SZEGÖ LIMIT THEOREM ON THE LATTICE

JITENDRIYA SWAIN AND M KRISHNA

Abstract. In this paper, we prove a Szegö type limit theorem on $\ell^2(\mathbb{Z}^d)$. We consider operators of the form $H = \rho\Delta + |\xi|^k, 0 \leq \rho \leq 1,\ 0 < k < 2$ on $\ell^2(\mathbb{Z}^d)$ and π_λ the orthogonal projection of $\ell^2(\mathbb{Z}^d)$ on to the space of eigenfunctions of H with eigenvalues $\leq \lambda$. We take A be a 0th order self adjoint pseudo-difference operator with symbol $a(\xi, x)$ satisfying $[A, H](H+1)^{-\sigma}$ bounded for some $0 < \sigma < 1$. Then for $f \in \mathcal{C}(\mathbb{R})$ and $(\xi, x) \in \mathbb{Z}^d \times \mathbb{T}^d$,

$$\lim_{\lambda\to\infty} \frac{\text{tr f}(\pi_\lambda \text{A} \pi_\lambda)}{\text{rank } \pi_\lambda} = \lim_{\lambda\to\infty} \frac{1}{(2\pi)^d} \frac{1}{vol(H(\xi, x) \leq \lambda)} \sum_{(\xi,x):H(\xi,x)\leq\lambda} \int f(a(\xi, x))dx$$

assuming one of the limits exists. The limits are invariant under compact perturbation of A.

1. Introduction

Let Δ be the discrete Laplacian $(\Delta u)(n) = \sum_{|i|=1} u(n+i)+2du(n)$ on $\ell^2(\mathbb{Z}^d)$ and V be a positive function on $\mathbb{Z}^d$ such that $V(\xi) = |\xi|^k,\ 0 < k < 2$ for large $|\xi|$. We denote by V the operator of multiplication by the function $V(\xi)$ on $\ell^2(\mathbb{Z}^d)$. Our choice of the normalisation in the definition of Δ makes it a positive operator with purely absolutely continuous spectrum in $[0, 4d]$.

We take $V(\xi)$ as above with some $0 < k < 2,\ 0 \leq \rho \leq 1$ fixed and consider

$$H = \rho\Delta + V. \tag{1.1}$$

It turns out that H is a pseudo difference operator with symbol $H(\xi, x) = 2\rho \sum_{k=1}^d \cos(x_k) + V(\xi) + 2d\rho$ where $(\xi, x) \in \mathbb{Z}^d \times \mathbb{T}^d$.

The H is self adjoint on the domain $\{u \in \ell^2(\mathbb{Z}^d) : Vu \in \ell^2(\mathbb{Z}^d)\}$, since Δ is bounded. It is easy to see that the resolvent $(H - z)^{-1}$ is compact for some (hence for all) $z \in \mathbb{C}^+$, so the spectrum of H is discrete and the multiplicity of each eigen value is finite.

Let $\{(\lambda_i, f_i)\}$ be the set of eigenvalues and eigen functions of H counted with multiplicity. Let π_λ denote the (finite rank) orthogonal projection of

Date: April 20, 2010.
We thank M N Namboodiri for introducing us to the Szegö type theorems.

$\ell^2(\mathbb{Z})$ on to span$\{f_j : \lambda_j \leq \lambda\}$. In this paper, we consider a Szegö type theorem associated with H.

The classical theorem of Szegö is stated as follows: Let P_n be the orthogonal projection of $L^2[0, 2\pi]$ onto the linear subspace spanned by the functions $\{e^{im\theta} : 0 \leq m \leq n; 0 \leq \theta < 2\pi\}$. For a positive function $f \in \mathcal{C}^{1+\alpha}[0, 2\pi], \alpha > 0$ the operator T_f defined by the operator of multiplication by the function f on $L^2[0, 2\pi]$ the following result holds

$$\lim_{n\to\infty} \frac{1}{n+1} \log\det P_n T_f P_n = \frac{1}{2\pi} \int_0^{2\pi} \log f(\theta) d\theta.$$

The above result is well known as Szegö limit theorem. We refer to [8, 4] for details and related results. In fact, Szegö limit theorem is a special case of a more general result proved by Szegö (see [4]) in section 5.3 as follows. Let f be a bounded, real valued integrable function and $\{\lambda_i^n\}_{i=1}^n$ be the eigenvalues of $P_n T_f P_n$. Then for any continuous function F on $[\inf f, \sup f]$ it was proved in (see [4], sect. 5.3) that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_i^n) = \frac{1}{2\pi} \int_0^{2\pi} F(f(\theta)) d\theta.$$

Notice that $e^{im\theta}$ is an eigenfunction of $\Delta = -\frac{d^2}{dx^2}$, one can view the above results on $L^2[0, 2\pi]$ as a special cases of Szegö limit theorem for the Laplace-Beltrami operator or more generally one can consider such results for pseudo differential operators on compact manifolds.

We however consider such a result on $\ell^2(\mathbb{Z}^d)$ in this paper. Consider H, π_λ on $\ell^2(\mathbb{Z}^d)$ as stated earlier. Let A be a bounded, pseudo-difference operator on $\ell(\mathbb{Z}^d)$ with symbol $a(\xi, x)$. Then $\pi_\lambda A \pi_\lambda$ is a finite rank operator and hence its spectral measure μ_λ can be defined as the sum of δ-functions at its eigenvalues. (In [12], Zelditch considered a Schrödinger operator on $\mathbb{R}^n$ of the form $H = -\frac{1}{2}\Delta + V$, where V is a smooth positive function which grows like $V_0|x|^k,\ k > 0$.) To establish a Szegö type theorem we need to consider ratios of distribution functions $\phi_\mu,\ \phi_\nu$ of measures $\mu,\ \nu$ respectively and compute the limits $\dfrac{\phi_\mu(\lambda)}{\phi_\nu(\lambda)}$ as $\lambda \to \infty$.

Such limits are computed using Tauberian theorems where some transforms of these measures are considered and limits taken for such transforms. While Zelditch [12] used the Laplace transform (via Karamata's Tauberian theorem ([11],p-192), Robert [7] suggested the use of Stieltjes transform (via Keldy's Tauberian theorem[1]). The application of Keldy's theorem requires one of the measures μ or ν to be absolutely continuous. We don't have this feature in our problem, so we use the Tauberian theorem of Grishin-Poedintseva (theorem 8,[5]).

In addition, our proof also requires the use of an improved version of Laptev-Saffrov [3] type estimate. Their result requires that $[A, H]$ be bounded. If the symbol of A were a function of only ξ, then this requirement holds for any $k > 0$, however, when the symbol of A is a function of both x, ξ, then the commutator picks up a discrete derivative of the function $|\xi|^k$ which may not be bounded. The improved theorem is following.

Theorem 1.1. *Let S be a positive selfadjoint operator and T be a bounded selfadjoint operator in a Hilbert space. Let π_λ be the spectral projection of S corresponding to the interval $[0, \lambda]$ and $N(\lambda)$ be the counting eigenvalues function with*

$$N_\epsilon(\lambda) = \sup_{\beta \le \lambda}[N(\beta) - N(\beta - \epsilon)].$$

Assume that the commutator $\tilde{T} = [S, T]$ satisfies $[S, T](S + I)^{-\sigma}$ is bounded for some $0 < \sigma < 1$. Then for any $\epsilon > 0$ and for any function $f \in \mathcal{C}^2(K)$ the following inequality holds.

$$|\mathrm{tr}\ \pi_\lambda \mathrm{f}(\mathrm{T})\pi_\lambda - \mathrm{tr}\ \mathrm{f}(\pi_\lambda \mathrm{T} \pi_\lambda)| \le (2\|\mathrm{T}\|^2 + \mathrm{C}_\epsilon \|\tilde{\mathrm{T}}\|^2)\mathrm{N}_\epsilon(\lambda) \max_{\mathrm{K}} |\mathrm{f}''|,$$

where $K = [-\|T\|, \|T\|]$ and the constant C_ϵ depends on ϵ only.

In this paper, in our main Theorem 1.2, we present a Szegö type theorem for H given in the equation 1.1 and A a zeroth order pseudo difference operator. We need the restriction on k to be in the interval $(0, 2)$ owing to our use of the extended Laptev-Saffrov [2] result.

The main theorem of the paper is the following, for which we set $\tilde{d}x(= dx/(2\pi)^d)$ to be the normalized invariant measure on $\mathbb{T}^d$. Let $0 \leq \rho \leq 1,\ 0 < k < 2$ be fixed.

Theorem 1.2. *Consider the positive self adjoint operator H as in equation (1.1) on $\ell^2(\mathbb{Z}^d)$. Let A be a zeroth order bounded pseudo difference operator such that $[A, H]$ is relatively bounded with respect to H in the sense $[A, H](H + I)^{-\sigma}$ is bounded for some $0 < \sigma < 1$. For $f \in \mathcal{C}(\mathbb{R})$ define $\mu_\lambda(f), \mu(f)$ as in corollary 3.5. Then the sequence $\nu_\lambda(f) = \dfrac{\text{tr } (\text{f}(\pi_\lambda \text{A} \pi_\lambda))}{\text{rank } \pi_\lambda}$ also has the same limit $\mu(f)$. That is*

$$\begin{aligned}\lim_{\lambda\to\infty} \frac{\text{tr } (\text{f}(\pi_\lambda \text{A} \pi_\lambda))}{\text{rank } \pi_\lambda} &= \lim_{\lambda\to\infty} \nu_\lambda(f) = \lim_{\lambda\to\infty} \frac{\mu_\lambda(f)}{\text{rank } \pi_\lambda} = \mu(f) \\ &= \lim_{\lambda\to\infty} \frac{1}{vol(H(\xi, x) \leq \lambda)} \sum_{H(\xi,x)\leq\lambda} \int f(a(\xi, x))\tilde{d}x.\end{aligned}$$

2. Preliminaries

In this section we will introduce some notations which we will be using frequently.

Definition 2.1. *let $m \in \mathbb{R}$. Then $S^m_H(\mathbb{Z}^d, \mathbb{T}^d)$, the space of symbols of order m relative to (H, λ) consisting of those functions $a(\xi, x)$ which are smooth in x for all $\xi \in \mathbb{Z}^d$ satisfying*

$$|\partial^\alpha_x \Delta^\gamma a(\xi, x)| \leq C_{a\alpha\gamma m}(H(\xi, x) + |\lambda|)^{\frac{m-|\gamma|}{2}}$$

for every $x \in \mathbb{T}^d, \alpha, \gamma \in \mathbb{N}^d_0$ and $\xi \in \mathbb{Z}^d$.

Now we discuss some of the properties of pseudo-difference calculus which is developed by Ruzhansky and Turunen in [10]. The pseudo-difference operators are closed under composition.

Theorem 2.2. *(Theorem 4.3, [10]) Let P and Q be pseudo difference operators with symbols $p(\xi, x) \in S^{m_1}_H$ and $q \in S^{m_2}_H$. Then PQ is a pseudo difference operator with symbol $r(\xi, x) \in S^{m_1+m_2}_H$ and*

$$r(\xi, x) \sim \sum_\alpha \frac{1}{\alpha!} \Delta^\alpha_\xi p(\xi, x) \frac{\partial^\alpha}{\partial x^\alpha} q(\xi, x)$$

where Δ_ξ is the forward difference operator in d-dimension.

Consider the pseudo difference operator $H+\lambda$ with symbol $(H+\lambda)(\xi,x) = 2\rho\sum_{j=1}^{d}\cos x_j + |\xi|^k + 2d\rho + \lambda$ where H is the discritized Schröedinger operator and λ be a positive real number. Let $(H+\lambda)^m(\xi,x)$ denote the symbol of the operator $(H+\lambda)^m$ (composed m times with itself) by theorem 2.2 . Then we have

$$\begin{aligned}(H+\lambda)^2(\xi,x) &= \sum_{\alpha}\frac{1}{\alpha!}\Delta_\xi^\alpha(H+\lambda)(\xi,x)\frac{\partial^\alpha}{\partial x^\alpha}(H+\lambda)(\xi,x)\\ &= (H(\xi,x)+\lambda)^2 + \sum_{|\alpha|=1}^{\infty}\frac{1}{\alpha!}\Delta_\xi^\alpha(H+\lambda)(\xi,x)\frac{\partial^\alpha}{\partial x^\alpha}(H+\lambda)(\xi,x)\end{aligned}$$

But

$$\begin{aligned}\frac{(H+\lambda)^2(\xi,x)}{(H(\xi,x)+\lambda)^2} &\leq 1+2d\sum_{|\alpha|=1}^{\infty}\frac{1}{\alpha!}\frac{|\Delta_\xi^\alpha(H+\lambda)(\xi,x)|}{(H(\xi,x)+\lambda)^2}\\ &= 1+2d\sum_{|\alpha|=1}^{\infty}\frac{1}{\alpha!}|\sum_{\gamma\leq\alpha}(-1)^{|\alpha-\gamma|}\binom{\alpha}{\gamma}\frac{|H(\xi+\gamma,x)|}{(H(\xi,x)+\lambda)^2}\\ &\leq 1+2d\frac{(e^2-1)^d}{\lambda+|\xi|^k}\end{aligned}$$

By induction we extend the above argument for higher order composition of the operator $H+\lambda$ with itself.

Similarly if $q(\xi,x) = (H(\xi,x)+\lambda)^m$ and $p(\xi,x) = (H(\xi,x)+\lambda)^{-m}$ then by theorem 2.2 the symbol of the composition is given by $1+r_0(\xi,x)$, where $r_0(\xi,x) = \sum_{|\alpha|=1}^{\infty}\frac{1}{\alpha!}\Delta_\xi^\alpha q(\xi,x)\frac{\partial^\alpha}{\partial x^\alpha}p(\xi,x)$. Then it can be shown by using Proposition 3.1 of [10] that

$$|\sum_{|\alpha|=1}^{\infty}\frac{1}{\alpha!}\Delta_\xi^\alpha q(\xi,x)\frac{\partial^\alpha}{\partial x^\alpha}p(\xi,x)| \leq \frac{C}{\lambda}\sum_{|\alpha|=1}^{\infty}\frac{2^\alpha}{\alpha!}.$$

Therefore for large λ, we have the following remark.

Remark 2.3. For large λ and $m\in\mathbb{Z}^+$ we have

(i) $(H+\lambda)^m(\xi,x)\sim(H(\xi,x)+\lambda)^m + O(\lambda^{-1})$

(ii) $(H+\lambda)^{-m}(\xi,x)\sim(H(\xi,x)+\lambda)^{-m} + O(\lambda^{-1})$

We denote $p^w(\xi,x)$ as the pseudo-difference operator associated to the symbol $p(\xi,x)$.

By using the above facts, we have the following proposition.

Proposition 2.4. *Let* $H=\rho\Delta+|\xi|^k, 0\le\rho\le 1,\ k>0$ *on* $\ell(\mathbb{Z}^d)$ *and let* $H(\xi,x)$ *be the symbol of* H. *Let* A *be a 0th order pseudo-difference operator. Let* m *be a positive integer. Then*

(i) $\dfrac{|\mathrm{tr}\,((\mathrm{H}+\lambda)^{-\mathrm{m}})-\mathrm{tr}\,((\mathrm{H}(\xi,\mathrm{x})+\lambda)^{-\mathrm{m}})^{\mathrm{w}}|}{|\mathrm{tr}\,((\mathrm{H}(\xi,\mathrm{x})+\lambda)^{-\mathrm{m}})^{\mathrm{w}}|}\to 0$ *as* $\lambda\to\infty$

(ii) $\dfrac{|\mathrm{tr}\,(\mathrm{A}(\mathrm{H}+\lambda)^{-\mathrm{m}})-\mathrm{tr}\,(\mathrm{A}(\mathrm{H}(\xi,\mathrm{x})+\lambda)^{-\mathrm{m}})^{\mathrm{w}}|}{|\mathrm{tr}\,(\mathrm{A}(\mathrm{H}(\xi,\mathrm{x})+\lambda)^{-\mathrm{m}})^{\mathrm{w}}|}\to 0$ *as* $\lambda\to\infty$,

Proof. Form remark ¿From remark 2.3(ii) we have

$$q^w(\xi,x)op^w(\xi,x)=I+B^w_\lambda,$$

where $q(\xi,x)=(H(\xi,x)+\lambda)^m,\ p(\xi,x)=(H(\xi,x)+\lambda)^{-m}$ and $B^w_\lambda=r^w_0(\xi,x)$.

Thus we have $\|B^w_\lambda f\|_2\le\dfrac{C}{\lambda}\|f\|_2$ for $f\in\mathcal{S}(\mathbb{Z}^d)$. So $\|B^w_\lambda f\|_2\to 0$ as $\lambda\to\infty$ and hence $\|B^w_\lambda\|\to 0$ as $\lambda\to\infty$.

Since for large λ, $q^w(\xi,x)=(H+\lambda)^m$, we have

$$q^w(\xi,x)op^w(\xi,x)=I+B^w_\lambda \;=\; (H+\lambda)^m(H+\lambda)^{-m}+B^w_\lambda$$

which can be written as

$$p^w(\xi,x)-(H+\lambda)^{-m}=(H+\lambda)^{-m}B^w_\lambda \tag{2.1}$$

Now applying trace and dividing by $\mathrm{tr}\,((\mathrm{H}+\lambda)^{-\mathrm{m}})$ both sides we have

$$\left|\frac{\mathrm{tr}\,(\mathrm{p}^{\mathrm{w}}(\xi,\mathrm{x}))-\mathrm{tr}\,(\mathrm{H}+\lambda)^{-\mathrm{m}})}{\mathrm{tr}\,(\mathrm{H}+\lambda)^{-\mathrm{m}})}\right|=\left|\frac{\mathrm{tr}\,[(\mathrm{H}+\lambda)^{-\mathrm{m}})\mathrm{B}^{\mathrm{w}}_\lambda]}{\mathrm{tr}\,(\mathrm{H}+\lambda)^{-\mathrm{m}})}\right|\le\|B^w_\lambda\|\to 0$$

as $\lambda\to\infty$. This proves (i). For (ii), composing the operator A from the left in equation 2.1, applying trace and dividing by $\mathrm{tr}\,\mathrm{A}(\mathrm{H}+\lambda)^{-\mathrm{m}})$ we have

$$\left|\frac{\mathrm{tr}\,\mathrm{A}(\mathrm{p}^{\mathrm{w}}(\xi,\mathrm{x}))-\mathrm{tr}\,\mathrm{A}(\mathrm{H}+\lambda)^{-\mathrm{m}})}{\mathrm{tr}\,\mathrm{A}(\mathrm{H}+\lambda)^{-\mathrm{m}})}\right|=\left|\frac{\mathrm{tr}\,[\mathrm{A}(\mathrm{H}+\lambda)^{-\mathrm{m}})\mathrm{B}^{\mathrm{w}}_\lambda]}{\mathrm{tr}\,\mathrm{A}(\mathrm{H}+\lambda)^{-\mathrm{m}})}\right|\le\|B^w_\lambda\|\to 0$$

as $\lambda\to\infty$. □

Lemma 2.5. *Let* $H(\xi,x)=2\rho\sum_{j=1}^{d}\cos x_j+|\xi|^k+2d\rho,\ 0\le\rho\le 1,\ k>0,\ x_j\in\mathbb{T}$ *and* $\xi\in\mathbb{Z}^d$*. Define* $\varphi_2(\lambda)=\frac{1}{(2\pi)^d}\sum\int_{H(\xi,x)\le\lambda}dx$ *with* $|\xi|^2=\sum_{i=1}^{d}\xi_i^2$ *and* $\varphi_\infty(\lambda)=\frac{1}{(2\pi)^d}\sum\int_{H(\xi,x)\le\lambda}dx$ *with* $|\xi|=\max_{1\le i\le d}|\xi_i|$*. Then*

$$d^{-\frac{d}{2}}\varphi_\infty(\lambda)\le\varphi_2(\lambda)\le\varphi_\infty(\lambda)$$

for large λ*.*

Proof. Let

$$\begin{aligned}\frac{1}{(2\pi)^d}\sum\int_{H(\xi,x)\le\lambda}dx &= \frac{1}{(2\pi)^d}\sum\int_{2\rho\sum_{j=1}^{d}\cos x_j+|\xi|^k+2d\rho\le\lambda}dx\\ &= \frac{1}{(2\pi)^d}\sum\int_{|\xi|\le(\lambda-2\rho\sum_{j=1}^{d}\cos x_j-2d\rho)_+^{\frac{1}{k}}}dx\end{aligned}$$

where $(\lambda-2\rho\sum_{j=1}^{d}\cos x_j-2d\rho)_+=\max(\lambda-2\rho\sum_{j=1}^{d}\cos x_j-2d\rho,0)$. But one has following relation between the Euclidean metric and the maximum metric.

$$d_\infty(a,b)\le d_2(a,b)\le\sqrt{d}d_\infty(a,b)\tag{2.2}$$

where $d_2(a,b)$ is the Euclidean metric and $d_\infty(a,b)=\max_{1\le i\le d}|a_i-b_i|$. By making use of the above relation we have $\frac{1}{\sqrt{d}}B_\infty(0,a)\subset B_2(0,a)\subset B_\infty(0,a)$, where $B_2(0,a)$ and $B_\infty(0,a)$ is the disk with centre at origin and radius a with Euclidean metric and maximum metric respectively.

¿From the above facts we have

$$\frac{d^{-\frac{d}{2}}}{(2\pi)^d}\sum\int_{B_\infty(0,a)}dx\le\frac{1}{(2\pi)^d}\sum\int_{B_2(0,a)}dx\le\frac{1}{(2\pi)^d}\sum\int_{B_\infty(0,a)}dx.$$

Hence the lemma. □

The above lemma tells us that up to a multiplicative constant Euclidean metric and maximum metric are equivalent. Hereafter we will work on maximum metric for convenience.

Before going in to Szegö theorem, we will introduce few definitions and theorems which can be found in [5].

Definition 2.6. *Let f be a positive function on the half line $[0, \infty)$. Let S denote the set of numbers α for which there exist numbers M and R such that $f(tr) \leq Mt^{\alpha}$ for $t \geq 1$ and $r \geq R$. Then $\alpha(f) = \inf S$ is called The upper Matushevskaya index of f.*

Let G denote the set of numbers ζ for which there exist numbers M and R such that $f(tr) \geq Mt^{\zeta}$ for $t \geq 1$ and $r \geq R$. Then $\beta(f) = \sup G$ is called The lower Matushevskaya index of f.

Theorem 2.7. *([5], Theorem 2) Let $m > -1$. Assume that φ is positive measurable function on $[0, \infty)$ that does not vanish identically in any neighbourhood of infinity. Let $\Phi(r) = \int_0^{\infty} \frac{1}{(1 + \frac{t}{x})^m} d\varphi(t)$. Then the functions φ and Φ have same growth at infinity if and only if $\beta(\varphi) > -1$ and $\alpha(\varphi) < m$.*

Definition 2.8. *A function φ is said to be multiplicatively continuous at infinity if it satisfies*

$$\lim_{\substack{r \to \infty \\ \tau \to 1}} \frac{\varphi(\tau r)}{\varphi(r)} = 1 \text{ and } \lim_{\substack{\tau \to 1 \\ r \to \infty}} \frac{\varphi(\tau r)}{\varphi(r)} = 1.$$

Theorem 2.9. *([5], Theorem 8) Let φ and ψ be positive functions on $[0, \infty)$ satisfying the following conditions:*

1. *the functions φ and ψ do not vanish identically in any neighbourhood of infinity;*
2. *the function φ is multiplicatively continuous at infinity and $\beta(\varphi) > -1$;*
3. *the function ψ is increasing;*
4. *at least one of the inequalities $\alpha(\varphi) < m$ and $\alpha(\phi) < m$ holds, where $m > -1$;*
5. *the functions*

$$\Phi(r) = \int_0^{\infty} \frac{1}{(1 + \frac{u}{r})^m} d\varphi(u) \text{ and } \Psi(r) = \int_0^{\infty} \frac{1}{(1 + \frac{u}{r})^m} d\psi(u)$$

are finite and $\lim_{r \to \infty} \frac{\Psi(r)}{\Phi(r)} = 1$ *then* $\lim_{r \to \infty} \frac{\psi(r)}{\varphi(r)} = 1$.

Lemma 2.10. *For $\xi \in \mathbb{Z}^d$, let $|\xi| = \max_{1\le i\le d} |\xi_i|$. Let $H(\xi,x) = 2\rho\sum_{j=1}^{d}\cos x_j + |\xi|^k + 2d\rho$, $0 \le \rho \le 1$, $k > 0$, $x_j \in \mathbb{T}$ and $\xi \in \mathbb{Z}^d$. Define $\varphi(\lambda) = \frac{1}{(2\pi)^d}\sum\int_{H(\xi,x)\le\lambda} dx$. Then φ satisfies the following conditions:*

1. *the function φ do not vanish identically in any neighbourhood of infinity;*
2. *the function φ is multiplicatively continuous at infinity and $\beta(\varphi) > 1$;*
3. *$\alpha(\varphi) < m$, where $m > -1$.*

Proof. By imitating first few steps of lemma 2.5 leads

$$\begin{aligned}
\varphi(\lambda) &= \frac{1}{(2\pi)^d}\sum\int_{|\xi|\le(\lambda-2\rho\sum_{j=1}^{d}\cos x_j-2d\rho)_+^{\frac{1}{k}}} dx \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}(2[(\lambda-\rho\sum_{j=1}^{d}\cos x_j-2d\rho)_+^{\frac{1}{k}}]+1)^d dx \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}(2(\lambda-\rho\sum_{j=1}^{d}\cos x_j-2d\rho)_+^{\frac{1}{k}} \\
&\quad -2\{(\lambda-\rho\sum_{j=1}^{d}\cos x_j-2d\rho)_+^{\frac{1}{k}}\}+1)^d dx \\
&= \frac{\lambda^{\frac{d}{k}}}{(2\pi)^d}\int_{\mathbb{T}^d}(2(1-\frac{\rho}{\lambda}\sum_{j=1}^{d}\cos x_j-\frac{2d\rho}{\lambda})_+^{\frac{1}{k}} \\
&\quad -\frac{1}{\lambda}\{(\lambda-\rho\sum_{j=1}^{d}\cos x_j-2d\rho)_+^{\frac{1}{k}}\}+\frac{1}{\lambda^{\frac{1}{k}}})^d dx,
\end{aligned}$$

where $[p]$ denote the greatest integer function and $\{p\}$ is the fractional part of p.

Since the integrand is bounded for large λ and the integration is over a compact set, it can be realised that $\varphi(\lambda)$ behaves like constant times $\lambda^{\frac{d}{k}}$. We need to show $\beta(\varphi) > 1$ and $\alpha(\varphi) < m$, where $m > -1$. It is enough to show φ and Φ have same growth at infinity. A straight forward computation gives

$$\lim_{r\to\infty}\frac{\Phi(r)}{\varphi(r)} = C\int_0^{\infty}\frac{u^{\frac{d}{k}}}{(1+u)^{m+1}}du.$$

We notice that $\dfrac{u^{\frac{d}{k}}}{(1+u)^{m+1}}$ converges if $m \geq \frac{d}{k}+1$. So if we choose $m = \frac{d}{k}+1$, we have $0 < \lim_{r\to\infty} \dfrac{\Phi(r)}{\varphi(r)} < \infty$. Thus φ and Φ have same growth at infinity. $\square$

Corollary 2.11. *Let* $\psi(\lambda) = \sum_{\lambda_j \leq \lambda} \delta_{\lambda_j}$ *and* $\varphi(\lambda) = \dfrac{1}{(2\pi)^d} \sum \int_{H(\xi,x)\leq\lambda} dx.$ *Then* $\lim_{\lambda\to\infty} \dfrac{\psi(\lambda)}{\varphi(\lambda)} = 1$

Proof. By lemma 2.10, it follows that $\varphi(\lambda)$ satisfies all the assumptions of theorem 2.9. Also $\Psi(\lambda) = \mathrm{tr}\,((\mathrm{H}+\lambda)^{-\mathrm{m}}) < \infty$ and $\Phi(\lambda) = \mathrm{tr}\,([(\mathrm{H}(\xi, \mathrm{x}) + \lambda)^{-\mathrm{m}})]^{\mathrm{w}}) < \infty$. It follows from proposition 2.4 (ii) that $\lim_{r\to\infty} \dfrac{\Psi(r)}{\Phi(r)} = 1$ implying $\lim_{\lambda\to\infty} \dfrac{\psi(\lambda)}{\varphi(\lambda)} = 1$ $\square$

3. Proof of Main Theorem

Our aim in this section is to prove the averaging theorem and then deduce the Szegö Theorem from it. Before proving the averaging theorem, we need to prove the following lemma.

Lemma 3.1. *Let A be a bounded, positive self adjoint operator and H be positive self adjoint operator with discrete spectrum. Let $E_H(\cdot)$ be the spectral measure of H. Then*

(i) $\mathrm{tr}\,(E_H(\cdot)A)$
(ii) $\mathrm{tr}\,(E_H(\cdot))$

are σ-finite measures.

Proof. The second item is obvious, the first item follows by writing $\mathrm{tr}\,(\mathrm{E_H}(\cdot)\mathrm{A})$ as $\mathrm{tr}\,(\mathrm{A}^{\frac{1}{2}}\mathrm{E_H}(\cdot)\mathrm{A}^{\frac{1}{2}})$ using the properties of trace and the positivity of A. $\square$

Lemma 3.2. *Let $a(\xi, x)$ be a non-negative bounded function and A the associated 0th order pseudo-difference operator. Let H be the pseudo difference operator with symbol $H(\xi, x) = 2\rho \sum_{j=1}^{d} \cos x_j + |\xi|^k + 2d\rho,\ 0 \leq \rho \leq 1,\ 0 < k < 2$. Then*

(i) $\phi(\lambda) = \sum \int_{\{(\xi,x):H(\xi,x)\leq\lambda\}} a(\xi, x)\ dx$
(ii) $\psi(\lambda) = \sum \int_{\{(\xi,x):H(\xi,x)\leq\lambda\}}\ dx$

are distribution functions of σ finite positive Borel measures.

Proof. We note that both ϕ and ψ are non-decreasing positive functions on $[0, \infty)$ with $\phi(0) = \psi(0) = 0$ and hence there are σ-finite measures with ϕ and ψ as distribution functions. □

Now we are in a position to prove the averaging theorem.

Theorem 3.3. *Let A be a bounded pseudo difference operator with symbol $a(\xi, x)$ and H the operator given in equation (1.1) with k fixed. Suppose*

$$\lim_{\lambda\to\infty} \frac{\mathrm{tr}\,(\mathrm{A}(\mathrm{H}+\lambda)^{-\mathrm{m}})}{\mathrm{tr}\,(\mathrm{H}+\lambda)^{-\mathrm{m}}}$$

exists (and nonzero) then the following limits exist and assumes the same value:

(i) $\lim_{\lambda\to\infty} \dfrac{\mathrm{tr}\,(\pi_\lambda \mathrm{A} \pi_\lambda)}{\mathrm{rank}\ \pi_\lambda}$

(ii) $\lim_{\lambda\to\infty} \dfrac{\sum \int_{H(\xi,x)\leq\lambda} a(\xi, x)\tilde{d}x}{vol(\{(\xi, x) : H(\xi, x) \leq \lambda\})}$ *where* $\tilde{d}x = \dfrac{dx}{2\pi}$.

Proof. We first note that since $a(\xi, x)$ is a bounded function, we can add a constant c so that $a(\xi, x) + c$ is positive and since the limits in items (i) (respectively (ii)) exist iff the corresponding limits exit with A replaced by $A + c$ (respectively $a(\xi, x) + c$), we can take i without loss of generality $a(\xi, x)$ to be a positive function and hence A to be a positive self adjoint bounded pseudo difference operator in the argument below.

Assume that $\lim_{\lambda\to\infty} \frac{\mathrm{tr}\,(A(H+\lambda)^{-m})}{\mathrm{tr}\,(H+\lambda)^{-m}}$ exists ($l \neq 0$). By writing the spectral theorem for H and using lemma 3.1, proposition 2.4 we have

$$\begin{aligned}\lim_{\lambda\to\infty} \frac{\mathrm{tr}\ (A(H+\lambda)^{-m})}{\mathrm{tr}\ (H+\lambda)^{-m}} &= \lim_{\lambda\to\infty} \frac{\mathrm{tr}\ (A^{\frac{1}{2}}(H+\lambda)^{-m}A^{\frac{1}{2}})}{\mathrm{tr}\ (H+\lambda)^{-m}} \\ &= \lim_{\lambda\to\infty} \frac{\int_0^\infty \frac{1}{(x+\lambda)^m} d(\mathrm{tr}\ (\mathrm{A}^{\frac{1}{2}}\mathrm{E_H(x)}\mathrm{A}^{\frac{1}{2}})}{\int_0^\infty \frac{1}{(x+\lambda)^m} d(\mathrm{tr}\ (\mathrm{E_H(x)}))} = l\end{aligned}$$

Then by using theorem 2.9 we have

$$\lim_{\lambda\to\infty} \frac{\mathrm{tr}\ (\pi_\lambda A \pi_\lambda)}{\sum \int_{H(\xi,x)\leq\lambda} \tilde{d}x} = l. \tag{3.1}$$

Again using proposition 2.4 we have

$$\begin{aligned}\lim_{\lambda\to\infty} \frac{\mathrm{tr}\ (A(H+\lambda)^{-m})}{\mathrm{tr}\ (H+\lambda)^{-m}} &= \lim_{\lambda\to\infty} \frac{\mathrm{tr}\ (A[(H(\xi,x)+\lambda)^{-m}]^w)}{\mathrm{tr}\ [(H(\xi,x)+\lambda)^{-m}]^w} \\ &= \lim_{\lambda\to\infty} \frac{\sum\int \frac{a(\xi,x)}{(H(\xi,x)+\lambda)^m} dx}{\sum\int \frac{1}{(H(\xi,x)+\lambda)^m} dx} \\ &= \lim_{\lambda\to\infty} \frac{\int_0^\infty \frac{1}{(u+\lambda)^m} aoH^{-1}(u) d(\overline{\mu}oH^{-1})(u)}{\int_0^\infty \frac{1}{(u+\lambda)^m} d(\overline{\mu}oH^{-1})(u)} = l,\end{aligned}$$

where $\overline{\mu}$ is the product measure of counting measure on $\mathbb{Z}^d$ and the normalised invariant measure on $\mathbb{T}^d$. Again by using theorem 2.9 we have

$$\lim_{\lambda\to\infty} \frac{\sum \int_{H(\xi,x)\leq\lambda} a(\xi,x)dx}{\sum \int_{H(\xi,x)\leq\lambda} dx} = l. \tag{3.2}$$

By using equation (3.1) and (3.2), one has

$$\lim_{\lambda\to\infty} \frac{\mathrm{tr}\ (\pi_\lambda A \pi_\lambda)}{\sum \int_{H(\xi,x)\leq\lambda} a(\xi,x)dx} = 1.$$

Consider

$$\begin{aligned}
&\lim_{\lambda\to\infty} \frac{tr(\pi_\lambda A\pi_\lambda)}{\text{rank } \pi_\lambda} \\
= &\lim_{\lambda\to\infty} \frac{\text{tr}\,(\pi_\lambda \text{A}\pi_\lambda)}{\sum\int_{H(\xi,x)\le\lambda} a(\xi,x)dx} \frac{\sum\int_{H(\xi,x)\le\lambda} a(\xi,x)dx}{\text{rank } \pi_\lambda} \frac{\text{rank } \pi_\lambda}{\sum\int_{H(\xi,x)\le\lambda} dx} \\
= &\lim_{\lambda\to\infty} \frac{\sum\int_{H(\xi,x)\le\lambda} a(\xi,x)dx}{\sum\int_{H(\xi,x)\le\lambda} dx} \\
= &\lim_{\lambda\to\infty} \frac{\sum\int_{H(\xi,x)\le\lambda} a(\xi,x)\tilde{d}x}{vol(\{(\xi,x) : H(\xi,x)\le\lambda\})}.
\end{aligned}$$

□

Corollary 3.4. *Let $P(\lambda)$ be a polynomial on $\mathbb{R}$. Then*

$$\lim_{\lambda\to\infty} \frac{\text{tr } \pi_\lambda \text{P(A)}\pi_\lambda}{\text{rank } \pi_\lambda} = \lim_{\lambda\to\infty} \frac{1}{vol(H(\xi,x)\le\lambda)} \sum_{H(\xi,x)\le\lambda} \int P(a(\xi,x))\tilde{d}x$$

Proof. Notice that by the composition rule, $P(A)$ is a pseudo difference operator with symbol $P(a(\xi,x)) + r_{-1}(\xi,x), r_{-1} \in S_H^{-1}$. Since any $r \in S_H^{-1}$ satisfies $|r(\xi,x)| \le C(H(\xi,x)+\lambda)^{-\frac{1}{2}}$, it is clear that $r(\xi,x)$ vanishes for large λ. Therefore we have

$$\lim_{\lambda\to\infty} \frac{\text{tr } \pi_\lambda \text{P(A)}\pi_\lambda}{\text{rank } \pi_\lambda} = \lim_{\lambda\to\infty} \frac{1}{vol(H(\xi,x)\le\lambda)} \sum_{H(\xi,x)\le\lambda} \int P(a(\xi,x))\tilde{d}x$$

□

Corollary 3.5. *Consider $f \in \mathcal{C}(\mathbb{R})$ and let $\mu_\lambda(f) = \sum_{\lambda_j\le\lambda}(f(A)f_j, f_j)$. Then $\dfrac{\mu_\lambda(f)}{\text{rank } \pi_\lambda}$ has a $\mu(f)$ as $\lambda\to\infty$, where*

$$\mu(f) = \lim_{\lambda\to\infty} \frac{1}{vol(H(\xi,x)\le\lambda)} \sum_{H(\xi,x)\le\lambda} \int f(a(\xi,x))\tilde{d}x.$$

Proof. We note that $rank(\pi_\lambda) = \text{tr}\,(\pi_\lambda)$. The eigen-values of $\pi_\lambda A\pi_\lambda$ are bounded by $\|A\|$ for all λ. Also the values of $a(\xi,x)$ are bounded by some constant C (say). Therefore any continuous function $f \in \mathcal{C}(\mathbb{R})$ can be approximated uniformly on $I = \{|x| \leq \max(\|A\|, C)\}$ by a polynomial. Then

$$\begin{aligned}
\lim_{\lambda\to\infty} \frac{\mu_\lambda(f)}{\text{rank } \pi_\lambda} &= \lim_{\lambda\to\infty} \frac{\sum_{\lambda_j \leq \lambda} (f(A)f_j, f_j)}{\text{rank } \pi_\lambda} \\
&= \lim_{n,\lambda\to\infty} \frac{\sum_{\lambda_j \leq \lambda} (P_n(A)f_j, f_j)}{\text{rank } \pi_\lambda} \\
&= \lim_{n,\lambda\to\infty} \frac{1}{vol(H(\xi,x) \leq \lambda)} \sum_{H(\xi,x)\leq\lambda} \int P_n(a(\xi,x))\tilde{d}x \\
&= \lim_{\lambda\to\infty} \frac{1}{vol(H(\xi,x) \leq \lambda)} \sum_{H(\xi,x)\leq\lambda} \int f(a(\xi,x))\tilde{d}x.
\end{aligned}$$

□

We now prove theorem 1.1 before taking up the proof of the main theorem.

Proof of Theorem 1.1: The proof is almost identical to that of the Laptev-Saffrov proof in [3] with mild modification to accommodate for the relative boundedness of $\tilde{T}$. We shall indicate the main steps of the proof.

Assume that $[S, T]$ is relatively bounded with respect to S . To prove the above result it is sufficient to estimate $\|(I - \pi_\lambda)T\pi_\lambda\|_{HS}$, where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. Since

$$\|(I - \pi_\lambda)T\pi_\lambda\|_{HS}^2 \leq 2(\|(I - \pi_\lambda)T\pi_{\lambda-\epsilon}\|_{HS}^2 + \|(I - \pi_\lambda)T(\pi_\lambda - \pi_{\lambda-\epsilon}\|_{HS}^2)$$

and

$$\|(I - \pi_\lambda)T(\pi_\lambda - \pi_{\lambda-\epsilon}\|_{HS} \leq \|T\|^2 N_\epsilon(\lambda),$$

we need to estimate $\|(I - \pi_\lambda)T\pi_{\lambda-\epsilon}\|_{HS}$ only.

So by definition

$$\|(I - \pi_\lambda)T\pi_{\lambda-\epsilon}\|_{HS}^2 = \sum_{\lambda_k \geq \lambda} \sum_{\lambda_j < \lambda-\epsilon} |\langle Tf_j, f_k\rangle|^2.$$

Since

$$\langle Tf_j, f_k\rangle = (\lambda_k - \lambda_j)^{-1}(\lambda_j + 1)^\sigma \langle \tilde{T}(S + I)^{-\sigma} f_j, f_k\rangle,$$

where $\tilde{T} = [T, S]$, we have,

$$
\begin{aligned}
\|(I-\pi_\lambda)T\pi_{\lambda-\epsilon}\|^2_{HS} &= \sum_{\lambda_k\ge\lambda}\sum_{\lambda_j<\lambda-\epsilon}|\langle Tf_j,f_k\rangle|^2\\
&= \sum_{\lambda_k\ge\lambda}\sum_{\lambda_j<\lambda-\epsilon}|(\lambda_k-\lambda_j)^{-2}(\lambda_j+1)^{2\sigma}||\langle\tilde{T}(S+I)^{-\sigma}f_j,f_k\rangle|^2\\
&\le \sum_{k}\sum_{\lambda_j<\lambda-\epsilon}|(\lambda_k-\lambda_j)^{-2}[(\lambda_k-\lambda_j)^{2\sigma}\\
&\quad +(\lambda_j+1)^{2\beta}]|\langle\tilde{T}(S+I)^{-\sigma}f_j,f_k\rangle|^2\\
&\le \|\tilde{T}(S+I)^{-\sigma}\|^2\sum_{\lambda_j<\lambda-\epsilon}|(\lambda-\lambda_j)^{-2}[(\lambda-\lambda_j)^{2\sigma}+(\lambda_j+1)^{2\beta}]|\\
&= \|\tilde{T}(S+I)^{-\sigma}\|^2\int_0^{\lambda-\epsilon}[\frac{1}{|\lambda-\beta|^{2(1-\sigma)}}+\frac{(\beta+1)^{2\sigma}}{|\lambda-\beta|^2}]dN(\beta)\\
&\le \|\tilde{T}(S+I)^{-\sigma}\|^2N_{\frac{\epsilon}{2}}(\lambda)\sum_{k=0}^{K^*}\left(\frac{1}{|\lambda-\frac{k\epsilon}{2}|^{2(1-\sigma)}}+\frac{(\frac{k\epsilon}{2}+1)^{2\sigma}}{|\lambda-\frac{k\epsilon}{2}|^2}\right)
\end{aligned}
$$

where $\lambda-\frac{\epsilon}{2}\ge K^*>\lambda-\epsilon$. The sum in the right hand side converges if $0<\sigma<1$ and is independent of λ.

Proof of Theorem 1.2 : We prove the theorem for the case $\rho=1$, the other cases are similar. Assume that the commutator $[A,H]$ is relatively bounded with respect to H. Take S to be the discritized Schrödinger operator and T to be the 0th order bounded selfadjoint pseudo difference operator. To prove the Szegö theorem we make use of theorem 1.1 and therefore we only have to show that $\frac{N_\epsilon(\lambda)}{\text{rank }\pi_\lambda}\to 0$ as $\lambda\to\infty$. Notice that $N(\beta)=\text{tr P}_{\text{H}\le\beta}$, where $P_{H\le\beta}$ is the projection of $\ell(\mathbb{Z}^d)$ onto $\{u\in\mathbb{Z}^d:\langle Hu,u\rangle_{\mathbb{Z}^d}\le\beta\|u\|^2_{\mathbb{Z}^d}\}$. We show that $P_{H\le\beta}\le P_{V\le\beta}$ and $P_{P\le\beta-\epsilon}\ge P_{V\le\beta-4d-\epsilon}$. To show $P_{H\le\beta}\le P_{V\le\beta}$ enough to show $H_\beta:=P_{H\le\beta}\ell^2(\mathbb{Z}^d)\subset P_{V\le\beta}\ell^2(\mathbb{Z}^d):=\tilde{H}_\beta$. Let $u\in H_\beta$. Then $\langle Vu,u\rangle_{\mathbb{Z}^d}\le\langle Hu,u\rangle_{\mathbb{Z}^d}\le\beta\|u\|^2_{\mathbb{Z}^d}$. So $V\in\tilde{H}_\beta$. Similarly $P_{P\le\beta-\epsilon}\ge P_{V\le\beta-4d-\epsilon}$ can be shown.

Consider

$$\begin{aligned}\frac{\text{tr }(\mathrm{P_{H\leq\beta}}-\mathrm{P_{H\leq\beta-\epsilon}})}{\text{rank }(\pi_\lambda)} &\leq \frac{\text{tr }(\mathrm{V_{H\leq\beta}})-\text{tr }(\mathrm{P_{V\leq\beta-4d-\epsilon}})}{\text{rank }(\pi_\lambda)}\\ &\leq \frac{\beta^{\frac{d}{k}}-(\beta-4d-\epsilon)^{\frac{d}{k}}}{\text{rank }(\pi_\lambda)}\\ &= \frac{\beta^{\frac{d}{k}}-(\beta-4d-\epsilon)^{\frac{d}{k}}}{\lambda^{\frac{d}{k}}}\end{aligned}$$

Taking supremum over all β less or equal to λ and allowing $\lambda\to\infty$ we have $\frac{N_\epsilon(\lambda)}{\text{rank }\pi_\lambda}\to 0$ as $\lambda\to\infty$.

Remark 3.6. The above limits are the same if A is replaced by $A+B$ for any compact operator B on $\ell^2(\mathbb{Z}^d)$.

Proof. To prove the above result, enough to show $\lim_{\lambda\to\infty}\dfrac{\text{tr }(\pi_\lambda \mathrm{A^n}\pi_\lambda)}{\text{tr }(\pi_\lambda)} = \lim_{\lambda\to\infty}\dfrac{\text{tr }(\pi_\lambda(\mathrm{A+B})^{\mathrm{n}}\pi_\lambda)}{\text{tr }(\pi_\lambda)}$ for any compact operator B on $\ell^2(\mathbb{Z}^d)$. Notice that $(A+B)^n = A^n+$ terms with factor A^pB^{n-p} or B^pA^{n-p} where $p\in\{1,2\cdots,n\}$. Since the class of compact operators form a two sided ideal of the class of bounded operators $(A+B)^n = A^n+$ a compact operator. We show that for any compact operator T, $\lim_{\lambda\to\infty}\frac{\text{tr }(\mathrm{T(H+\lambda)^{-m}})}{\text{tr }((\mathrm{H+\lambda})^{-\mathrm{m}})}=0$.

Since T is a compact operator, for given $\epsilon>0$ there exist a finite rank operator T_n such that $\|T_n-T\|<\epsilon$ for $n\geq N_0$. Consider

$$\begin{aligned}\frac{\text{tr }(\mathrm{T(H+\lambda)^{-m}})}{\text{tr }((\mathrm{H+\lambda})^{-\mathrm{m}})} &= \frac{\text{tr }(\mathrm{T_n(H+\lambda)^{-m}})}{\text{tr }((\mathrm{H+\lambda})^{-\mathrm{m}})}+\frac{\text{tr }((\mathrm{T-T_n})(\mathrm{H+\lambda})^{-\mathrm{m}})}{\text{tr }((\mathrm{H+\lambda})^{-\mathrm{m}})}\\ &\leq \frac{\text{tr }(\mathrm{T_n(H+\lambda)^{-m}})}{\text{tr }((\mathrm{H+\lambda})^{-\mathrm{m}})}+\|T-T_n\|\end{aligned}$$

Since T_n is a finite rank operator, without loss of generality we show $\lim\limits_{\lambda\to\infty}\dfrac{\text{tr }(\mathrm{T_n(H+\lambda)^{-m}})}{\text{tr }((\mathrm{H+\lambda})^{-\mathrm{m}})}=0$ for a rank one operator. Assume that T_n is a rank one operator. Therefore

$$\text{tr }(\mathrm{T_n(H+\lambda)^{-m}}) = \langle(\mathrm{H+\lambda})^{-\mathrm{m}}\mathrm{f},\mathrm{f}\rangle = \int_{\mathrm{Spec(H)}}\frac{1}{(\mathrm{x}+\lambda)^{\mathrm{m}}}\mathrm{d}\mu_{\mathrm{f,f}}(\mathrm{x}) \leq \frac{\mu_{\mathrm{f,f}}(\mathbb{R})}{\lambda^{\mathrm{m}}}.$$

So

$$\begin{aligned}
\frac{\mathrm{tr}\ (\mathrm{T}(\mathrm{H}+\lambda)^{-\mathrm{m}})}{\mathrm{tr}\ ((\mathrm{H}+\lambda)^{-\mathrm{m}})} &\leq C\frac{1}{\lambda^m \mathrm{tr}\ ((\mathrm{H}+\lambda)^{-\mathrm{m}})} \\
&= C\frac{\lambda^{-m}}{\displaystyle\sum_{i\in\mathbb{N}}\frac{1}{(\lambda_i+\lambda)^m}} \\
&\leq C\frac{\lambda^{-m}}{\displaystyle\sum_{\lambda_i\leq\lambda}\frac{1}{(\lambda_i+\lambda)^m}} \\
&\leq C\frac{\lambda^{-m}}{\displaystyle\sum_{\lambda_i\leq\lambda}\frac{1}{(2\lambda)^m}} = O(\lambda^{-1}).
\end{aligned}$$

Therefore $\lim_{\lambda\to\infty}\frac{\mathrm{tr}\ (\mathrm{A}(\mathrm{H}+\lambda)^{-\mathrm{m}})}{\mathrm{tr}\ ((\mathrm{H}+\lambda)^{-\mathrm{m}})} = \lim_{\lambda\to\infty}\frac{\mathrm{tr}\ (\tilde{\mathrm{A}}(\mathrm{H}+\lambda)^{-\mathrm{m}})}{\mathrm{tr}\ ((\mathrm{H}+\lambda)^{-\mathrm{m}})}$, where $\tilde{A} = A+B$. Now applying theorem lemma 3.1, proposition 2.4 and theorem 2.9 individually to both the limits, we have $\lim_{\lambda\to\infty}\dfrac{\mathrm{tr}\ (\pi_\lambda \mathrm{A}^\mathrm{n}\pi_\lambda)}{\mathrm{tr}\ (\pi_\lambda)} = \lim_{\lambda\to\infty}\dfrac{\mathrm{tr}\ (\pi_\lambda(\tilde{\mathrm{A}})^\mathrm{n}\pi_\lambda)}{\mathrm{tr}\ (\pi_\lambda)}$. □

Example 3.7. Let $V(\xi)$ be a bounded function on $\mathbb{Z}^d$. Let A be the operator of multiplication by the function $V(\xi)$ on $\ell^2(\mathbb{Z}^d)$ and $H = \rho\Delta + |\xi|^k, 0 \leq \rho \leq 1,\ k > 0$ with Δ defined as in the equation 1.1. Then it is clear that H (being a sum of two positive operators) is a positive operator and a simple estimate of the resolvent shows that it has purely discrete spectrum.

The commutator $[A, H] = [\Delta, V]$ is bounded and hence also $[A, H](H+I)^{-\sigma}$ for any $\sigma > 0$. Therefore in this case the conclusions of theorems 1.1 and 1.2 are valid.

Example 3.8. Let V and Δ be defined as in the previous example. Let $A = \Delta + V$ and $H = \rho\Delta + |\xi|^k$ for $0 \leq \rho \leq 1,\ 0 < k < 2$. Then the commutator $[A, H] = [A, \rho\Delta] + [A, |\xi|^k]$ turns out to be $[\rho\Delta, |\xi|^k]$ up to an addition of a bounded operator. This term behaves like $C|\xi|^{k-1}$. Therefore $[A, H](H+I)^{-\sigma} \sim |\xi|^{k-1}(H+I)^{-\sigma}+$ a bounded operator.

But it is a simple estimate to see that $|\xi|^{k-1}(H + I)^{-\sigma}$ is bounded and $|\xi|^{k-1}(H + I)^{-\sigma}$ is bounded iff $k \in (0, 2)$.

Example 3.9. Let V be defined as before. Let $A = P(\Delta) + V$ and $H = \rho\Delta + |\xi|^k$ for $0 \leq \rho \leq 1,\ 0 < k < 2$, where $P(\Delta)$ is a real polynomial in Δ. Then by using the previous argument we have $[A, H](H + I)^{-\sigma} \sim |\xi|^{k-1}(H + I)^{-\sigma}+$ a bounded operator iff $k \in (0, 2)$.

References

1. Keldys, M. V. On a Tauberian theorem, Trudy Mat. Inst. Stekloc. **38**, (1951), 77-86.
2. Laptev, A and Saffrov, Yu. Szegö type limit theorems, J. Func. Anal. **138**, (1996), 544-559.
3. Laptev, A and Saffrov, Yu. Error estimates in the generalized Szegö theorem, Journees Equations aux derivees partieleles, (1991), 1-7.
4. U. Grenander and G. Szegö,Toeplitz Forms and their applications, 2nd edn. Chelsea, New Tork (1984).
5. A. F. Grishin and I. V. Poedintseva, Towards the Keldys Tauberian theorem, J. Math. Sci. no. 4 **134** (2006), 2272-2287.
6. L. Hormander, The analysis of linear partial differential operators - III, Pseudo-differential operators, Springer-Verlag, Berlin, 1994.
7. D. Robert, Remarks on a paper of S. Zelditch: ‘’Szego limit Theorems in Quantum Mechanics”, *J. Funct. Anal.* **53**(1983), 304–308.
8. G. Szegö, On certain Hermitian forms associated with the Fourier series if positive function, *comm. Sem. Math. Univ. Lund.* (1952), 228–238.
9. M. Taylor, ‘’Pseudo differential operators,“Princeton univ. press, Princeton, 1980.
10. M. Ruzhansky and V. Turunen, Quantisation of Pseudo-differential operators on the torus, arxiv 0805.2892v1[math.FA], 19 May 2008.
11. D. Widder, The Laplace Transform, *Princeton Univ. press* (1941).
12. S. Zelditch, Szegö limit theorems in quantum mechanics, *J. Funct. Anal.* **50**(1983), 67–80.

Institute of Mathematical Sciences, Taramani, Chennai 600113
E-mail address: jitumath@imsc.res.in, krishna@imsc.res.in