Matchings in Graphs

Jan-May 2010

Problem Set: 3 | Assigned on: 22 Apr 2010
Submission Deadline: 29 Apr 2010

1. (a) Describe an $O(n)$ algorithm to find a maximum matching in a tree.

   (b) Design a dynamic algorithm to maintain a maximum matching in a tree under the operations

   delete: delete a pendant vertex,

   insert: insert a new vertex and make it a leaf

   Analayse the complexity of your algorithm.

2. Consider the following variant of the popular matching problem. The input is an instance G of the popular matching problem, and an arbitrary matching M_0 (which can be empty). The task is to find a popular matching by consensus: find a sequence $M_0, M_1, \ldots, M_k = M$ where each M_i is more popular than M_{i-1} and where the final M is popular. Even if G admits a popular matching, it is not obvious that there is a popular matching more popular than M_0. And even if there is one, it is not obvious that the k in the sequence above is small. It turns out that in fact k is at most 2, but this requires non-trivial proof. Show that a polynomially large k suffices.

3. The Hadamard inequality for the determinant states that if N is an $n \times n$ matrix with column vectors v_i for $1 \leq i \leq n$, then

$$|\det(N)| \leq \prod_{i=1}^{n} \|v_i\|,$$

   Here, $\|v\|$ denotes the Euclidean length of the vector v: $\|v\| = (v[1]^2 + \ldots + v[n]^2)^{1/2}$.

   (a) Using Pfaffian orientations and the Hadamard bound, show that a planar graph with n vertices (where n is even) and at most kn edges can have at most $(2k)^{n/4}$ perfect matchings. Hence conclude that any planar graph can have at most $6^{n/4} \leq (1.57)^n$ perfect matchings.

   (b) Using the above, show that a planar graph on n vertices (where n is even) can have at most $30^{n/4}$ distinct Hamiltonian cycles.

4. Reading exercise: If you have not already seen this, then find, read and understand the proof of the following statement:
   Counting perfect matchings in bipartite graphs is #P-hard.

5. Optional Reading: Pareto-optimal matchings are similar to popular matchings, but with a slight variation. There is a bipartite graph $G = (A \cup H, E)$ with a set of agents $A = \{a_1, a_2, \ldots, a_r\}$ and a set of houses $H = \{h_1, h_2, \ldots, h_s\}$, and an edge $(a, h) \in E$ means that house h is acceptable to agent a. Each agent has a preference list ranking all the hosues acceptable to a. Agent a prefers matching M' to matching M if

   - agent a is matched to some house in M', but not in M, or
   - agent a is mathed in both, but likes the house in M' better than the house in M.

   A matching M is Pareto-optimal if there does no exist a matching M' such that

   - there is at least one agent who prefers the house he got in M' to the house he got in M, and
   - no agent prefers the house he got in M to the house he got in M'.

   A maximum size Pareto-optimal matching, if one exists, can be found in $O(\sqrt{n}m)$ time, see [ACMM].

References

[ACMM] D. J. Abraham, K. Cechlarova, D. F. Manlove, and K. Mehlhorn. *Pareto optimality in house allocation problems.* Proc. 15th ISAAC 2004.