Quantum Theory of Dark Particles

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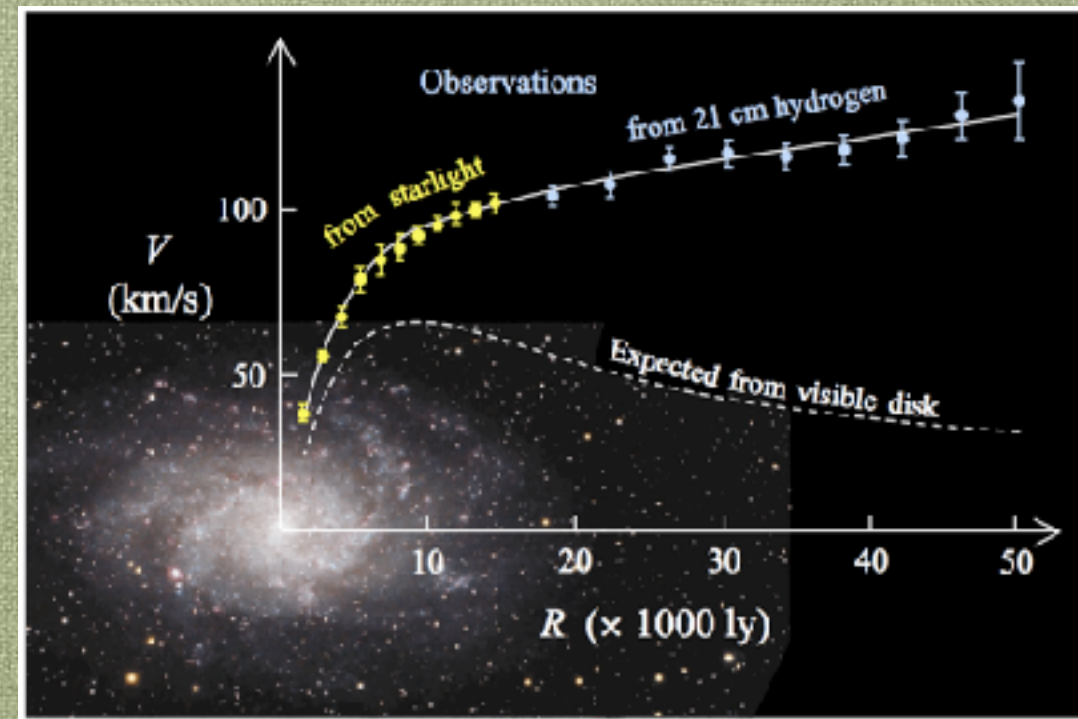
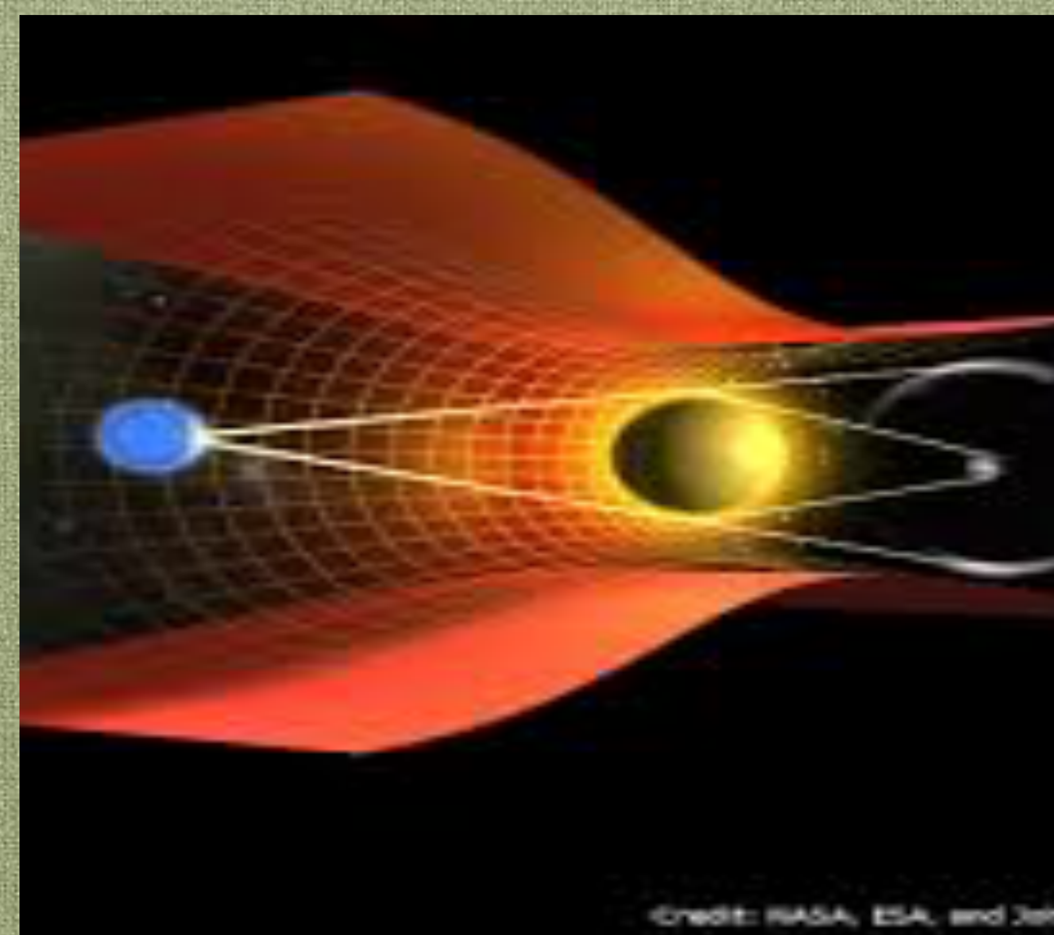
(Part of this work jointly with Werner Kirsch)

There is increasing evidence that there is matter out there that is not ordinary Baryons and Leptons.

Estimates based on observations show that this ‘matter’ constitutes about 27% of all known matter-energy in the Universe.





The images are from NASA

In addition to these there is the expansion of the Universe which which was well known, governed by the Hubble law by which the velocity of motion of a galaxy is proportional to the distance of the galaxy (on cosmological scales)

Several of these facts lead to the conclusion that there are particles that do not interact with ordinary matter as we know it but influence it through Gravity

This makes one wonder why these particles, if they exist, do not interact with others except through Gravity and what their Quantum Theory could be.

We therefore start with a postulate:

`The configuration space of dark particles is a set which is the support of a measure singular with respect to the Lebesgue measure.`

The reason for this postulate is to ensure that the Hilbert spaces associated with the two classes of particles are automatically orthogonal with respect to each other, so there is no possibility of interaction between them. Gravity is of course beyond this.

We consider a Hilbert space with respect to a measure μ which is singular with respect to the Lebesgue measure.

We have to formulate the Hamiltonian of a free particle with the state space given by such a Hilbert space. The problem then is to define a notion of 'derivative' , since for all the functions here which have non-zero derivatives belong to the equivalence class 0 in this space.

So let us look at how it is done for the Lebesgue measure.

$$\frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f'(x)dx((x, x+h))}{m((x, x+h))}$$

Thus, the derivative of ‘f’ is a Radon-Nykodym derivative, where ‘m’ is the Lebesgue measure. This definition was used for finite measures by others to define derivative for functions in Hilbert spaces of singular measures.

We note that every function f in $L^2(X,\mu)$ is also in $L^1(X,\mu)$ so gives rise to a finite complex measure $f d\mu$ whose Radon-Nikodym derivative with respect to μ is f .

When X is $\mathbb{R}$ the Randon-Nikodym derivative is also computed using the distribution function of f , namely $\int_{-\infty}^{x} f(s)\, d\mu(s)$ which we call $\phi_f(x)$

We therefore define the kinetic energy operator to be the unique self-adjoint operator associated to the quadratic form given by

$$\mathcal{E}(\phi_f, \phi_g) = \int \overline{f}(x) g(x) \; d\mu(x) = \langle \phi_f, \Delta_\mu \phi_g \rangle.$$

We call this operator the (AK) operator. For finite measures there is no ambiguity in defining the distribution function so this operator is defined on distribution functions of elements of the space $L^2(\mathbb{R}, \mu)$.

In the case when the measure is not finite but σ finite, we define distribution functions based on partitioning the space as a disjoint union of compact intervals. Thus if $\mathcal{P}$ is a partition

$$\mathcal{P} = \{(a_n, b_n] : \sqcup_n (a_n, b_n] = \mathbb{R}\}$$

Then we define the distribution function associated with a measure and this partition by

$$\phi_{\nu,\mathcal{P}}(x) = \begin{cases} \nu((a_n, x]), & \text{if } \ x \in (a_n, b_n] \\ 0 \ \ \text{otherwise.} \end{cases}$$

The idea is to use the Radon-Nikodym derivative in the place of usual derivative in the case of singular measures. However we need to take derivatives of functions and the only reasonable function associated to a measure is its distribution function.

We can then talk about derivatives of these functions with respect to singular measures. Thus

$$f \in L^1_+(\mathbb{R}, \mu), \quad \text{then} \quad f d\mu \quad \text{is a measure.}$$

We define

$$\frac{d\phi_{f\mu,\mathcal{P}}}{d\mu}(x) = f(x), \quad a.e. x.$$

Henceforth we will write

$$\phi_{f\mu,\mathcal{P}} \text{ as simply } \phi_{f,\mathcal{P}}.$$

Then as before we define a positive definite quadratic form on $L^2(\mathbb{R},\mu)$

$$\mathcal{E}(\phi_{f,\mathcal{P}},\phi_{g,\mathcal{P}}) = \langle f,g\rangle$$

This quadratic form is shown to be sesquilinear and closed. Therefore has a unique self-adjoint operator associated with it. We call it $\Delta_{\mu,\mathcal{P}}$

At this stage we note that a measure is recovered from the distribution function uniquely. It turns out that the measure ν is recovered from $\phi_{\nu,\mathcal{P}}$ uniquely independent of the partition $\mathcal{P}$. However the operator $\Delta\nu,\mathcal{P}$ depends on the partition.

We will comment on this towards the end of the talk.

Henceforth we will consider only atomic measures and since their support is countable, we will take the support to be integers and to make the problem interesting, we will assume that the measure is non zero at all integers.

For such a model we find explicitly the operator $\Delta\mu, \mathcal{P}$

We take the standard basis for $\ell^2(\mathbb{Z}, \mu)$

$$e_x(y) = \frac{1}{\sqrt{\mu(x)}}\delta_{xy}, \quad x \in \mathbb{Z}.$$

We want to find

$$f_x \in \ell^2(\mathbb{Z}, \mu) : e_x = \phi_{f_x, \mathcal{P}} \quad \text{for} \quad x \in \mathbb{Z},$$

so that we can get the matrix elements

$$\langle e_x, \Delta_{\mu,\mathcal{P}} e_y \rangle = \langle f_x, f_y \rangle.$$

To do this we fix different partitions and based on the partition we will get a different operator.

$$\mathcal{P}_N = \{\{mN, mN+1, \ldots, mN+N-1\} : m \in \mathbb{Z}\},$$
$$N \in \mathbb{N},$$

$$\mathcal{P}_\infty = \mathbb{Z}^- \sqcup \mathbb{Z}^+, \quad \mathbb{Z}^+ = \mathbb{N} \cup \{0\} \text{ and } \mathbb{Z}^- = \mathbb{Z} \setminus \mathbb{Z}^+.$$

$$\mathcal{P}_0 = \{\mathbb{Z}\}$$

We will do this for a finite N and for the infinite case separately. We need start with a recurrence relation coming from the definition of the distribution function. We set

$$S_{m,N} = \{mN, mN+1, \ldots, mN+N-1\}.$$

Lemma : We have

$$\phi_{f,\mathcal{P}_N}(x) = \phi_{f,\mathcal{P}_N}(x+1) = 0 \implies f(x+1) = 0.$$

Proof : It is easy to see since by definition

$$\phi_{f,\mathcal{P}_N}(w) = \sum_{y=mN}^{w} f(y)\mu(y), \ w \in S_{m,N}$$

$$\phi_{f,\mathcal{P}_N}(x+1) = \phi_{f,\mathcal{P}_N}(x) + f(x+1)\mu(x+1)$$

As a corollary we see that every finitely supported function has a derivative with respect to the measure μ .

Therefore the standard basis vectors having support at one point have derivatives, so one can explicitly write down the operators $\Delta_{\mu,\mathcal{P}}$ because its matrix elements are the inner product of the corresponding derivatives.

$$\langle e_x, \Delta_{\mu,\mathcal{P}} e_y \rangle = \langle f_x, f_y \rangle$$

From the previous lemma and induction we get:

$$f_x(y) = \begin{cases} 0, & \text{if } x \in S_{m,N}, y \in S_{n,N}, n \neq m, \\ 0, & \text{if } x < y, \quad \text{or} \quad y < x+1 \end{cases}$$

$$\frac{1}{\sqrt{\mu(x)}} = e_x(x) = f_x(y)\mu(x), \quad \text{and}$$
$$0 = e_x(x+1) = f_x(x)\mu(x) + f_x(x+1)\mu(x+1).$$

So

$$f_x(x) = \frac{1}{\sqrt{\mu(x)^3}}$$
$$f_x(x+1) = -\frac{1}{\mu(x+1)\sqrt{\mu(x)}}.$$

Theorem : The AK operators are given by,

$$\Delta_{\mu,\mathcal{P}_N} = \bigoplus_m \Delta_{\mu,m,N}$$

$$\Delta_{\mu,m,N} =$$

$$\begin{bmatrix} \alpha(mN) & -\beta(mN) & 0 & \dots & \dots & 0 \\ -\beta(mN) & \alpha(mN+1) & -\beta(mN+1) & 0 & \dots & 0 \\ 0 & -\beta(mN+1) & \alpha(mN+2) & -\beta(mN+3) & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & -\beta(mN+N-1) & \alpha(mN+N-1) \end{bmatrix}$$

where

$$\alpha(x) = \frac{1}{\mu(x)^2} + \frac{1}{\mu(x)\mu(x+1)}$$

$$\beta(x) = \frac{1}{\sqrt{\mu(x)\mu(x+1)^3}}$$

We similarly have for the infinite case which we will not write down, but give the operators for a special case of the counting measure, in which case we call the operators $\Delta_{\mu,\mathcal{P}_N}$and $\Delta_{\mu,\mathcal{P}_\infty}$ as Δ_N and Δ_∞ respectively. Then

Theorem : The AK operators on $\ell^2(\mathbb{Z})$ are :

$$\Delta_N = \bigoplus_m \chi_{S_{m,N}} \Delta \chi_{S_{m,N}}$$

$$\Delta_\infty = -\chi_{\mathbb{Z}^-} \Delta \chi_{\mathbb{Z}^-} \oplus \chi_{\mathbb{Z}^+} \Delta \chi_{\mathbb{Z}^+}$$

where Δ is the usual Laplacian on $\ell^2(\mathbb{Z})$ and agrees with $\Delta_{\mathcal{P}_0}$.

Specializing to the case when N =1, the AK operator is just the identity. This is the model that was chosen initially to model the Dark Particles.

In this model, the kinetic energy commutes with the potential energy if we take a pair of particles subjected to pair coulomb potential, the nature of which is unknown but repulsive, then we obtain the two particle spectrum in 3 dimensions given by

$$H = \frac{\tilde{h}}{2m}(I \otimes I + I \otimes I)(I - \sum_{x} P_{x,x}) + \sum_{x \neq y x, y \in \mathbb{Z}^3} \frac{D}{|x-y|} P_{x,y}$$

$$\sigma(H) = \left\{ \frac{\tilde{h}}{m} + \frac{D}{\sqrt{L}} : L \neq 4^a(8b+7),\ a, b \in \mathbb{N} \right\}.$$

Comments:

I think of this as an analog of spin in the quantum theory of ordinary matter. The representations of the orthogonal group, which are the group of symmetries for the system, give rise to super selection sectors one of the quantum numbers of which is the spin. You cannot transition from one spin state to another.

Similarly the N here is to be seen as a quantum number which gives different types of ‘dark particles’.

One also sees an exclusion principle operating here.