**Matchings in Graphs**
Jan-May 2010

Problem Set: 1 | Assigned on: 28 Jan 2010
Submission Deadline: 11 Feb 2010

1. (a) Give an example bipartite graph with 2k left nodes and 2k right nodes with a maximal matching of size k and a maximum matching of size 2k.

   (b) The ratio of the size of the maximum matching to the maximal matching is 2:1 in the previous construction. Prove that this is the largest possible ratio. We can conclude from this that a maximal matching is an approximation (to within a factor of 2) of the maximum matching.

2. In any graph, consider the following parameters:

   - edge covering number $\rho(G) =$ $\min\{|F| \mid F \subseteq E(G),$ for every vertex of $G$, at least one incident edge is in $F\}$.
   - matching number $\nu(G) =$ $\max\{|F| \mid F \subseteq E(G),\ F$ is a matching$\}$.

   Show that if $G$ has no isolated vertex, then $\rho(G) + \nu(G) = n$.

3. Reading exercise: If you have not already seen these, then find, read and understand proofs of the following.

   (a) König-Evergáry Theorem: In a bipartite graph, the size of a maximum matching is equal to the size of a minimum vertex cover.
   Remark: With the new terminology we have seen, this theorem says that in a bipartite graph, every minimum-size vertex cover is a witness set.

   (b) Hall's Theorem: Let $G = (U, V, E)$ be a bipartite graph, with $E \subseteq U \times V$. For $S \subseteq U$, let $N(S)$ denote the set of neighbours of $S$. Then $G$ has a matching in which no vertex of $U$ is free if and only if for every subset $S$ of $U$, $|N(S)| \geq |S|$.

4. Show that any nontrivial regular bipartite graph has a perfect matching.

5. Use the Tutte-Berge theorem (the existence of witness sets) to show that every graph has an odd-set-cover whose cost equals the size of a maximum matching.

6. We showed in class that when Edmonds' blossom-shrinking algorithm terminates, the set of vertices labelled Odd forms a witness set. Show that in fact it is the Tutte set $A(G)$. Also show that the set of vertices labelled Even is $D(G)$, and the set of unlabelled vertices is $C(G)$.