &= 2 \times (|\text{ODD}| + \frac{|\text{Rest}|}{2}) \\ \implies \frac{|\text{ODD}| + |V| - \circ(H \setminus \text{ODD})}{2} &= |N| \end{aligned} \tag{2}$$

Hence, the set ODD is a witness set. ■

Theorem 9 *(Tutte-Berge's Theorem)*

$$\max_M |M| = \min_{U \subseteq V} \frac{|V| + |U| - \circ(G \setminus U)}{2}$$

Proof: Equation 1 is valid $\forall$ matching M and $\forall U \subseteq V$. Hence taking maximum on left and minimum on right gives,

$$\max_M |M| \leq \min_{U \subseteq V} \frac{|V| + |U| - \circ(G \setminus U)}{2} \tag{3}$$

Claim 8 shows that the equality holds for a witness set. ■

Definition 10 *An Odd set cover $\mathcal{C} = \{C_1, C_2, \ldots, C_l\}$ is a set of subset of vertices, V, where C_i's are disjoint and,*

1. *each $|C_i|$ is odd*
2. *each edge (u, v) has*
   either $\{u\} = C_i$ for some i
   or $\{v\} = C_i$ for some i
   or $\{u, v\} \subseteq C_i$ for some i