$W_0^{1,1}$ SOLUTIONS IN SOME BORDERLINE CASES OF CALDERON-ZYGMUND THEORY

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1. 20th Century Results, in collaboration with Thierry Gallouet

Let Ω be a bounded open set in $I\!R^N$, $N \geq 2$. The simplest example of nonlinear (and variational) boundary value problem is the Dirichlet problem for the p–Laplace operator, with $1 < p < N$,

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.1}$$

so that the growth of the differential operator is $p-1$. The classical theory of nonlinear elliptic equations states that $W_0^{1,p}(\Omega)$ is the natural functional spaces framework to find weak solutions of (1.1), if the function f belongs to the dual space of $W_0^{1,p}(\Omega)$.

This approach fails if $p = 1$.

On the one hand, if $p > 1$, for the model problem (1.1), the existence of $W_0^{1,p}(\Omega)$ solutions also fails if the right hand side is a function $f \in L^m(\Omega)$ $(m \geq 1)$ which does not belong to the dual space of $W_0^{1,p}(\Omega)$: it is possible to find distributional solutions in function spaces "larger" than $W_0^{1,p}(\Omega)$, but contained in $W_0^{1,1}(\Omega)$ (see [1], [2]).

To be more precise, in this paper, we will present some existence results of $\mathbf{W_0^{1,1}(\Omega)}$ distributional solutions (not so usual in elliptic problems) for nonlinear elliptic boundary value problems of the type

$$\begin{cases} A(u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.2}$$

where

$$f \in L^m(\Omega),\ m \geq 1, \tag{1.3}$$

and A is the operator, acting on $W_0^{1,p}(\Omega)$, $A(v) = -\mathrm{div}\,(a(x,v,\nabla v))$. We assume the standard hypotheses on $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$. The simplest example is given by the differential operator $A(v) = -\mathrm{div}(|\nabla v|^{p-2}\nabla v)$, appearing in (1.1).

The existence of $W_0^{1,1}(\Omega)$ solutions, instead of $W_0^{1,p}(\Omega)$ or $W_0^{1,q}(\Omega)$, with $1 < q < p$, solutions of the boundary value problem (1.2) is a consequence of the poor summability of the right hand side, even if the "growth" of the operator A is not zero, but $p - 1 > 0$.

Existence of solutions for problem (1.2) with nonregular right hand side, for general nonlinear problems, are contained in [1], [2]; in particular, we recall the following results.

THEOREM 1.1. *Let* $m = 1$ *and* $2 - \frac{1}{N} < p < N$. *Then there exists a distributional solution* $u \in W_0^{1,q}(\Omega)$, $q < \frac{N(p-1)}{N-1}$, *of (1.2); that is*

$$\int_\Omega a(x, u, \nabla u)\nabla v = \int_\Omega f\, v, \quad \forall v \in W_0^{1,\infty}(\Omega)\,.$$

Observe that $\frac{N(p-1)}{N-1} > 1$ if and only if $p > 2 - \frac{1}{N}$.

THEOREM 1.2. *Let* $2 - \frac{1}{N} < p < N$. *If*

$$\int_\Omega |f| \log(1 + |f|) < \infty, \tag{1.4}$$

then there exists a distributional solution $u \in W_0^{1,\frac{N(p-1)}{N-1}}(\Omega)$ *of (1.2).*

THEOREM 1.3 (Calderon-Zygmund theory for infinite energy solutions). *If* $f \in L^m(\Omega)$, $\frac{N}{N(p-1)+1} < m < \frac{Np}{pN+p-N} = (p^\star)'$, $p > 1 + \frac{1}{m} - \frac{1}{N}$, *then there exists a distributional solution* $u \in W_0^{1,(p-1)m^\star}(\Omega)$ *of (1.2).*

We will show the existence of $W_0^{1,1}(\Omega)$ distributional solutions as consequence of the fact that we improve the existence results of Theorem 1.2 and Theorem 1.3 in some borderline cases.

2. NEW EXISTENCE RESULTS IN $W_0^{1,1}(\Omega)$ IN COLLABORATION WITH THIERRY GALLOUET ([3])

THEOREM 2.1. *Let* $f \in L^m(\Omega)$, $m = \frac{N}{N(p-1)+1}$, $1 < p < 2 - \frac{1}{N}$. *Then there exists a distributional solution* $u \in W_0^{1,1}(\Omega)$ *of (1.2).*

THEOREM 2.2. *Assume (1.4) and* $p = 2 - \frac{1}{N}$. *Then there exists a distributional solution* $u \in W_0^{1,1}(\Omega)$ *of (1.2).*

REFERENCES

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