Matchings in Graphs

Jan-May 2010

Problem Set: 2 | Assigned on: 12 Mar 2010 | Submission Deadline: 25 Mar 2010

1. Complete the proof of Edmond's Theorem characterizing the perfect matching polytope by proving Claim 11 in the (edited) lecture notes.

2. We consider a variant of the stable matching problem discussed in class. Let (M, W, P) be a preference pattern in the stable marriage instance. The preference pattern for $m \in M$ need not be a total order on all of W; it is a total order on a (possibly empty) subset of W. What this means is that m would prefer remaining single to being matched with any w not in his preference list. One could think of this as a total order on $W \cup \{m\}$, where m appears before any element not in $P(m)$ but after all elements in $P(m)$. (Similarly for $w \in W$.)

   In a matching, some people could be left unmatched; they are considered matched to themselves. So a matching μ is a bijection of $M \cup W$ onto itself, of order 2 (that is, $\mu \circ \mu = id$), where if $m \in M$ or $w \in W$ is not self-matched, then $\mu(m) \in W$ and $\mu(w) \in M$.

   A stable matching is now defined exactly as for complete preference lists: a matching μ is stable if there is no pair (m, w) such that (1) $\mu(m) \neq w$, (2) m prefers w to $\mu(m)$, and (3) w prefers m to $\mu(w)$.

   There could be many stable matchings for a given preference pattern.

   (a) Show that the set of people left unmatched (or self-matched) is the same in all stable matchings.

   (b) Let μ and μ' be two matchings. Say that $\mu \succ_M \mu'$ if for every $m \in M$, m prefers $\mu(m)$ to $\mu'(m)$. Show that this relation $\succ_M$ on stable matchings is a partial ordering, and further, that it forms a lattice.

3. Let G be a graph on $2n$ vertices. An even length cycle C is said to be nice if the graph obtained by deleting all vertices on C from G has a perfect matching.

   (a) Show that if G has at most n^k perfect matchings, then G has at most n^{2k+1} nice cycles.

   (b) Show that there is no analogous converse; there are graphs with $n^{O(1)}$ nice cycles but exponentially many perfect matchings.

4. Show that if a bipartite graph has a unique perfect matching, then it has at least one pendant (degree one) vertex.