Asymptotic Solutions to Hyperbolic Equations - a brief overview

(M. Vanninathan, TIFR Centre, P.O.Box 1234, Bangalore - 560 012)

The purpose of this write-up is to discuss certain aspects of linear and nonlinear wave propagation in homogeneous and nonhomogeneous media. It is going to be a brief over-view of problems, methods and ideas rather than technicalities. This presentation contains both classical material and moden developments. Bibliographic material is by no means meant to be complete. The goal is merely to understand the significance of some recent developments viewed in the light of classical problems of wave propagation. Most of the arguments are heuristic and so hopefully accessible also to non-specialists in the field.

Talking of wave propagation in non-homogeneous media, two essential parameters come to our mind : wavelength of the waves (denoted as ε) and wavelength of the medium (denoted by λ). We can distinguish qualitatively three different cases.

(a) $\lambda << \varepsilon$: An instructive example of this case would be homogenization of elliptic problems with periodically oscillating coefficients. There is a huge literature on this topic presenting a wide variety of methods : method of asymptotic expansion [1], Tartar's method [1], two-scale convergence method [2], [3], Γ-convergence method [4], Bloch-wave method [3],[5]. We are not concerned with this important development in this write-up.

(b) $\lambda \approx \varepsilon$: In this case, there is resonant interaction between the medium and the propagating waves and as a consequence new waves are produced (e-g : Bloch waves). Since such waves and their perturbation were widely discussed in this meeting, we do not intend to pursue further these lines.

(c) $\varepsilon << \lambda$: in this case, oscillation of the waves is much more compared with the variations of the medium. Heuristically for such short waves, the medium looks almost homogeneous and so a perturbation theory from the homogeneous case is possible. This is the basic idea behind the method of Geometrical Optics and we will be concerned with this aspect in the sequel. Laws governing the propagation of waves are quite often described by Partial Differential Equations (PDE) which may be linear or nonlinear and are usually complicated. The task at hand is to get a simplified picture of propagation under the above asymptotic limit. A classical example is Newton's theory of light which predicted straight line motion of light waves.

In carrying out the above task, our discussions centre around Maxwell type classical waves rather than Schrodinger type Quantum Mechanical waves. The essential characteristics of the first type of waves (referred to as hyperbolic) are finiteness of the speed of propagation, boundedness of group velocity w.r.t the wave numbers etc. In contrast, the second family of waves has infinite propagation speed and their behaviour was the main point of discussion in the contribution of W. Craig to this volume. Some of the recent advances which we will be discussing below have taken place only in the hyperbolic set-up ; similar developments in the understanding of other types of waves should be hotly pursued in future.

Linear waves - Homogeneous case

By this we mean a situation where the PDE involved is linear and with constant coefficients. Moreover, the initial profile of the wave is a plane wave. More precisely, we have the problem

$$\left\{ \begin{array}{rl} P(\partial_t, \partial_x)u = & 0\ ,\ x \in \mathrm{IR}^{\mathrm{d}},\ \mathrm{t} > 0\ , \\ u(x,0) = & e^{i\xi x}, x \in \mathrm{IR}^{\mathrm{d}}\ . \end{array} \right. \tag{1}$$

Here $\xi \in \mathrm{IR}^{\mathrm{d}} \backslash (0)$ is given and P is a partial differential operator in (x,t) variables. Assuming that $\{t = 0\}$ is non-characteristic w.r.t P, we demand that there are m (= order of P) real and distinct roots $w = w(\xi)$ for the polynomial equation $P(-iw, i\xi) = 0 \quad \forall \quad \xi \in \mathrm{IR}^{\mathrm{d}}$. This is called hyperbolicity condition. For each such root, we can generate a plane wave $u(x,t) = \exp\left[i(x\cdot\xi - wt)\right]$ which solves the above problem. Thus we get a very satisfying picture : each initial plane wave bifurcates into m-families of plane waves, each one is identified by its phase speed $\frac{w(\xi)}{|\xi|}$. General wave patterns may be obtained via super position.

Linear waves - Nonhomogeneous case

Inhomogenities may be introduced into (1) by making the coefficients of the operator in (1) to depend on x or by considering initial condition which is not a plane wave. In such situations, there is a new phenomenon which was not present in homogeneous case and the purpose of this section is to highlight it in an example. It is quite clear that plane wave solutions will not work anymore and the idea is to get a propagation picture which is a perturbation from the homogeneous case when inhomogenities vary slowly. How to quantify this statement ? By introducing a small parameter $\varepsilon > 0$, it is reasonable to propose the following Ansatz for the approximate solution $u^{\varepsilon} = a(\varepsilon x, \varepsilon t) e^{i\varepsilon^{-1}\phi(\varepsilon x, \varepsilon t)}$ where the amplitude a is no more a constant and a slowly varying complex-valued function of (x,t) and the phase function ϕ is no more linear but slowly varying and real-valued. This is the fundamental idea in Linear Geometrical Optics (LGO) and in mathematics literature, it has been pioneered by LAX [6]. Theory of Pseudo Differential Operators and Fourier Integral Operators formalizes his ideas.

Below, we shall see how this Ansatz works in the example of the wave equation which governs propagation of waves in an isotropic medium with speed unity :

$$u_{tt} - \Delta u = 0\ , \qquad x \in \mathbb{R}^d,\ t > 0\ . \tag{2}$$

Even though this has constant coefficients, we will consider an initial wave front $\sum_0$ which is not necessarily planar. Since the equation is homogeneous, our previous Ansatz reduces to

$$u^\varepsilon = a(x,t)e^{i\varepsilon^{-1}\theta(x,t)}\ . \tag{3}$$

What are then the equations governing the evolution of a and θ ? They are obtained by simply inserting (3) into (2) and equating the coefficient of ε^{-2} and ε^{-1} to zero. This procedure yields

$$\theta_t^2 = |\nabla\theta|^2\ , \tag{4}$$

$$2\theta_t a_t - 2\nabla\theta\cdot\nabla a + (\theta_{tt} - \Delta\theta)\, a = 0\ . \tag{5}$$

Discussion of (4)

In fact (4) consists of two equations $\theta_t + |\nabla\theta| = 0$ and $\theta_t - |\nabla\theta| = 0$. This simply signifies that there are two families of waves generated by (2). We will deal with one of them in the sequel :

$$\theta_t + w(\nabla\theta\ ) = 0 \quad \text{with} \quad w(\xi) = -|\xi|\ . \tag{6}$$

This is a scalar, first order equation for the phase θ called Eikonal equation. Even though, the original equation (2) is linear, the equation (6) obtained from it is nonlinear. Hamilton-Jacobi theory reduces the resolution of (6) to a system of ordinary differential equations (ODE) by the method of characteristics. They are defined by

$$\frac{dx}{dt} = \frac{\partial w}{\partial\xi}(x(t),\xi(t)) \quad , \quad \frac{d\xi}{dt} = -\frac{\partial w}{\partial x}(x(t),\xi(t))\ . \tag{7}$$

Since the equation (2) has constant coefficients, we have $\frac{\partial w}{\partial x} \equiv 0$ and so we obtain

$$\xi(t) \equiv \xi(0), \quad x(t) = \frac{\xi(0)}{|\xi(0)|}t + x(0)\ . \tag{8}$$

Further we have $\xi(t) = \nabla\theta(x(t),t)$ and (6) reduces to $\frac{d}{dt}\theta(x(t),t) \equiv 0$. These informations enable us to construct the wave-front at time t :

$$\Sigma_t = \{x/\theta(x,t) = \text{ constant }\}\ .$$

Indeed $x(t)$ lies on Σ_t and $\xi(t)$ is normal to Σ_t at $x(t)$ and hence defines an infinitesimal piece of Σ_t at $x(t)$. From this picture, it is clear that we are not only able to follow the evolution of Σ_t as a whole but also that of individual points on it. It is also worth to note that even though (6) is an equation in x-space, it is necessary to work in (x, ξ) space to solve it. We refer to $x(t)$ as rays and $(x(t), \xi(t))$ as bicharacteristic strip.

Discussion of (5)

Given θ, (5) is a linear, first order equation for a (called transport equation) which can easily be solved once again by method of characteristics. We note that the characteristics of (5) are precisely the rays of (4) given by (7). (5) is reduced to a linear ODE along the rays $x(t)$ which can be integrated.

Caustics

The classical picture of propagation described above is, in general, valid only locally. In this paragraph, we are going to see what the obstacles are for a global theory. First of all, given initial condition $(x(0), \xi(0))$, unique solution to the Hamiltonian system (7) exists in general only locally in time. Secondly, even if global solution exists as in (8), we have the formation of caustics ($\equiv$ envelope of rays) due to the focussing effect of rays. To illustrate the point, let us consider a spherical wave front $\theta(x,t) = t + |x|$ which solves (4). As time evolves, one can easily see the associated rays $x(t)$ tend to focus at the origin. What happens to the amplitude at the origin ? In fact, the transport equation (5) is reduced to $a_s + \frac{d-1}{2s} a = 0$ along rays. Explicit solution is $a = s^{-\frac{1}{2}(d-1)}$ which shows blow-up of the amplitude at the focus. Thus our Ansatz breaks down in such cases and we need to modify it to overcome the difficulties arising out of caustics. Without going into details, let us remark that this can be overcome by the introduction of multi-phases and by assuming some apriori knowledge on the nature of caustics. One example which is intuitive has been worked out in LUDWIG [7].

Justification

In this paragraph, we will give some indications as to why the solution constructed above is an approximate solution. To this end, we need to talk about initial conditions. We have already seen that there are two families of waves generated. Let us denote their phases by $\theta^{\pm}$ and the corresponding amplitudes by $a^{\pm}$. Since the equation is linear, it is reasonable to propose a general Ansatz by super position : $u^{\varepsilon} = a^{+} e^{i\varepsilon^{-1}\theta^{+}} + a^{-} e^{i\varepsilon^{-1}\theta^{-}}$. It is then reasonable to take initial conditions for the exact solutions of (2) of the following form :

$$\left\{ \begin{array}{ll} u(x,0) = & f(x) e^{i\varepsilon^{-1}\tilde{\theta}(x)} , \\ u_t(x,0) = & i\varepsilon^{-1} g(x) e^{i\varepsilon^{-1}\tilde{\theta}(x)} , \end{array} \right. \tag{9}$$

where $f, g, \tilde{\theta}$ are all given functions. To see what are the initial conditions for $\theta^{\pm}$ and $a^{\pm}$, let us calculate u^{ε}_t :

$$u^{\varepsilon}_t = \left(a^{+}_t + i\varepsilon^{-1} \theta^{+}_t a^{+} \right) e^{i\varepsilon^{-1}\theta^{+}} + \left(a^{-}_t + i\varepsilon^{-1} \theta^{-}_t a^{-} \right) e^{i\varepsilon^{-1}\theta^{-}} .$$

Comparing u^ε with u and u^ε_t with u_t, we are led to the following conditions :

$$\begin{cases} \theta^\pm(x,0) = \tilde{\theta}(x) \ , \\ a^+(x,0) + a^-(x,0) = f(x), \quad \left(\theta^+_t a^+ + \theta^-_t a^-\right)(x,0) = g(x) \ . \end{cases}$$

Thanks to (4), these determine $a^\pm(x,0)$ uniquely.

Thus, in a formal manner, it follows from the very construction of u^ε, that $u - u^\varepsilon$ satisfies

$$\begin{cases} (u-u^\varepsilon)_{tt} - \Delta(u-u^\varepsilon) = O(1) \ , \\ (u-u^\varepsilon)(x,0) = 0 \ , \\ (u-u^\varepsilon)_t(x,0) = O(1) \ . \end{cases} \tag{10}$$

It is therefore natural to measure the error $(u - u^\varepsilon)$ in the energy norm. For a function $u = u(x,t)$, the energy at time t is defined by

$$E(u;t) = \frac{1}{2}\int_{\mathbb{R}^d} \left(|u_t|^2 + |\nabla u|^2\right) dx \ .$$

If u satisfies (2), it is well-known that $E(u;t)$ is independent of t. However $u - u^\varepsilon$ satisfies (10) where there is a non-zero right hand side denoted by F. In such situations, it is common to use the energy inequality which can be written as follows :

$$E(u-u^\varepsilon;t) \le c\left[E(u-u^\varepsilon;0) + \int_0^t \|F(\cdot,s)\|^2_{L^2(\mathbb{R}^d)}\, ds\right]$$

Because of (10), we get

$$E(u-u^\varepsilon;t) \le ct \ . \tag{11}$$

Certain remarks are in order regarding the above estimate. Recall that $E(u,;t) = E(u;0) = 0(\varepsilon^{-2})$. Thus (11) is indeed a non-trivial estimate. We can refine (11) by calculating approximate solution of the form $v^\varepsilon = u^\varepsilon + \varepsilon u^\varepsilon_1 + \varepsilon^2 u^\varepsilon_2 + \ldots + \varepsilon^N u^\varepsilon_N$ in which case, we can derive $E(u - v^\varepsilon;t) \le c\varepsilon^{2N} t$. It is also to be noted that these estimates are not uniform as $t \to \infty$. Another inference one can make from the estimate (11) is that the two families of waves with phases $\theta^\pm$ decouple and they do not interact to the highest order.

Energy Propogation in LGO limit

Having addressed the problem of the validity of LGO approximation of the entire wave field u, let us now pay attention to a particular functional namely energy functional of the field. In this paragraph, we are going to derive a law of propagation for the limiting energy density as $\varepsilon \to 0$. Since the two families of waves decouple, it is enough to consider only the family described by (6). Energy density of u is defined by

$$E^\varepsilon(x,t) = \frac{1}{2}\left(|\partial_t u|^2 + |\nabla u|^2\right) \ .$$

The corresponding energy flux density is defined by

$$F^{\varepsilon}(x,t) = \frac{1}{2}\left(\partial_t u \nabla \overline{u} + \partial_t \overline{u} \nabla u\right)$$

Since u satisfies the wave equation (2), we have the following conservation law :

$$\partial_t E^{\varepsilon} + \text{ div } F^{\varepsilon} = 0 \; .$$

If, as shown in the previous paragraph, there is an approximation in the energy norm then we have

$$u = a e^{i\varepsilon^{-1}\theta} + O(\varepsilon)$$

and this gives

$$\begin{aligned} E^{\varepsilon} &= \varepsilon^{-2}|\theta_t|^2|a|^2 + O(\varepsilon^{-1}) + R \\ F^{\varepsilon} &= \varepsilon^{-2}\theta_t|a|^2\nabla\theta + O(\varepsilon^{-1}) + R \end{aligned}$$

where R is a remainder consisting of ε^0 terms and $e^{i\varepsilon^{-1}\theta}$ terms. Thus, to the highest order in ε, we must have the following conservation law :

$$\partial_t(|\theta_t|^2|a|^2) + \text{ div}_x \left(\theta_t|a|^2\nabla\theta\right) = 0 \; .$$

which can be rewritten as (using (6))

$$\partial_t\left(|\theta_t|^2|a|^2\right) + \text{ div}_x \left(\frac{\nabla\theta}{|\nabla\theta|}|\theta_t|^2|a|^2\right) = 0 \; . \tag{12}$$

This is a first order equation whose characteristics is again the rays $x(t)$ defined by (7). When this equation is restricted to the rays, it becomes an ODE which shows that the energy at LGO limit propagates with velocity $\frac{\xi(0)}{|\xi(0)|}$ along rays.

Microlocal Energy measures

We have already seen the difficulties involved in the justification of the full wave field under LGO approximation. Can we justify at least the energy field ? Of course, caustics is the main problem. To see the difficulties more closely, let us consider general sequences of solutions of the wave equation :

$$\left\{ \begin{array}{l} \partial_t^2 u_n - \Delta u_n = 0 \; , \; x \in \mathrm{I\!R}^{\mathrm{d}}, \; \mathrm{t} > 0 \; , \\ u_n(x,0) = u_n^0(x), \; \partial_t u_n(x,0) = u_n^1(x), \; x \in \mathrm{I\!R}^{\mathrm{d}} \; . \end{array} \right. \tag{13}$$

For simplicity, assume that u_n^0, u_n^1 have compact support. Because of finite speed of propagation, $u_n(\cdot,t)$ will have compact support $\forall\, t$. We assume further that

$$u_n^0 \rightharpoonup 0 \text{ in } H^1(\mathrm{I\!R}^{\mathrm{d}}) \text{ weak }, \; \mathrm{u_n^1} \rightharpoonup 0 \text{ in } \mathrm{L}^2(\mathrm{I\!R}^{\mathrm{d}}) \text{ weak }. \tag{14}$$

Energy considerations yield very easily that $\forall\, T > 0$

$$u_n \rightharpoonup 0 \text{ in } L^\infty\left(0,T;H^1(\mathbb{R}^d)\right) \quad \text{weak } * ,$$
$$\partial_t u_n \rightharpoonup 0 \text{ in } L^\infty\left(0,T;L^2(\mathbb{R}^d)\right) \quad \text{weak } * .$$

The question on the energy field in LGO approximation is related to the behaviour of the energy $E(u_n;t)$ as $n \to \infty$. (Indeed a simple comparison between (13) and (9) shows that $u = \varepsilon^{-1} u_n$ and so $E(u;t) = \varepsilon^{-2} E(u_n;t)$. Using non-stationary phase method, one can easily verify that the conditions (14) are satisfied by G.O. data (9) if we assume that $\tilde{\theta}$ has no stationary point). Since $E(u_n;t)$ is a quadratic, weak convergence of u_n will not suffice to pass to the limit. It is well-known that oscillations and concentrations are the sources of non-compactness. In this particular case, it is already seen that too much of energy is concentrated at caustics. One idea which was fruitful in elliptic problems (see LIONS [8]), is the consideration of the weak limit of the energy density $E_n(x,t) = \frac{1}{2}\left(|\partial_t u_n(x,t)|^2 + |\nabla u_n(x,t)|^2\right)$ in the sense of measures. However, we have here a hyperbolic problem where there is the effect of propagation ; in particular, we have seen the need to work in the phase space instead of physical space.

The introduction of micro-local energy measures formalizes the above ideas. These measures are in some sense weak limits of energy densities against test functions which depend on (x,ξ). With this at the background, let us introduce a micro-localized version of the energy

$$E(u_n;t,A) = \frac{1}{2}\int_{\mathbb{R}^d} A u_n(x,t)\overline{u}_n(x,t)dx + \frac{1}{2}\int_{\mathbb{R}^d} A\nabla u_n(x,t)\nabla\overline{u}_n(x,t)dx$$

whose A is a pseudo differential operator of order zero. The introduction of the above object is due to GERARD [9] and TARTAR [10] who prove the following compactness result :

Theorem *Fix $t > 0$. Then there is a subsequence of u_n (denoted again by n) and a positive Radon measure $\mu^t(x,\xi)$ on $\mathbb{R}^d \times \mathrm{S}^{d-1}$ such that*

$$E(u_n;t;A) \longrightarrow \int_{\mathbb{R}^d\times \mathrm{S}^{d-1}} \sigma_0(A)(x,\xi)d\mu^t(x,\xi)$$

for all pseudo differential operators A of order 0. *(Its principal symbol is denoted by $\sigma_0(A)$).*
■

This is the basic compactness Theorem giving the existence of micro-local energy measure μ^t at time t. μ^t describes the energy field of $u_n(\cdot,t)$ as $n \to \infty$ in a weak sense with all its concentrations and oscillations. Our question therefore is to know the law of propagation of μ^t as t varies. That this is the same as the one found (via heuristics) at the end of the previous paragraph is the content of the following result due to FRANCFORT & MURAT [11] and GERARD [12] :

Theorem *Fix $T > 0$. Assume that $\mu_{\pm}^0$ are the micro-local measures associated to $u_n^1 \pm i|D|u_n^0$. Then one can choose a subsequence of u_n independent of $t \in [0, T]$ and A such that*

$$E(u_n; t; A) \longrightarrow \int_{\mathbb{R}^d \times S^{d-1}} \sigma_0(A)(x,\xi) d\mu^t(x,\xi)$$

uniformly on $[0, T\ ]$. Further we have

$$\mu^t(x,\xi) = \frac{1}{2}\left[\mu_+^0(x+t\xi,\xi) + \mu_-^0(x-t\xi,\xi)\right]$$

on $\mathbb{R}^d \times S^{d-1}$. ∎

Above result gives a very satisfying picture of energy propagation overcoming the caustics problem. It shows that total energy is the sum of the energies of two families of moves propagating with velocities $\pm\xi$ along bicharacteristics strips.

We end this paragraph by giving references to other "weak" theories which ought to be pointed out in the present context. On one hand, there is a basic work of CRANDALL & LIONS [13] on viscosity solutions to (4) and on the other hand DIPERNA & LIONS [14] establish the existence of a global flow associated with (7). The possible interconnection between these view-points are to be investigated.

Nonlinear waves-Ansatz

From now on, we will be concerned with the task of generalizing the topics discussed so far to nonlinear equations. Since this is a growing field, many problems are open or under development. So we have decided to highlight mainly some basic issues connected with the works of the author [15], [16] in this field.

We consider quasi-linear equations. A simple but instructive example is the Hopf equation :

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad x \in \mathbb{R},\ t > 0\ . \tag{15}$$

The first task we face is to find a suitable Ansatz. For this purpose, it will be helpful to have a knowledge of the waves generated by (15). It is well-known that discontinuities develop in the solution at a finite time even if the initial condition is smooth. This is because of the focussing of characteristic curves associated with (15). (This is, in some sense, worse than focussing of rays described in LGO). In terms of Fourier analysis, this signifies the generation of higher harmonics in the solution starting from a single harmonic in the initial condition. Hence exponential dependence on the phase will not be a good idea here ; we must allow arbitrary dependence apriori if we want a self-consistent Ansatz. Secondly, since the system depends on u, we cannot, in general, hope to develop a big amplitude theory. Since ours is a perturbation theory from homogeneous medium to cover slowly varying medium, the hope is that an Ansatz containing a small amplitude perturbation from a known smooth function u^0 will work.

Accordingly, we set out with a constant $u^0 \in \mathbb{R}$ and propose an Ansatz of the form

$$u^\varepsilon = u^0 + \varepsilon u_1(\varepsilon^{-1}\phi, x, t) + O(\varepsilon^2) \tag{16}$$

where $\phi = \phi(x,t)$ is the phase function to be determined. The dependence of $u_1 = u_1(\theta, x, t)$ on θ cannot obviously be arbitrary and it will be decided by the equation satisfied by u^ε. The point we would like to stress is that it is not going to be exponential.

Let us consider the equation

$$u_t + f(u)_x = 0 \ , \ x \in \mathbb{R}, \ \mathrm{t} > 0 \ , \tag{17}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function. The solution $u = u(x,t)$ is a scalar function. Assuming $u^0 = 0$ without loss of any generality, we develop by Taylor expansion

$$f(u) = f(u^0) + \lambda^0 u + bu^2 + cu^3 + \ldots$$

where $\lambda^0 = f'(u^0), b = \frac{1}{2}f''(u^0), c = \frac{1}{6}f^{(3)}(u^0)$. Several simplifications occur because (17) is an equation in one-space dimension. First of all, substituting (16) into (17) and equating the coefficient of ε^0 to zero, we find

$$\left(\phi_t + \lambda^0\phi_x\right)\frac{\partial u_1}{\partial\theta} = 0 \ .$$

By the very principle of nonlinear geometrical optics (NGO), u_1 depends on θ and so $\phi_t + \lambda^0\phi_x = 0$ which means that the phase is linear $\phi = x - \lambda^0 t$.

Secondly, the equation (17) remains invariant by dilations $(x,t) \mapsto (\alpha x, \alpha t) \quad \forall\, \alpha > 0$. Thus (16) can be rewritten as

$$u^\varepsilon = u^0 + \varepsilon u_1(x - \lambda^0 t, \ \varepsilon x, \varepsilon t) + O(\varepsilon^2)$$

where $u_1 = u_1(\theta, y, \tau)$ is a function to be determined.

Next, we impose initial conditions for (17) which are of the form

$$u^\varepsilon(x,0) = u^0 + \varepsilon v(x) \ . \tag{18}$$

This choice eliminates the dependence of u_1 on the variable $y = \varepsilon x$. Thus, we arrive at

$$u^\varepsilon = u^0 + \varepsilon u_1(x - \lambda^0 t, \varepsilon t) + O(\varepsilon^2) \ . \tag{19}$$

Our next step is to substitute this expansion into (17) which yields the following equation for u_1 :

(20)
$$\begin{cases} \frac{\partial u_1}{\partial \tau} + b\frac{\partial}{\partial \theta}(u_1^2) = 0, \quad \theta \in \mathbb{R}\ ,\ \tau > 0\ , \\ u_1(\theta, 0) = v(\theta\ )\ . \end{cases}$$

This system is OK provided $b \neq 0$. If $b = 0$ then this shows that $\frac{\partial u_1}{\partial \tau} = 0$ and so u_1 does not depend on τ. This indicates that (19) is not a good Ansatz in case $b = 0$, i.e., when $f''(u^0) = 0$.

It is well-known from the theory of equations of type (17), (18) that the cases $f''(u^0) \neq 0$ and $f''(u^0) = 0$ are qualitatively different which means that the structure of corresponding solutions are not the same. Thus it is no surprise that the same is reflected in our asymptotic analysis.

Assuming $f''(u^0) = 0$ but $f^{(3)}(u^0) \neq 0$, we modify (19) as follows :

(21)
$$u^\varepsilon = u^0 + \varepsilon u_2(x - \lambda^0 t,\ \varepsilon^2 t) + O(\varepsilon^2)\ .$$

In the above expansion, we have introduced a slower time variable, viz $\varepsilon^2 t$, which can be heuristically justified as follows : Due to the fact that $f''(u^0) = 0$, the characteristic speed $f'(u)$ gets lowered when u is near u^0. If we do not introduce a slower time variable as in (21), it is clear that there will be important phase differences which will make the approximation bad. Once we accept (21), it remains to use it in (17), (18) and deduce the following equation for u_2 :

(22)
$$\begin{cases} \frac{\partial u_1}{\partial \tau} + c\frac{\partial}{\partial \theta}(u_2^3) = 0, \quad \theta \in \mathbb{R}\ ,\ \tau > 0\ , \\ u_2(\theta, 0) = v(\theta\ )\ . \end{cases}$$

Justification

The purpose of this paragraph is to give elements of proof of validity of the Ansatz suggested above and make a comparison with the linear case.

First of all, the right space for solutions of the equations we have been considering is $L^\infty \cap BV$ on $[0, T] \times \mathbb{R}$ for any $T < \infty$. Recall that BV is defined as the space functions whose distributional derivatives are measures. Such a space is required to incorporate solutions which exhibit discontinuities. Secondly, solutions in this space are not uniquely determined by the equation and the initial condition. This is because the focusing of characteristics makes the solution overdetermined at the points of discontinuity. We must stipulate other conditions to make a suitable selection. They are called entropy inequalities in the literature. We discuss them briefly here. An entropy is convex function $\eta(u)$. We associate to η, the corresponding flux function $q(u)$ specified by $q'(u) = \eta'(u)f'(u)$. If u is a solution of (17), we demand that

(23)
$$\eta(u)_t + q(u)_x \leq 0 \qquad \forall \text{ convex } \eta\ .$$

If $u \in L^\infty \cap BV$ then the left side of (23) is a measure and (23) is required to hold in the sense of measures. It is also clear that the support of this measure is contained in the set where u exhibits discontinuities. Thus (23) offers a selection criterion among possible discontinuities. The maximal entropy measure $m(u)$ is defined to be

$$m(u) = \sup_{\eta \text{ convex}} \{\eta(u)_t + q(u)_x\} \ .$$

The important result which we will not discuss further here is that entropy solutions u (i.e., solutions for which $m(u) \leq 0$) are unique.

Apart from uniqueness questions, these measures play important role in stability analysis also. Let $u_1, u_2 \in L^\infty \cap BV$ satisfy

$$\frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x} F(u_i) = \mu_i \quad , \quad i = 1, 2,$$

where μ_i are (signed) measures. Then one has

$$\int_{\mathbb{R}} |u_1(x,t) - u_2(x,t)| dx \leq \int_{\mathbb{R}} |u_1(x,0) - u_2(x,0)| dx + \Sigma_{i=1}^2 m_i(S_t) + \Sigma_{i=1}^2 |\mu_i|(S_t) \tag{24}$$

where m_1, m_2 are maximal entropy measures associated with u_1, u_2 respectively and $S_t = \{(x,s); 0 \leq s \leq t\}$. Above estimate has to be compared with the energy inequality written down earlier in the linear case. The only new phenomenon is the appearance of the measures m_i which we have discussed just above.

With this preparatory material out of our way, we can now proceed with the justification of the Ansatz introduced in the earlier paragraph. We introduce

$$\begin{aligned} U_1(x,t) &= u^0 + \varepsilon u_1(x - \lambda^0 t, \varepsilon t) \quad \text{if } b \neq 0 \ , \\ U_2(x,t) &= u^0 + \varepsilon u_2(x - \lambda^0 t, \varepsilon^2 t) \quad \text{if } b = 0, c \neq 0 \ . \end{aligned}$$

They satisfy

$$\begin{cases} \frac{\partial U_i}{\partial t} + \frac{\partial}{\partial x} f(U_i) = \frac{\partial}{\partial x}(f(U_i) - Pf(U_i)) \\ U_i(x,0) = u^0 + \varepsilon v(x) \ , \end{cases} \tag{25}$$

where

$$\begin{aligned} Pf(v) &= u^0 + f'(u^0)v + \frac{1}{2} f''(u^0) v^2 \quad \text{if } b \neq 0 \ , \\ Pf(v) &= u^0 + f'(u^0)v + \frac{1}{3!} f^{(3)}(u^0) v^3 \quad \text{if } b = 0, c \neq 0 \ . \end{aligned}$$

It remains to compare each of the solution of (25) with the solution of (17) and apply the stability estimate (24). We get

Theorem *Assume $v \in L^\infty(\mathbb{R}) \cap \mathrm{BV}(\mathbb{R})$ has compact support. Then* *(a) if $b \neq 0$ we have*

$$\sup_{0<t<\infty} \|u(\cdot,t) - U_1(\cdot,t)\|_{L^1(\mathbb{R})} = O(\varepsilon^2) \ .$$

(b) if $b = 0, c \neq 0$, we have

$$\sup_{0<t<\infty} \|u(\cdot,t) - U_2(\cdot,t)\|_{L^1(\mathbb{R})} = O(\varepsilon^2) \ . \blacksquare$$

The proof of the above result is essentially reduced to estimating the maximal entropy measures and the measures occuring in the right hand side of (25). The details of these estimates can be found in VANNINATHAN [16]. We finish this paragraph by pointing out one important difference from the linear case. In LGO, error estimate was not uniform as $t \to \infty$ contrary to the above result which assets that the error estimate is uniform as $t \to \infty$. This is because nonlinear equations under our discussion define dissipative systems and the waves decay as $t \to \infty$. These decay estimates, essential ingredient in the proof of above Theorem, are also discussed in VANNINATHAN [16].

Further discussion on NGO

In this final paragraph, we attempt to define very briefly the problems in the analysis of Nonlinear Geometrical Optics (NGO). We also give references to works where some progress is achieved. The nonlinear propagation laws can be in the form of semi-linear system or quasi-linear system which are strictly hyperbolic :

$$u_t + \sum_{j=1}^{d} A_j(x,t)\frac{\partial u}{\partial x_j} = F(x,t,u) \ , \ x \in \mathbb{R}^d, \ t > 0 \ , \tag{26}$$

$$u_t + \sum_{j=1}^{d} A_j(x,t,u)\frac{\partial u}{\partial x_j} = F(x,t,u) \ , \ x \in \mathbb{R}^d, \ t > 0 \ . \tag{27}$$

Here $u = u(x,t)$ is a N-vector, A_j is a $N \times N$ matrix depending smoothly on its arguments. F is also a N-vector depending smoothly on its arguments. Appropriate initial conditions in NGO are respectively

$$u(x,0) = H(x, \varepsilon^{-1}\phi^0(x)), \quad x \in \mathbb{R}^d \ , \tag{28}$$

$$u(x,0) = H_0(x) + \varepsilon H(x, \varepsilon^{-1}\phi^0(x)), \quad x \in \mathbb{R}^d \ . \tag{29}$$

where H, H_0 are given functions. The corresponding Ansatz are respectively of the form

$$u^\varepsilon = \sum_j U_j(x,t,\varepsilon^{-1}\phi_j(x,t)) + o(1) \ , \tag{30}$$

$$u^\varepsilon = v(x,t) + \varepsilon \sum_j U_j(x,t,\varepsilon^{-1}\phi_j(x,t)) + o(\varepsilon) \ . \tag{31}$$

Unlike linear equations, even the existence of solution u cannot be taken for granted. Even if it exists locally, the domain of existence may shrink as $\varepsilon \to 0$. Thus a modest formulation of the task in NGO can be put down along the following lines :

(T1) Prove the existence of solutions u satisfying the initial condition in a domain independent of ε.

(T2) Find equations satisfied by phases ϕ_j, profiles U_j and the mean flow V for the Ansatz (30), (31). Solve these equations.

(T3) Prove that the Ansatz (30), (31) indeed provide approximation to the exact solution as $\varepsilon \to 0$ in a suitable norm.

In the previous paragraph, we have carried out these tasks in a global way in one example where the existence and the stability of solutions are known. The general case is difficult as it can involve several instabilities.

Each phase ϕ_j usually satisfies the Eikonal equation of LGO. The resulting caustics, well-understood in linear regime, pose grave difficulties in the nonlinear case. Above programme with single phase expansion is carried out in [17], [18]. Multi-phase expansions are more difficult because typically there is a resonant interaction between the waves respresented by these phases and new phases may be created. The resulting wave pattern may be very complicated as demonstrated by [19]. Hence in order to have a reasonable picture, one is forced to impose hypotheses on the nature of the phases that can be generated by the system. By means of such assumptions, it is shown in [20] that the profiles U_j satisfy a coupled system of integro-differential equations. These equations were earlier obtained via heuristic arguments by [21] who also showed that they are greatly simplified in some physical models of compressible fluid flows. If there is no resonant interaction, then these equations become simple and decoupled.

The next task is to analyze the behaviour of the NGO waves after the formation of caustics. Depending on the nature of the nonlinearities, amplitude may blow-up, decay or can be continued. For examples, let us refer to [22], [23].

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