Counting resonances of Schrödinger operators

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0. Introduction

In the past week we have heard many talks about Schrödinger operators whose spectral measure is singular. In my lectures, I want to examine Schrödinger operators whose spectral measures are analytic. For these operators, the Green's function and the scattering operator have meromorphic continuations into the complex plane. Singularities in these continuations are called resonances, or scattering poles. I will discuss their asymptotic distribution.

In this talk I will consider Schrödinger operators

$$H = -\Delta + V$$

acting in $L^2(\mathbb{R}^n)$, n odd, whose potentials V are super-exponentially decreasing. By definition, this means that for every N there is a constant C such that

$$|V(x)| \leq Ce^{-N|x|}$$

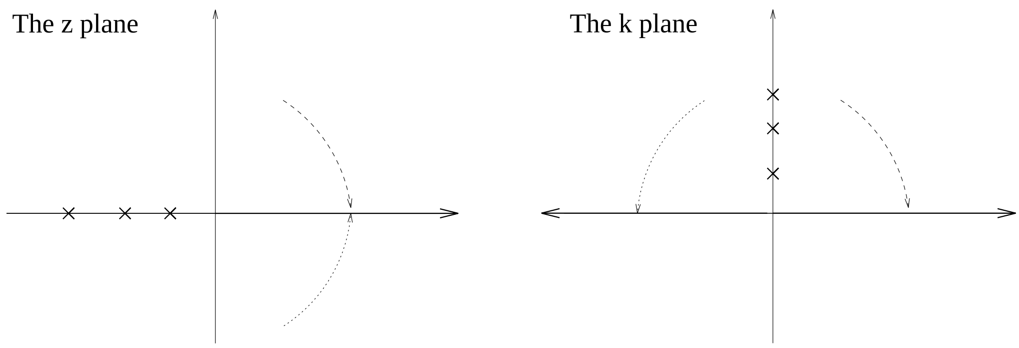
I had originally planned also to talk about some work with Peter Hislop on Laplace operators for a class of infinite volume hyperbolic manifolds. This was my original motivation for thinking about this topic. In that setting it seems more convenient to work with the scattering operator rather than with weighted resolvents. To illustrate this method, I will use the scattering operator in this paper too. I believe that this method might give better constants.

The results for Schrödinger operators that I will talk about are due to Melrose, Zworski, Vodev and others. Two general references that I can recommend are Melrose's book *Geometric Scattering Theory* [Me1] and Zworski's review article [Z1].

1. Definition of resonances

The resolvent $(H - z)^{-1}$ of a selfadjoint operator H blows up in the operator topology as z approaches the spectrum. However, when considered in a different topology, the resolvents of certain operators not only have limits onto a line of continuous spectrum, but have meromorphic continuations across this line. Poles in this continuation are called resonances. Since poles in the resolvent in the original region of definition correspond to eigenvalues, resonances can be thought of as generalized eigenvalues.

The spectrum of a Schrödinger operator with compactly supported potential consists of the positive axis together with finitely many eigenvalues on the negative axis. Thus the resolvent $(H - z)^{-1}$ is a meromorphic function on the complex plane cut along the positive real axis. The poles occur at the eigenvalues. Since the Riemann surface for Schrödinger operators in odd dimensions turns out to be the surface of $\sqrt{z}$, it is convenient to introduce the uniformizing variable $k^2 = z$. We make the convention that the upper half plane $\mathrm{Im}\, k > 0$ corresponds to the original cut plane (the "physical sheet"). On the k plane, eigenvalues lie on the positive imaginary axis. The two sides of the cut on the z plane become the postive and negative real axes on the k plane.



Define the resolvent $R(k) = (H - k^2)^{-1}$ and the free resolvent $R_0(k) = (-\Delta - k^2)^{-1}$. The Greens functions $G(x, y, k)$ and $G_0(x, y, k)$ are defined to be the integral kernels for the resolvent operators. As a first definition, we will say that a resonance is a pole in the meromorphic continuation of $G(x, y, k)$.

To get a feeling for this, let us compute the free Greens function.

$$\begin{aligned} G_0(x,y,k) &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-i\langle\xi,x-y\rangle}\frac{1}{|\xi|^2-k^2}d^n\xi \\ &= \frac{i}{4}\left(\frac{k}{2\pi|x-y|}\right)^{(n-2)/2} H^{(1)}_{(n-2)/2}(k|x-y|) \end{aligned}$$

Here $H^{(1)}_{(n-2)/2}$ denotes a Hankel function of the first kind. It has a logarithmic singularity if n is even. From now on, we will assume that n is odd. For $n=1$ and $n=3$ the expression above simplifies and we obtain

$$G_0(x,y,k)=\begin{cases} \dfrac{i}{2k}e^{ik|x-y|} & \text{if} \quad n=1 \\ \dfrac{1}{4\pi|x-y|}e^{ik|x-y|} & \text{if} \quad n=3 \end{cases}$$

This calculation shows that there are no resonances when $V=0$.

It is convenient to adopt a slightly different definition of resonances. Let χ_1 and χ_2 be functions with compact support. Then the operator $\chi_1 R_0(k)\chi_2$ has integral kernel $\chi_1(x)G_0(x,y,k)\chi_2(y)$. This operator can be analytically continued into the lower half plane, using the continuation of the kernel. Note however that in the lower half plane the middle factor grows exponentially in $|x-y|$, so the continuation cannot be interpreted as an operator product. We can now define resonances as poles in the continuation of $\chi_1 R(k)\chi_2$, whenever this exists. It is convenient to take the potential itself as a weight. Define $V^{\frac{1}{2}}=\mathrm{sgn}(V)|V|^{\frac{1}{2}}$.

Definition: A *resonance* is a pole in the meromorphic continuation of the compact operator valued function

$$R_V(k)=V^{\frac{1}{2}}R(k)|V|^{\frac{1}{2}}.$$

As we will see, when V has compact support (or is super-exponentially decreasing) $R_V(k)$ has a continuation to the whole complex plane. If V is decreasing at some fixed exponential rate, the continuation can be made to a strip but (probably) no futher. Given a connected set containing the upper half space, is there a potential V with this set as the domain of meromophicity of $R_V(k)$? It seems like one can use inverse theory to show the existence of such a potential [Ma]. However doesn't seem easy to find out much the potential's properties, such as the decay at infinity.

There are many other definitions of resonances in the literature. The theory of dilation analyticity and translation analyticity gives rise to definitions that do not require the potentials to lie in such a restrictive class. They also correspond to certain exponentially growing solutions of the Schrödinger equation.

Resonances can also be defined as poles in the continuation of the scattering operator. We will show below why this definition co-incides with the one we made above.

2. The counting function

How are properties of V reflected in the locations of the resonances? This is a subject with a huge literature. However, since resonances close to the real axis are the ones that can be measured in experiments, interest has focussed on these. Results on the asymptotic number of resonances for large $|k|$ are comparitively more recent. They were perhaps motivated by analogous results for obstacle scattering, on which there is a large literature, or even by analogous questions for hyperbolic Laplacians, motivated by analytic number theory.

The counting function is defined by

$$n(r) = \#\{\text{resonances } k : |k| < r\}$$

The large r behaviour gives the asymptotic density of resonances. In this lecture I want to survey what is known for $n(r)$ for Schrödinger operators in odd dimensions. Here is an outline.

When $n = 1$ and V has compact support

$$n(r) = \frac{2}{\pi}\mathrm{diam}(\mathrm{supp}(V))r + o(r).$$

This result is due to Zworski [Z2]. For some super-exponentially decaying potentials there is a comparible result [F]

$$n(r) = Cr^{\rho} + o(r^{\rho}),$$

where ρ is the order of growth of the Fourier transform of V. For a class of radially symmetric potentials in dimensions greater than one Zworski [Z3] proves

$$n(r) = C_n \mathrm{radius}(\mathrm{supp}(V))r^n + o(r^n).$$

This exhausts the examples of Schrödinger operators for which the first term in an asymptotic expansion is known.

For a general C_0^∞ potential in dimensions greater than one, only upper bounds are known:

$$n(r) \leq Cr^n.$$

The first polynomial bound (r^{n+1}) was obtained by Melrose [Me2]. The sharp bound (r^n) was first obtained by Zworski [Z4], with additional work by Vodev [V]. A variation on the techniques used for these bounds can also give bounds on super-exponentially decaying potentials in terms of the growth of the Fourier transform of V. We will show that if $\widehat{V}(k)$ grows like $e^{\Phi(|k|)}$, then

$$n(r) \leq C\Phi^n(cr).$$

There are no lower bounds for a general C_0^∞ potential in dimensions greater than one. For $n = 3$ it is known that there are infinitely many resonances [SZ]. This is also known for potentials with some positivity conditions [CPS]. However, for a general C_0^∞ potential in dimensions other than one and three it is not been proved that a single resonance exists!

3. Birman–Schwinger identities

Now we will use some Birman-Schwinger identities to prove that $R_V(k)$ has a meromorphic continuation. For $\operatorname{Im} k > 0$, begin with the resolvent formula

$$R_0(k) - R(k) - R(k)VR_0(k) = 0$$

and multiply on the left with $V^{\frac{1}{2}}$ and on the right with $|V|^{\frac{1}{2}}$. Recall that $R_V(k) = V^{\frac{1}{2}}R(k)|V|^{\frac{1}{2}}$ and define $R_{0V}(k) = V^{\frac{1}{2}}R_0(k)|V|^{\frac{1}{2}}$. Then this equation reads

$$R_{0V}(k) - R_V(k) - R_V(k)R_{0V}(k) = 0.$$

Thus

$$(1 - R_V(k))(1 + R_{0V}(k)) = 1$$

For $\operatorname{Im} k$ large norm of $R_{0V}(k)$ is small, so $(1 + R_{0V}(k))$ is invertible and

$$1 - R_V(k) = (1 + R_{0V}(k))^{-1}$$

The important feature of this equation is that left side only involves the operator $R_{0V}(k)$ which has an explicit integral kernel with an explicit analytic continuation. This analytic continuation defines a compact operator on the whole complex plane (except for a pole at zero in dimension 1). Thus, by analytic Fredholm theory, the left side of this formula defines a meromorphic continuation for $R_V(k)$.

From this formula we can see that the resonances are precisely those values of k for which $R_{0V}(k)$ has an eigenvalue -1. Equivalently, resonances are precisely the zeros of the analytic function $\det(1+R_{0V}(k))$—provided $R_{0V}(k)$ is trace class. Unfortunately, this only happens when $n=1$.

In higher dimensions, it turns out that $R_{0V}^p(k)$ for $p > n/2$ is trace class. If -1 is an eigenvalue for $R_{0V}(k)$, then it is also an eigenvalue for $R_{0V}^p(k)$ for p odd. Thus the set of resonances is contained in the set of zeros of the function $\det(1+R_{0V}^p(k))$ for $p > n/2$ and odd. This function is entire, except for poles arising from eigenvalues. One can therefore estimate the number of resonances by estimating the growth of this function. This is the approach of previous work. Instead of this we will consider the determinant of the scattering operator. I suspect that if one kept track of the constants this method would give a better result, since by taking the pth power one is forced to overcount.

4. Green's function identities

We will need the classical Green's function identity

$$G_0(x,y,k) - G_0(x,y,-k) = \frac{i\pi k^{n-2}}{(2\pi)^n} \int_{S^{n-1}} e^{ik\langle \omega, x-y\rangle} d\omega. \tag{4.1}$$

Here is a formal derivation starting with the expression for G_0 as a Fourier transform, written in polar co-ordinates. For $\operatorname{Im} k > 0$

$$G_0(x,y,k) = (2\pi)^{-n} \int_{S^{n-1}} \int_0^\infty e^{i\rho\langle \omega, x-y\rangle} \frac{1}{\rho^2 - k^2} \rho^{n-1} d\rho d\omega.$$

When k is just above the positive real axis, we deform the ρ contour into the lower half plane close to k. Then we may move k right onto the positive real axis. Similarly, if k is just above the negative real axis, we deform the contour upwards near $-k$ and move k onto the negative real axis. Subtracting this two expressions, we that for k real,

$$G_0(x,y,k) - G_0(x,y,-k) = (2\pi)^{-n} \int_{S^{n-1}} \oint_\Gamma e^{i\rho\langle \omega, x-y\rangle} \frac{1}{\rho^2 - k^2} \rho^{n-1} d\rho d\omega,$$

where Γ is a contour enclosing k. This is easily evaluated by residues to give the desired formula.

The reason why this derivation is formal is that the integral defining G_0 is divergent for large ρ. This can be remedied by thinking of it as a distributional Fourier transform, or by starting with powers of the resolvent (see [Me1]).

Let π_k denote the operator that takes a function on $\mathbb{R}^n$ to its Fourier transform restricted to a sphere of radius k. Using this notation, the identity (4.1) for real k can be written as an operator equation

$$R_0(k) = R_0(-k) + c(k)\pi_k^*\pi_k.$$

where

$$c(k) = \frac{i\pi k^{n-2}}{(2\pi)^n}.$$

This has to be interpreted as an equation involving operators between Besov spaces, since none of the operators are bounded on $L^2(\mathbb{R}^n)$ for real k. However, if we multiply on the left by $V^{\frac{1}{2}}$ and on the right by $|V|^{\frac{1}{2}}$ then we do obtain an equation involving operators on $L^2(\mathbb{R}^n)$, namely

$$R_{0V}(k) = R_{0V}(-k) + c(k)F_V^T(k)F_{|V|}(k) \tag{4.2}$$

where the operator $F_{|V|}(k) : L^2(\mathbb{R}^n) \to L^2(S^{n-1})$ is the formal product $\pi_k|V|^{\frac{1}{2}}$ is given by

$$\left(F_{|V|}(k)\psi\right)(\omega) = \int_{\mathbb{R}^n} e^{ik\langle x,\omega\rangle}|V|^{\frac{1}{2}}(x)\psi(x)d^nx,$$

the operator $F_V^T(k) : L^2(S^{n-1}) \to L^2(\mathbb{R}^n)$ is the formal product $V^{\frac{1}{2}}\pi_k^*$ given by

$$\left(F_V^T(k)\phi\right)(x) = V^{\frac{1}{2}}(x)\int_{S^{n-1}} e^{ik\langle x,\omega\rangle}\phi(\omega)d\omega$$

Since all the operators in the equation (4.2) have analytic continuations to complex k, the equation remains valid for all $k \in \mathbb{C}$.

5. Connection with the scattering operator

Recall that resonances are exactly the values of k in the lower half plane for which $1 + R_{0V}(k)$ has a non-trival kernel. However in dimensions higher than one we can't take the determinant because $R_{0V}(k)$ is not trace class. One possibility to remedy this is to multiply $1 + R_{0V}(k)$ by a suitable invertible operator and then take the determinant. For k in the lower half plane, $-k$ is in the upper half plane, and so $(1 + R_{0V}(-k))$ is invertible except at the finitely many eigenvalues of the $-\Delta+V$. We will therefore consider

$(1+R_{0V}(-k))^{-1}(1+R_{0V}(k))$. (The disadvantage of this regularization is that we have introduced infinitely many poles in the upper half plane, and will be forced to work in a half plane.) Using (4.2) this can be rewritten

$$\begin{aligned}(1+R_{0V}(-k))^{-1}(1+R_{0V}(k)) &= (1+R_{0V}(-k))^{-1}(1+R_{0V}(-k)+c(k)F_V^T(k)F_{|V|}(k)) \\ &= 1+c(k)(1+R_{0V}(-k))^{-1}F_V^T(k)F_{|V|}(k)) \\ &= 1+c(k)(1-R_V(-k))F_V^T(k)F_{|V|}(k)\end{aligned}$$

It follows from the estimates on singular values below that the second term on the left side is trace class, so we may take the determinant. Using the identity $\det(1+AB)=\det(1+BA)$ gives

$$\det((1+R_{0V}(-k))^{-1}(1+R_{0V}(k))) = \det(1+c(k)F_{|V|}(k)(1-R_V(-k))F_V^T(k)).$$

The operators on the left are now operators on $L^2(S^{n-1})$. Now

$$c(k)F_{|V|}(k)(1-R_V(-k))F_V^T(k) = c(k)\pi_k(V-VR(-k)V)\pi_k^*,$$

which is exactly the expression from stationary scattering theory for the T matrix defined by $1-S(k)$ where $S(k)$ is the scattering operator. Thus

$$\det(1+R_{0V}(-k))^{-1}(1+R_{0V}(k)) = \det(S(-k)),$$

and we find that resonances correspond to zeros of $S(k)$ in the upper half plane.

We can see from these formulas that zeros of $S(k)$ in the upper half plane correspond to poles in the lower half plane. This also follows from the equation $S(k)S(-k)=1$.

6. Upper bounds

We now need to count the zeros of the determinant of the scattering operator in a half-disk in the upper half plane. Thus we are counting the zeros of a function of the form

$$\phi(k) = \det(S(k)) = \det(1+T(k))$$

where

$$T(k) = c(-k)F_{|V|}(-k)(1-R_V(k))F_V^T(-k)$$

We will show below that $T(k)$ is a trace-class operator valued analytic function in the upper half plane. When s is real, $1+T(s)$ is unitary. Moreover $T(0)=0$.

Lemma 6.1 *Let $\phi(k) = \det(1+T(k))$ where $T(k)$ is a trace-class operator valued analytic function in the closed upper half plane, where $1 + T(s)$ is unitary for $s \in \mathbb{R}$, and where $T(0) = 0$. Let $n(t)$ denote the number of zeros of $\phi(k)$ in a half disk in the upper half plane of radius t. Define*

$$N(r) = \int_0^r \frac{n(t)}{t} dt.$$

Then

$$N(r) \leq \frac{1}{2\pi} \int_0^r t^{-1} \int_{-t}^{t} \|T'(s)\|_1 ds dt + \frac{1}{2\pi} \int_0^{\pi} \ln |\phi(re^{i\theta})| d\theta$$

Here $\|\cdot\|_1$ denotes the trace norm.

Proof: Integrating along a contour enclosing the half disk, we have

$$\begin{aligned} n(t) &= \frac{1}{2\pi i} \oint \frac{\phi'(k)}{\phi(k)} dk \\ &= \frac{1}{2\pi} \operatorname{Im} \int_{-t}^{t} \frac{\phi'(s)}{\phi(s)} ds + \frac{1}{2\pi} \int_0^{\pi} t \frac{d}{dt} \ln |\phi(te^{i\theta})| d\theta \\ &\leq \frac{1}{2\pi} \int_{-t}^{t} |\phi'(s)| ds + \frac{1}{2\pi} \int_0^{\pi} t \frac{d}{dt} \ln |\phi(te^{i\theta})| d\theta \end{aligned}$$

We used the fact that $|\phi(s)| = 1$ for real s. We are ignoring the finitely many poles, as these don't affect the counting function. Dividing by t and integrating, we find

$$N(r) \leq \frac{1}{2\pi} \int_0^r t^{-1} \int_{-t}^{t} |\phi'(s)| ds dt + \frac{1}{2\pi} \int_0^{\pi} \ln |\phi(re^{i\theta})| d\theta$$

We used $\phi(0) = 1$ to evaluate the second term. Since $\phi'(s) = \phi(s)\mathrm{tr}((1+T(s))^{-1}T'(s))$, and $|\phi(s)| = \|(1+T(s))^{-1}\| = 1$ for real s, it follows that for real s

$$\begin{aligned} |\phi'(s)| &\leq \|((1+T(s))^{-1}T'(s)\|_1 \\ &\leq \|(1+T(s))^{-1}\| \|T'(s)\|_1 \\ &= \|T'(s)\|_1. \end{aligned}$$

This completes the proof. $\square$

This lemma shows that to estimate $N(r)$, we must estimate the trace norm of T' along the real axis, and the growth of ϕ in the upper half plane.

Lemma 6.2 *Let V be a super-exponentially decaying potential. Then for $s \in \mathbb{R}$*

$$\|T'(s)\|_1 \leq C|s|^{n-2}$$

Proof: The operator $T(s)$ is a product $c(s)F_{|V|}(s)(1-R_V(-s))F_V^T(s)$. We estimate each term and its derivative. To begin, we have

$$|c(s)| \leq C|s|^{n-2}$$
$$|c'(s)| \leq C|s|^{n-3}$$

It follows from the (generalized) limiting absorption principle [JMP] that

$$\|1-R_V(-s)\| \leq C$$
$$\|R_V'(-s)\| \leq C$$

Using the explicit integral kernels for $F_V^T(s)$ and $F_{|V|}(s)$ it easy to estimate the Hilbert-Schmidt norms

$$\|F_V^T(s)\|_2^2 = \int_{S^{n-1}}\int_{\mathbb{R}^n} |e^{is\langle\omega,x\rangle}V^{\frac{1}{2}}(x)|^2 dx d\omega \leq C$$
$$\|F_V^{T\prime}(s)\|_2^2 = \int_{S^{n-1}}\int_{\mathbb{R}^n} |i\langle\omega,x\rangle e^{is\langle\omega,x\rangle}V^{\frac{1}{2}}(x)|^2 dx d\omega \leq C.$$

The same estimates hold for $\|F_{|V|}(s)\|_2$ and $\|F_{|V|}'(s)\|_2$. The proof is completed by using the Leibnitz rule to write $T'(s)$, and the estimate

$$\|AB\|_1 \leq \|A\|_2^{\frac{1}{2}}\|B\|_2^{\frac{1}{2}}$$

□

It remains to estimate the growth of $\phi(k)$ for complex k.

Lemma 6.3 *Suppose that the Fourier transform of V satisfies the growth estimate*

$$\widehat{V}(z) \leq Ce^{\Phi(|z|)}$$

for some positive, increasing function $\Phi(x)$. Then

$$|\phi(k)| \leq e^{\delta^{-(n-1)}\Phi^n((2+\epsilon)|k|)+O(\Phi^{n-1}((2+\epsilon)|k|))}$$

for any $\epsilon > 0$, and some constant C.

Proof: We will use Weyl's estimate

$$|\phi(k)| = |\det(1+T(k))| \leq \prod_j \left(1+\mu_j(T(k))\right)$$

and therefore must estimate the singular values of $T(k)$. Using the estimate

$$\mu_j(AB) \leq \|A\|\mu_j(B)$$

and

$$\mu_j(AB) = \mu_j(BA)$$

(twice) we find

$$\begin{aligned}\mu_j(T(k)) &\leq \|c(-k)(1-R_V(k)\|\mu_j(F_V^T(-k)F_{|V|}(-k)) \\ &\leq C|k|^{n-2}\mu_j(\mathbf{V}_k).\end{aligned} \tag{6.1}$$

Where the operator $\mathbf{V}_k = F_{|V|}(-k)F_V^T(-k)$ is an integral operator on $L^2(S^{n-1})$ with integral kernel $\widehat{V}(-k(\omega-\omega'))$. To estimate the singular values we will use the following bound without proof. (It follows from the analyticity of $\widehat{V}$.) Let L_ω denote the positive Laplacian on S^{n-1} in the variables ω. Then for any $\epsilon > 0$ there is a constant C such that

$$|L_\omega^{p/2}\widehat{V}(k(\omega-\omega'))| \leq C^p p! e^{\Phi((2+\epsilon)|k|)}$$

Summing the Taylor expansion for the exponential, this gives, for $\delta < C^{-1}$,

$$|e^{\delta L_\omega^{\frac{1}{2}}}\widehat{V}(k(\omega-\omega'))| \leq (1-\delta C)^{-1}e^{\Phi((2+\epsilon)|k|)}$$

Since the left side is the integral kernel for the operator $e^{\delta L_\omega^{\frac{1}{2}}}\mathbf{V}_k$, this implies that

$$\|e^{\delta L_\omega^{\frac{1}{2}}}\mathbf{V}_k\| \leq Ce^{\Phi((2+\epsilon)|k|)}$$

(The constants C may change from line to line.) Thus

$$\begin{aligned}\mu_j(\mathbf{V}_k) &= \mu_j(e^{-\delta L_\omega^{\frac{1}{2}}}e^{\delta L_\omega^{\frac{1}{2}}}\mathbf{V}_k) \\ &\leq \|e^{\delta L_\omega^{\frac{1}{2}}}\mathbf{V}_k\|\mu_j(e^{-\delta L_\omega^{\frac{1}{2}}}) \\ &\leq Ce^{\Phi-\delta j^{1/(n-1)}}\end{aligned}$$

Using (6.1), we get the same bound for $\mu_j(T(k))$, if we increase ϵ slightly. Thus

$$|\det(1+T(k))| \leq \prod(1+Ce^{\Phi-\delta j^{1/(n-1)}}).$$

This product is easily estimated by breaking it into two pieces. For $j \leq (\Phi/\delta)^{n-1}$ we obtain

$$\begin{aligned}\prod_{j\leq(\Phi/\delta)^{n-1}}(1+Ce^{\Phi-\delta j^{1/(n-1)}}) &\leq \prod_{j\leq(\Phi/\delta)^{n-1}}(C+1)e^{\Phi-\delta j^{1/(n-1)}} \\ &= (C+1)^{(\Phi/\delta)^{n-1}}e^{\Phi^n/\delta^{n-1}}\exp(\sum_{j=1}^{(\Phi/\delta)^{n-1}}-\delta j^{1/(n-1)}) \\ &= e^{\Phi^n/\delta^{n-1}+O(\Phi^{n-1})}\end{aligned}$$

For $j > (\Phi/\delta)^{n-1}$ we have

$$\begin{aligned}\prod_{j>(\Phi/\delta)^{n-1}} (1 + Ce^{\Phi-\delta j^{1/(n-1)}}) &\leq \exp(\sum_{j>(\Phi/\delta)^{n-1}} Ce^{\Phi-\delta j^{1/(n-1)}}) \\ &\leq \exp(Ce^{\Phi} \sum_{j>(\Phi/\delta)^{n-1}} e^{-\delta j^{1/(n-1)}})\end{aligned}$$

The sum appearing in this formula can be estimated by an integral.

$$\begin{aligned}\sum_{j>(\Phi/\delta)^{n-1}} e^{-\delta j^{1/(n-1)}} &\leq e^{-\delta(\Phi/\delta)} + \int_{(\Phi/\delta)^{n-1}}^{\infty} e^{-\delta x^{1/(n-1)}} dx \\ &\leq e^{-\Phi} + Ce^{-\Phi}\Phi^{n-2}\end{aligned}$$

Thus the product for large j satisfies the bound

$$\prod_{j>(\Phi/\delta)^{n-1}} (1 + Ce^{\Phi-\delta j^{1/(n-1)}}) \leq e^{C\Phi^{n-2}}.$$

Combining the estimates for small and large j completes the proof.

□

We can now prove the promised upper bound on the counting function.

Theorem 6.4 *Suppose that V is a super-exponentially decaying potential with*

$$\widehat{V}(z) \leq Ce^{\Phi(|z|)}$$

Then

$$n(r) \leq C\Phi^n(cr) + O(\Phi^{n-1}(cr))$$

for some constants c and C.

Proof: It follows from the previous lemmas that

$$\begin{aligned}&N(r) \\ &\leq C\int_0^r t^{-1}\int_{-t}^{t} |s|^{n-2} ds dt + (2\pi)^{-1}\int_0^{\pi} \left(\delta^{-(n-1)}\Phi^n((2+\epsilon)r) + O(\Phi^{n-1}((2+\epsilon)r))\right) d\theta \\ &\leq Cr^{n-1} + 2^{-1}\delta^{-(n-1)}\Phi^n((2+\epsilon)r) + O(\Phi^{n-1}((2+\epsilon)r))\end{aligned}$$

By looking at $\widehat{V}$ along the imaginary axis, we see that Φ must grow at least as fast as r. (For compactly supported V's we have $\Phi(r) = Cr$.) Thus we can ignore the first term. This gives

$$N(r) \leq 2^{-1}\delta^{-(n-1)}\Phi^n((2+\epsilon)r) + O(\Phi^{n-1}((2+\epsilon)r))$$

To get an estimate on $n(r)$ from this, note that for any $s > 1$, since $n(r)$ is monotone,

$$\begin{aligned} n(r) &= n(r)(\ln s)^{-1} \int_r^{sr} \frac{1}{t} dt \\ &\leq (\ln s)^{-1} \int_r^{sr} \frac{n(t)}{t} dt \\ &= (\ln s)^{-1} (N(sr) - N(r)) \\ &\leq (\ln s)^{-1} N(sr). \end{aligned}$$

This completes the proof. $\square$

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