1. The Lie algebra sl_2 and its finite dimensional representations [s:sl2]

1.1. $\mathrm{sl}_2(k)$ and $\mathrm{sl}_2(\mathbb{C})$. Let k be a commutative ring with identity. The Lie algebra $\mathrm{sl}_2(k)$ consists of traceless 2×2 matrices with entries in k, the Lie bracket being defined by $[X,Y] := XY - YX$, where XY and YX denote the usual matrix products. The elements [ss:sl2ksl2c]

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

generate $\mathrm{sl}_2(k)$ freely as a k-module. They constitute the *standard basis*. The Lie bracket with respect to the standard basis is given by:

$$[H,X] = 2X, \quad [X,Y] = H, \quad [H,Y] = -2Y \tag{1.1}$$

Thus if 2 is a unit in k then $\mathrm{sl}_2(k)$ is perfect, i.e., $[\mathrm{sl}_2(k), \mathrm{sl}_2(k)] = \mathrm{sl}_2(k)$. If k is a field in which $2 \neq 0$, then $\mathrm{sl}_2(k)$ is simple, i.e., it has no non-zero proper ideals and is not abelian.[1]

In particular $\mathrm{sl}_2(\mathbb{C})$ is a simple Lie algebra. It is the simplest *complex semisimple*[2] *Lie algebra* in that it is the unique such algebra of dimension 3 over $\mathbb{C}$, and 3 is the minimum possible dimension of any such algebra.

[ss:sl2crepshort]

1.2. Finite dimensional representation theory of $\mathrm{sl}_2(\mathbb{C})$: a short digest. We now consider FINITE DIMENSIONAL representations of $\mathrm{sl}_2(\mathbb{C})$. Stated below are some basic facts about these. The proofs follow in the later subsections below.

(a) ("complete reducibility") every such representation is a direct sum of irreducible sub representations.
(b) the expression as a direct sum of irreducible sub representations is unique: if $V_1 \oplus \cdots \oplus V_m$ and $W_1 \oplus \cdots \oplus W_n$ are two such expressions, then $m = n$ and there is a permutation σ of $\{1, \ldots, m\}$ such that $V_1 \simeq W_{\sigma 1}, \ldots, V_m \simeq W_{\sigma m}$.
(c) (isomorphism classes of finite dimensional) irreducible representations are "naturally" parametrized by the non-negative integers; the irreducible representation V_n corresponding to n has "highest weight" n.
(d) $\dim V_n = n + 1$.

The first statement holds verbatim for any complex semisimple Lie algebra. It is a fundamental theorem in the representation theory of such algebras and is called *Weyl's complete reducibility*. While uniqueness as in the second statement holds in a very general context (see, e.g., [**?**,]), it is in the present case (not just for $\mathrm{sl}_2(\mathbb{C})$ but also for any complex semisimple Lie algebra) a consequence of "linear independence of irreducible characters". Analogous to the third statement, there is, for any complex semisimple Lie algebra, a parametrization by "highest weights" due to Cartan of the irreducible representations. This goes by the name of *Cartan's highest weight theory*. Further, there are various famous formulae that elucidate the structure of a given irreducible representation: the fourth statement above, for example, is a special case of *Weyl's dimension formula* which computes the dimension of the irreducible representation corresponding to a highest weight in terms of the highest weight.

[1] One dimensional Lie algebras are not *simple* although they have no non-zero proper ideals.

[2] Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$. The *radical* of $\mathfrak{g}$ is the largest ideal in $\mathfrak{g}$ that is solvable as a Lie algebra. We call $\mathfrak{g}$ semisimple if it has trivial radical: the zero lie algebra is not considered semisimple by convention. Clearly $\mathfrak{g}$ is semisimple if it is simple.

[ss:der] 1.3. **Derivations constitute a Lie algebra.** Let k be a commutative ring with identity. A k-*algebra* A is a k-module A together with a bilinear map $A \otimes A \to A$ called "multiplication" (all tensor products are over k). Fix a k-algebra A and let us write $\mu(a,b)$ for the image of $a \otimes b$ under mulitiplication (a, b in A). A k-linear endomorphism D of A is a k-*derivation* or just *derivation* if it satisfies the "Liebniz rule": $D\mu(a,b) = \mu(Da,b) + \mu(a,Db)$.

- Derivations of A form a Lie subalgebra of the Lie algebra $\operatorname{End} A$ of all k-linear endomorphisms of A.
- If ϕ is a k-linear endomorphism such that $\phi(\mu(a,b)) = \mu(\phi(a),b) = \mu(a,\phi(b))$ and D a derivation, then ϕD is a derivation.
- Consider the algebra A of polynomials over k in commuting variables u_1, u_2, ... Given polynomials f_1, f_2, ..., the k-endomorphism $f_1\partial/\partial u_1 + f_2\partial/\partial u_2 + \cdots$ of A is a derivation. Moreover, every derivation of A is of this form. The "degree derivation", for which every homogeneous polynomial is an eigenvector with eigenvalue equal to its degree, is, for example, given by $u_1\partial/\partial u_1 + u_2\partial/\partial u_2 + \cdots$.
- The algebra A could in particular taken to be a Lie algebra. The endomorphisms $\operatorname{ad} X : Y \mapsto [X,Y]$ for X an element of A are then derivations (this is the content of the Jacobi identity axiom). Such derivations are *inner*. Derivations that are not inner are *outer*. As follows from Cartan's semisimplicity criterion[3], every derivation of a complex semisimple Lie algebra is inner.

[ss:sl2polyuv] 1.4. **Action of sl_2 on polynomials in two variables.** Let k be a commutative ring with identity. Let P denote the polynomial ring over k in two commuting variables u and v. The association

$$X \mapsto u\frac{\partial}{\partial v}, \quad H \mapsto u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \quad Y \mapsto v\frac{\partial}{\partial v}$$

defines an action of $\mathrm{sl}_2(k)$ on P by derivations, as can be readily checked. We have

$$X \cdot u^m v^n = n u^{m+1} v^{n-1}, \quad H \cdot u^m v^n = (m-n) u^m v^n, \quad Y \cdot u^m v^n = m u^{m-1} v^{n+1}$$

The action preserves degrees, so P splits as a direct sum $P_0 \oplus P_1 \oplus P_2 \oplus \cdots$ as an $\mathrm{sl}_2(k)$-module, where P_j is the submodule of homogeneous polynomials of degree j. The monomials of degree j form a basis for P_j, so P_j is a free k-module of rank $j+1$.

Let now k be a field. We claim that P_m is irreducible if the characteristic of k is either 0 or more than m: given $0 \neq v$ in an $\mathrm{sl}_2(k)$-submodule of P_m, we conclude, by repeated application of X that u^m belongs to the submodule; since $Y^j u^m = m(m-1)\cdots(m-j+1)u^{m-j}v^j$, it follows that the submodule is all of P_m. The condition on the characteristic is necessary for P_m to be irreducible: if $m \geq p$ where p denotes the characterisitc of k, then, as is easily seen, u^m, $u^{m-1}v$, $u^{m-2}v^2$, ..., $u^{m-j}v^j$ span a submodule where $0 \leq j < p$ is such that $m \equiv j \mod p$.

In particular, when $k = \mathbb{C}$, we have the following:

> the representations P_m, as m varies over the non-negative integers, are irreducible; they are pairwise non-isomorphic, since $\dim P_m = m+1$.

Exercise 1.4.1. The action of $\mathrm{sl}_2(k)$ on P extends also to the Laurent polynomial ring $\mathcal{L} := k[u,v,u^{-1},v^{-1}]$. We have $\mathcal{L} = \oplus_{m\in\mathbb{Z}}\mathcal{L}_m$, where $\mathcal{L}_m$ is the k-submodule spanned by homogeneous polynomials of degree m.

[3] This criterion says: a finite dimensional complex Lie algebra is semisimple if and only if its Killing form is non-degenerate (and the algebra is not zero).

Assuming k to be a field of characteristic 0, we have:

- for any m, the following are submodules of $\mathcal{L}_m$: $\mathcal{L}_m^> := \langle u^j v^{m-j} \mid j \geq 0\rangle$ and $\mathcal{L}_m^< := \langle u^{m-k} v^k \mid k \geq 0\rangle$. Their intersection is clearly $\langle u^j v^k \mid j+k = m, j \geq 0, k \geq 0\rangle$.
- For $m \geq -1$, the only proper submodules of $\mathcal{L}_m$ are $\mathcal{L}_m^>$, $\mathcal{L}_m^<$, and their intersection $\mathcal{L}_m^< \cap \mathcal{L}_m^<$; the sum of $\mathcal{L}_m^>$ and $\mathcal{L}_m^<$ is all of $\mathcal{L}_m$.
- For $m \leq -1$, the only non-zero submodules of $\mathcal{L}_m$ are $\mathcal{L}_m^>$, $\mathcal{L}_m^<$, and their sum $\mathcal{L}_m^> \oplus \mathcal{L}_m^<$; the intersection of $\mathcal{L}_m^>$ and $\mathcal{L}_m^<$ is zero.
- For $m = -1$, $\mathcal{L}_m^> \oplus \mathcal{L}_m^< = \mathcal{L}_m$.

Determine the submodule structure of $\mathcal{L}_m$ when k is a field of positive characteristic p.

[ss:eigenhighest]

1.5. **Action on eigenvectors of H; highest weight vectors.** Let V be an $\mathrm{sl}_2(k)$-module and v an element of V such that $Hv = \lambda v$ for some λ in k. Such an element v is called a *weight vector*, and λ its *weight*. The eigenspaces for H are called *weight spaces*. We have

$$H(Xv) = (\lambda+2)Xv \qquad \text{and} \qquad H(Yv) = (\lambda-2)Xv \tag{1.2}$$

Indeed, from Eq. (1.1) we get immediately the following commutation relations (among X, H, Y thought of as operators on V):

$$(H-(\lambda+2))^m X = X(H-\lambda)^m \qquad (H-(\lambda-2))^m Y = Y(H-\lambda)^m \tag{1.3}$$

From Eq.(1.2), we see the following:

(i) the sum W of eigenspaces of H in V for various eigenvalues is an invariant subspace.

(ii) Assume that k is an algebraically closed field and that V is finite dimensional. Then $W \neq 0$ (unless $V = 0$). In particular, if V is irreducible, then $V = W$, so V has a basis consisting of weight vectors.

(iii) Assume that k is a field of characteristic 0 and v is a weight vector of weight λ. If, for some non-negative integer n, the elements v, Xv, X^2v, …, $X^n v$ are all non-zero, then they are linearly independent, for they are all weight vectors with pairwise distinct weights; in particular, if V is finite dimensional, then $X^n v$ vanishes for some n.

(iv) Assume that k is an algebraically closed field of characteristic 0 and $V \neq 0$ is finite dimensional. Then V has a *highest weight vector*, i.e., a non-zero weight vector v such that and $Xv = 0$. Indeed, since k is algebraically closed, we can find a non-zero weight vector v for some λ in k; by the previous item, it follows that $X^n v = 0$ for some n; if n be the least such, then $H(X^{n-1}v) = (\lambda + (n-1)2)(X^{n-1}v)$ by Eq. (1.2) and of course $X(X^{n-1}v) = X^n v = 0$.

Exercise 1.5.1. Let k be a field of characteristic 0 and V a finite dimensional $\mathrm{sl}_2(k)$-module. Show that X and Y act nilpotently on V. Show that neither hypothesis can be omitted. (Hint: By extending scalars, may assume k is algebraically closed. Since an operator acts nilpotently on a vector space if it acts so on a subspace and the quotient thereof, we may assume that V is irreducible.)

Remark 1.5.2. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. An element S (respectively N) of $\mathfrak{g}$ is called *semisimple* (respectively *nilpotent*) if $\operatorname{ad} S$ (respectively $\operatorname{ad} N$) is semisimple (respectively nilpotent). It is a theorem, called "the preservation of Jordan decomposition", that, on any finite dimensional representation of $\mathfrak{g}$, semisimple elements act semisimply and nilpotent elements act nilpotently. It follows from this theorem that, on any finite dimensional $\mathrm{sl}_2(\mathbb{C})$-module, H acts semisimply and X, Y nilpotently. (We do not invoke the "preservation" theorem here, but instead prove this last conclusion directly. We have already seen in item (ii) above that H acts semisimply on irreducibles; that it acts semisimply on any V follows from complete reducibility. That X, Y act nilpotently is the previous exercise.)

[ss:divpower]

1.6. **The divided power notation.** Let k be a field of characteristic 0. If u is an element of an associative k-algebra with identity, and n a non-negative integer, we define the *divided power* $u^{(n)}$ to be the element $u^n/n!$. For commuting elements u and v, the usual binomial expansion becomes in this notation:

$$(u+v)^{(n)} = \sum_{k=0}^{n} u^{(k)} v^{(n-k)}$$

Example 1.6.1. Let P be the polynomial ring over k in two commuting variables u and v as in §1.4. Fix a non-negative integer d and let $V_{d,\mathbb{Z}}$ denote the $\mathbb{Z}$-span of $\{u^{(m)}v^{(n)} \mid m+n=d\}$. Since

$$X^{(j)}u^{(m)}v^{(n)} = \binom{m+j}{m} u^{(m+j)}v^{(n-j)} \qquad Y^{(j)}u^{(m)}v^{(n)} = \binom{n+j}{n} u^{(m-j)}v^{(n+j)}$$

it follows that $V_{d,\mathbb{Z}}$ is stable under the operators $X^{(j)}$ and $Y^{(j)}$. In particular, $V_{d,\mathbb{Z}}$ is an $\mathrm{sl}_2(\mathbb{Z})$-module. Given a commutative ring R with identity, we can, by extension of scalars, consider $V_{d,R} := V_d \otimes R$ as an $\mathrm{sl}_2(R)$-module.

Letting $P_{d,\mathbb{Z}}$ be the $\mathbb{Z}$-module generated by $\{u^m v^n \mid m+n=d\}$, we similarly define $P_{d,R} := P_{d,\mathbb{Z}} \otimes R$.

Since $P_{d,k} = P_d = V_{d,k}$ and both $P_{d,\mathbb{Z}}$ and $V_{d,\mathbb{Z}}$ are stable under $\mathrm{sl}_2(\mathbb{Z})$, we say that $P_{d,\mathbb{Z}}$ and $V_{d,\mathbb{Z}}$ are *admissible lattices* for the $\mathrm{sl}_2(k)$-module P_d.

It may happen that $P_{d,R} \not\simeq V_{d,R}$ when R is a field of positive characteristic: e.g., when $d=2$ and the characteristic of R is 2. $\square$

[ss:cdproof]

1.7. **Proofs of items (c) and (d) of §1.2.** Let k be a field of characteristic 0. Let V be a finite dimensional $\mathrm{sl}_2(k)$-representation and v be a highest weight vector: that is,

$$0 \neq v,\ Hv = \lambda v \text{ for some } \lambda \text{ in } k, \text{ and } Xv = 0$$

If k is algebraically closed and $0 \neq V$, then there are highest weight vectors in V, as seen in §1.5, item (iv).

Consider v, Yv, $Y^{(2)}v$, ... As seen in item (iii) of §1.5, these are all weight vectors with respective weights λ, $\lambda-2$, $\lambda-4$, ... In particular, the non-zero ones in the list are linearly independent, and there exists n such that $Y^{(n)}v=0$.

We have $XYv = [X,Y]v + Y(Xv) = Hv + 0 = \lambda v$; an easy calculation using induction and Eq. (1.2) gives the following:

$$X \cdot Y^{(n)}v = (\lambda - n + 1)Y^{(n-1)}v$$

We now draw several consequences from the above equation:

(A) Let n be least such that $Y^{(n)}v=0$. Then $\lambda = n-1$. In particular, λ is a non-negative integer, and $n = \lambda+1$.
(B) The span of v, Yv, $Y^{(2)}v$, ..., $Y^{(\lambda)}v$ is $\mathrm{sl}_2(k)$-invariant. In particular, if V is irreducible, the listed elements form a basis of V.

(C) Any irreducible representation $0 \neq V$ is determined by the weight λ of any highest weight vector v in it. Indeed if W is an irreducible that admits a highest weight vector, say w, of the same weight λ, then the association $Y^{(n)}v \leftrightarrow Y^{(n)}w$ defines evidently an isomorphism between V and W.