SIMPLE AND SEMISIMPLE FINITE DIMENSIONAL ALGEBRAS

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Let A be a ring (with identity). We call A *simple* if it is non-zero (that is, $1 \neq 0$) and has no proper non-zero two-sided ideals. We call A *semisimple* if it is semisimple as a (left) module over itself: see §1.3 for the definition of a semisimple module. Our goal here is to prove structure theorems for simple and semisimple finite dimensional (associative) algebras over a field. Recall that a ring A is an (associative) algebra over a field F if there is a ring homomorphism $F \to A$ (respecting identity as always) whose image lies in the centre of A; such an algebra is finite dimensional if the dimension of A as an F-vector space is finite.

1. Preliminaries

Let A be a ring. By an A-module, or just module if A is clear from the context, we mean a left A-module. We will have occasion to deal with right modules too, but in those we will invariably mention explicitly that the action of the ring is from the right. All modules are assumed to be *unital*: that is, the multiplicative identity 1 of A acts as the identity map.

If A is an algebra over a field F, then any A-module is naturally an F-vector space (via the ring homomorphism $F \to A$ that defines the algebra structure of A). Such a module is *finite dimensional* if its dimension as an F-vector space is finite.

1.1. **Simple modules and Schur's lemma.** A module is *simple* if it has precisely two submodules: 0 and itself. In particular, the zero module is not considered simple.

A module homomorphism from a simple module is either identically zero or injective, for the kernel of a homomorphism is a submodule; a module homomorphism to a simple module is either identically zero or surjective, for the image of a module homomorphism is a submodule. In particular:

> *A non-zero homomorphism between simple modules is an isomorphism. In particular, between non-isomorphic simple modules there is only the zero homomorphism.*

The above statement is the simplest version of *Schur's lemma*.

To derive another version of this lemma, suppose that we have a finite dimensional simple module M over an F-algebra A, where F is a field which we assume to be algebraically closed. Then any A-linear endomorphism φ of M is also F-linear, and as such it has an eigenvalue, say λ. Note that $\varphi - \lambda \cdot \text{identity}$ is also an A-linear endomorphism (for $F \to A$ has image in the centre of A). Its kernel being non-zero (since λ is an eigenvalue), it follows

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that the kernel is all of M (since M is simple). Thus $\varphi = \lambda \cdot$ identity. In other words, we have proved the following:

Lemma 1. (Schur's lemma, second version) *Let A be an algebra over an algebraically closed field F. Then any A-endomorphism of a finite dimensional simple A-module M is scalar multiplication by some element of F.*

1.2. **Simple modules as quotients of the ring as a left module over itself.** Given an arbitrary module M over a ring A and m an element of M, the map $\varphi_m : a \mapsto am$ is an A-linear map from ${}_AA$ to M, where ${}_AA$ denotes A thought of as a left module over itself. If M is a simple A-module, then for $0 \neq m$ in M (such an m exists since M is non-zero), φ_m is onto M. Thus every simple module is a quotient of ${}_AA$.

Now let A be an algebra of finite dimension over a field F. Fix a sequence ${}_AA = N_0 \supsetneq N_1 \supsetneq \ldots \supsetneq N_{k-1} \supsetneq N_k = 0$ of submodules of ${}_AA$ with all quotients N_j/N_{j+1} being simple. Such a sequence exists: indeed the length k of such a strictly decreasing sequence is bounded by $\dim_F A$. Given a simple A-module M, fix $\phi : {}_AA \to M$ a surjective A-module map—see the previous paragraph—and let r be the least integer such that $\phi|_{N_r} = 0$. Then $r \geq 1$ and $M \simeq N_{r-1}/N_r$. Thus any simple module is isomorphic to one of the quotients N_j/N_{j+1}. In particular:

> *There are only finitely many isomorphism classes of simple modules over a finite dimensional algebra.*

1.3. **Equivalent conditions for the semisimplicity of a module.** A module is *semisimple* if it satisfies (any and therefore all) the conditions in:

Proposition 2. *The following conditions are equivalent for an arbitrary module M over an arbitrary ring A:*

- *M is a sum of (some of its) simple submodules.*
- *M is a direct sum of (some of its) simple submodules.*
- *Every A-submodule N of M has a* complement*: that is, there exists an A-submodule P such that $M = N \oplus P$.*

For the proof, first observe that all the three conditions hold in case $M = 0$: there are no simple submodules in this case, but then the sum over an empty indexing set is zero by convention. We assume henceforth that $M \neq 0$.

The second condition clearly implies the first. Let us prove that the first implies the third. Let N be a submodule of M and let P be maximal with respect to the property that it intersects N trivially and is a sum of simple submodules. Such a P exists by a Zorn's lemma argument. We claim that $M = N \oplus P$. Since $N \cap P = 0$ by choice of P, it is enough to show $M = N + P$. Since M is a sum of its simple submodules, it is enough to show that each of its simple submodules K is contained in $N + P$. If $N \cap (K + P) = 0$, then $K \subseteq P$ by the maximality of P; if not, then $K \cap (N + P) \neq 0$, which implies by the simplicity of K that $K \subseteq N + P$, and the claim is proved.

We will show that the third condition implies the second assuming that (A is an algebra over a field and) M is a finite dimensional module. A proof in the general case is outlined in Exercise 4. Let us first observe that the third condition is inherited by submodules: if N' is a submodule of a submodule M', and P is a submodule such that $M = N' \oplus P$, then $M' = N' \oplus (P \cap M')$. Now proceed by induction on the dimension of M. Choose a simple submodule of N of M (which exists for dimension reasons). Let P be a complement, so that $M = N \oplus P$. Now P satisfies the third condition (by observation above) and so by induction the second condition holds for it. This in turn implies the second condition for M, and the proposition is proved.

Corollary 3. *Submodules and quotient modules of semisimple modules are semisimple. A sum (in particular, an arbitrary direct sum) of semisimple submodules is semisimple.*

Indeed, it is clear that the first condition passes to quotients: the image of a simple module is either zero or simple. By the third condition, every submodule of a semisimple module is isomorphic to a quotient and vice-versa. A sum of semisimple submodules is in turn a sum of simple modules and so the first condition holds for it.

1.4. **Isotypical components of a semisimple module.** For A-modules K and M, the set $\mathrm{Hom}_A(K, M)$ of A-homomorphisms from K to M has naturally the structure of a right $\mathrm{End}_A K$-module (by precomposition). There is a natural *evaluation* map $\mathfrak{ev} = \mathfrak{ev}_{K,M}$: $\mathrm{Hom}_A(K, M) \otimes K \to M$, where the tensor product is over $\mathbb{Z}$, and $\varphi \otimes k \mapsto \varphi(k)$. Observe that K has naturally the structure of a (left) $\mathrm{End}_A K$-module. The evaluation map factors through as $\mathrm{Hom}_A(K, M) \otimes_{\mathrm{End}_A K} K \to M$. Moreover the evaluation maps are A-homomorphisms if we make the tensor of $\mathrm{Hom}_A(K, M)$ and K (whether over $\mathbb{Z}$ or $\mathrm{End}_A K$) into an A-module by $a(\varphi \otimes k) := \varphi \otimes ak$.

Suppose now that K is simple. Then $\mathrm{Hom}_A(K, M) \otimes K$ (whether the tensor is over $\mathbb{Z}$ or over $\mathrm{End}_A K$) is a semisimple A-module. In fact, it is a sum of submodules all of which are isomorphic to K. Its image in M is therefore also such a sum and in fact the maximal submodule with that property. It is denoted $\mathrm{Iso}(K, M)$ and called the *K-isotypic component of M*.

Now let K_α, with α in some index set, be a complete list of simple A-modules no two of which are isomorphic. Then the sum over α of $\mathrm{Iso}(K_\alpha, M)$ is direct (item 2 of Exercise set 3). This sum equals the maximal semisimple submodule of M (which clearly exists because the sum of any two semisimple submodules is semisimple), called the *socle* of M. If M is semisimple, then it is its own socle, and

$$M = \bigoplus_\alpha \mathrm{Iso}(K_\alpha, M) \tag{1}$$

is called the *isotypic decomposition of M*.

Finally, we observe that, for K simple, the map $\mathrm{Hom}_A(K, M) \otimes_{\mathrm{End}_A K} K \to M$ is an injection: see item 3 of Exercise set 3. The dimension of $\mathrm{Hom}_A(K, M)$ over $\mathrm{End}_A K$ (which is a division ring by Schur's lemma) is the *multiplicity of K in M*, in other words, the number of times K appears in any decomposition of the socle of M into a direct sum of simples.

1.5. **Semisimple algebras: the case of (finite) group rings.** A ring A is *semisimple* (or, more correctly, *left semisimple*) if the module ${}_A A$ is semisimple.[1] To produce an important class of examples of semisimple algebras, consider the group ring FG of a group G with coefficients over a field F. This is an F-algebra:

- FG is finite dimensional if and only if the group G is finite.
- FG-modules are precisely linear representations of the group G over the field F.

Theorem 4. (Maschke's theorem) *Let F be a field and G a finite group. Then the group ring FG is semisimple if the characteristic of F is either zero or positive and not a divisor of the cardinality of G.*

Proof: We let N be a submodule of an FG-module M and construct a complementary submodule. This will show that any module is semisimple and in particular that FG as a left module over itself is semisimple.[2] Let φ be an F-linear projection from M onto N. Now consider the "average" $\tilde{\varphi} := (\sum_{g \in G} g\varphi g^{-1})/|G|$. This on the one hand is a FG-homomorphism and on the other a projection:

- For h in G, we have
  $h\tilde{\varphi} = h\tilde{\varphi}h^{-1}h = (\sum_{g \in G} hg\varphi g^{-1}h^{-1})h/|G| = (\sum_{g \in G} g\varphi g^{-1})h/|G| = \tilde{\varphi}h$
- For n in N, we have $g\varphi g^{-1}n = gg^{-1}n = 1n = n$ since φ is a projection onto N, and so $\tilde{\varphi}n = n$.

The kernel of $\tilde{\varphi}$ is then a complementary FG-submodule to N. □

Remark 5. The "if" in Theorem 4 could be changed to "if and only if". See Exercise 6.

2. Simple finite dimensional algebras

Let F be a field and A an (associative) algebra (with identity) over F with $\dim_F A < \infty$.

2.1. **An example.** The prototypical example of such an algebra A is $\mathrm{End}_F V$, the ring of F-linear endomorphisms of a finite dimensional (non-zero) F-vector space V.

There is a natural bijective inclusion-reversing corresponding between subspaces of V and left ideals of $\mathrm{End}_F V$: to a subspace W is associated the left ideal $W^\perp$ consisting of all endomorphisms that vanish on W. There is also a natural inclusion-preserving correspondence between subspaces and right ideals: given a subspace W, the set $\mathfrak{r}_W$ of all endomorphisms with range contained in W is a right ideal. In particular, there are only two left ideals that are also right ideals: $0^\perp$ which is the whole of $\mathrm{End}_F V$ and $V^\perp$ which is 0.

The F-dimension of $W^\perp$ is $\dim V \cdot \dim(V/W)$, that of $\mathfrak{r}_W$ is $\dim V \cdot \dim W$. If W_1 and W_2 are subspaces of the same dimension, then $W_1^\perp$ and $W_2^\perp$ are isomorphic as A-modules:

[1] As we will see in §3 below, the semisimplicity of finite dimensional algebra as a left module and as a right module over itself are equivalent.

[2] As is easily seen (Exercise **??**), any module over a semisimple ring is semisimple, so we are not proving any more than we set out to.

choosing g to be a linear isomorphism of V that maps W_1 to W_2, we have $\phi \mapsto \phi g$ an A-module isomorphism of $W_2^\perp$ onto $W_1^\perp$.

Any minimal (non-zero) left ideal ℓ of A arises as $H^\perp$ for some hyperplane H of V. It is isomorphic to V as A-modules. Indeed, choosing a non-zero element of V/H (two such elements are scalar multiples of each other), any element of $H^\perp$ is determined by the value at this element, and the resulting evaluation map from $H^\perp$ to V defines an A-isomorphism.

$\mathrm{End}_F V$ is semisimple: if we choose as many hyperplanes $H_1, \ldots, H_n$ in V as the dimension n of V such that they intersect trivially, then $\mathrm{End}_F V$ is the (internal) direct sum of the corresponding minimal left ideals.

The algebra $\mathrm{End}_F V$ is (non-canonically) isomorphic to its opposite. To see this, fix a non-degenerate bilinear form on V. Using this form, we can identity V with its dual V^* and so also $\mathrm{End}_F V$ with $\mathrm{End}_F V^*$. The association $\varphi \leftrightarrow \varphi^*$, where φ^* is the transpose of φ, defines an isomorphism between $(\mathrm{End}_F V)^{\mathrm{opp}}$ and $\mathrm{End}_F V^*$.

2.2. **The structure of a simple algebra.** Assume that F is algebraically closed and let ℓ be a minimal (non-zero) left ideal of a simple finite dimensional algerba A over F. The natural map $\rho : A \to \mathrm{End}_F \ell$ defining the action of A on ℓ is injective since the kernel is a proper two-sided ideal and A is simple. We claim that ρ is surjective and therefore an isomorphism. Indeed this follows from the following famous result.

Lemma 6. (Burnside) *Let B be an algebra (not necessarily finite dimensional) over an algebraically closed field F, and V a simple finite dimensional B-module. Then the canonical map $B \to \mathrm{End}_F V$ defining the action of B on V is an epimorphism.*

Proof: It is enough to show the following: given a F-basis $v_1, \ldots, v_n$ of V, and arbitrary elements $w_1, \ldots, w_n$ of V, there exists an element b in B such that $bv_1 = w_1, \ldots, bv_n = w_n$. Consider the B-module map $\psi : B \to V^{\oplus n}$ defined by $b \mapsto (bv_1, \ldots, bv_n)$. Since V is simple, $V^{\oplus n}$ is semisimple. The image of ψ being a submodule of $V^{\oplus n}$, it is isomorphic to $V^{\oplus r}$ with $r \leq n$. Any B-module map between $V^{\oplus r}$ and $V^{\oplus n}$ is given by $(u_1, \ldots, u_r) \mapsto (u_1, \ldots, u_r)M$, where M is an $r \times n$ matrix over $\mathrm{End}_B V$, and we now think of V and its direct sums naturally as right modules over $\mathrm{End}_B V$. Since 1 in B maps to $(v_1, \ldots, v_n)$ under ψ, it follows that there exists $(u_1, \ldots, u_r)$ in $V^{\oplus r}$ such that $(u_1, \ldots, u_r)M = (v_1, \ldots, v_n)$. Our hypothesis that F is algebraically closed now means that $\mathrm{End}_B V$ is F (by Schur's Lemma). Since $v_1, \ldots, v_n$ are linearly independent, this is possible only if $r = n$, and the lemma is proved. □

We've thus proved the following theorem:

Theorem 7. *A finite dimensional simple algebra over an algebraically closed field is isomorphic to the ring of linear endomorphisms of any of its minimal left ideals.*

Combining the theorem with the facts proved in §2.1, we conclude that such an algebra is semisimple: indeed, it is isomorphic to $\ell^{\oplus n}$ where ℓ is any minimal left ideal and n the F-dimension of ℓ. The F-dimension of such an algebra is the square of the dimension of any of its minimal left ideals.

Corollary 8. *Let F be an algebraically closed field. Let V, W be finite dimensional simple modules respectively for F-algebras B and C. Then $V \otimes W$ is a simple $B \otimes C$-module. Moreover, every simple finite dimensional $B \otimes C$-module arises thus.*

Proof: Since the canonical maps from B and C respectively to $\mathrm{End}_F V$ and $\mathrm{End}_F W$ are surjective (by Lemma 6), the image of $B \otimes C$ under their tensor product is $\mathrm{End}_F V \otimes \mathrm{End}_F W = \mathrm{End}_F(V \otimes W)$, which proves the first assertion. For the converse, given a finite dimensional $B \otimes C$-module X, let V be a simple B-submodule (this exists because $\dim_F X < \infty$). Now $\mathrm{Hom}_B(V, X)$ is naturally a C-module (by the action on X, since the actions of B and C commute), and we have an evaluation morphism $\mathfrak{ev} : V \otimes \mathrm{Hom}_B(V, X) \to X$, which is $B \otimes C$-linear. The image of $\mathfrak{ev}$ is non-zero, and therefore onto X since X is simple. Let W be any simple C-submodule of $\mathrm{Hom}_B(V, X)$. Since $\mathfrak{ev}(V \otimes \varphi) \neq 0$ for $\varphi \neq 0$ (by the nature of $\mathrm{Hom}_B(V, X)$), we have $\mathfrak{ev}(V \otimes W)$ is non-zero and therefore all of X. In other words, $\mathfrak{ev} : V \otimes W \to X$ is an isomorphism (that it is injective uses the first part). □

2.2.1. The case when F is not algebraically closed. If the base field is not algebraically closed, then the result above still holds in a slightly modified form. Let ℓ be a minimal left ideal and put $\mathbb{D} := \mathrm{End}_A \ell$. By Schur's lemma, $\mathbb{D}$ is a division subring of $\mathrm{End}_F \ell$. We now consider ℓ as a (finite dimensional) vector space over $\mathbb{D}$: observe that $\dim_{\mathbb{D}} \ell \cdot \dim_F \mathbb{D} = \dim_F \ell$. The claim now is that $A \to \mathrm{End}_F V$ maps onto $\mathrm{End}_{\mathbb{D}} \ell$.

To prove the claim, choose $v_1, \ldots, v_n$ to be a $\mathbb{D}$-basis of ℓ and proceed as before. The matrix M will now have entries over $\mathbb{D}$, and once again $r = n$ is forced by the linear independence of $v_1, \ldots, v_n$ (this time over $\mathbb{D}$). We've thus proved the following theorem:

Theorem 9. *A finite dimensional simple algebra over a field is isomorphic to* $\mathrm{End}_{\mathbb{D}} \ell$*, where ℓ is any minimal left ideal and $\mathbb{D}$ the division ring $End_A \ell$.*

2.3. **Revisiting the prototypical example.** In the light of §2.2.1, it is natural to look back at Example 2.1 and generalize it. To this end, let $\mathbb{D}$ be a division ring with F imbedded centrally in it and $\dim_{\mathbb{D}} F < \infty$. Let V be a finite dimensional (left) $\mathbb{D}$-vector space, and put $A = \mathrm{End}_{\mathbb{D}} V$.

The correspondence of left ideals of A with $\mathbb{D}$-subspaces of V works in a similar fashion. So does the correspondence of right ideals. In particular, A is simple as before. Subspaces of the same dimension correspond to isomorphic one-sided ideals. Any minimal left ideal of A is isomorphic to V as A-modules.

The dual $\mathrm{Hom}_{\mathbb{D}}(V, \mathbb{D})$ is naturally a right $\mathbb{D}$-vector space and so a left $\mathbb{D}^{\mathrm{opp}}$-vector space. The association $\varphi \leftrightarrow \varphi^*$, where φ^* is the transpose of φ, defines an isomorphism of $(\mathrm{End}_{\mathbb{D}} V)^{\mathrm{opp}}$ with $\mathrm{End}_{\mathbb{D}^{\mathrm{opp}}} V^*$.

3. Semisimple finite dimensional algebras

3.1. **Prototypical examples of semisimple algebras.** Let $V_1, \ldots, V_k$ be finite dimensional vector spaces over the (arbitrary) field F and A be the product $\mathrm{End}_F V_1 \times \cdots \times \mathrm{End}_F V_n$.

The vector spaces V_j are naturally A-modules: indeed, each V_j is an $\mathrm{End}_F V_j$-module and thus also an A-module via the projection $A \to \mathrm{End}_F V_j$. As observed in §2.1, each $\mathrm{End}_F V_j$ is semisimple as a module over itself. So each $\mathrm{End}_F V_j$ is a semisimple A-module and A being the direct sum of these as a module is also semisimple.

3.2. **The structure of a semisimple algebra.** Our goal is to show that every finite dimensional semisimple algebra over an algebraically closed field is of the prototypical form described in §3.1. The following simple, beautiful observation is used crucially in the proof:

> *Let R be a ring (with identity) and L denote R as a left module over itself. Then the ring $\mathrm{End}_R(L)$ of R-endomorphisms of L is naturally isomorphic to the opposite ring R^{opp} of R.*

Indeed, L being generated by 1, any R-endomorphism of L is determined by where it maps 1, say to r, but then it must be right multiplication by r.

Theorem 10. (Wedderburn) *Let A a finite dimensional semisimple algebra over an arbitrary field F. Let left ideals $\ell_1, \ldots, \ell_k$ be so chosen that no two of them are isomorphic and together they represent all isomorphism classes of simple modules. Let $\pi_i : A \to \mathrm{End}_F \ell_i$ for $1 \leq i \leq k$ be the algebra homomorphisms defining the actions of A on ℓ_i. Then their product*

$$\pi : A \to \mathrm{End}_F \ell_1 \times \cdots \times \mathrm{End}_F \ell_k \tag{2}$$

is an isomorphism onto $\mathrm{End}_{\mathbb{D}_1} \ell_1 \times \cdot \times \mathrm{End}_{\mathbb{D}_i} \ell_k$, where $\mathbb{D}_i$ are the division rings $\mathrm{End}_A \ell_i$. In particular, if F is algebraically closed, π is a bijection.

PROOF: That π is an injection is quite easy to see. Write $A = I_1 \oplus \cdots \oplus I_k$, where I_j is the isotypical components of A corresponding to ℓ_j. If a is in the kernel of π, then it kills all I_j and so all of A, and hence is zero (since A has identity). It remains only to show that the image of π is the product of the images of π_i: note that π_i is onto $\mathrm{End}_F \ell_i$ in case F is algebraically closed (Lemma 6) and that its image is $\mathrm{End}_{\mathbb{D}_i} \ell_i$ in general (§**??**). For this, it is enough to prove the claim:

$$\pi_i(I_j) = 0 \text{ for } j \neq i$$

Indeed, then $\pi_i(I_i) = \pi_i(A)$, and given arbitrary b_i in $\pi_i(A)$, choosing $a_i \in I_i$ such that $\pi_i(a_i) = b_i$, we get $\pi(a_1 + \cdots + a_k) = (\pi_1(a_1), \ldots, \pi_k(a_k)) = (b_1, \ldots, b_k)$.

To prove the claim, we use the observation made at the beginning of this subsection. Each I_j (defined as above) is preserved by $\mathrm{End}_A A$, and so under right multiplication by elements of A. In other words, the I_j are two-sided ideals (not just left ideals). Thus $I_j I_i = 0$ for $j \neq i$, so $I_j \subseteq \mathrm{Ann}\, \ell_i$ for $j \neq i$, which is precisely the claim. □

We list some consequences:

- (A criterion for simplicity) A semisimple finite dimensional algebra is simple if and only if it admits precisely one simple module.
- (Density) Assume F to be algebraically closed. Let $V_1, \ldots, V_n$ be pairwise non-isomorphic simple modules for a semisimple finite dimensional algebra A over F.

Given arbitrary linear transformations $\varphi_i \in \mathrm{End}_F V_i$, there exists a in A such that φ_i is left multiplication of a on V_i.

- (Left semisimple is right semisimple) The opposite of a finite dimensional semisimple algebra is semisimple. In particular, the notions of left and right semisimplicity coincide for finite dimensional algebras.

3.3. **Applications to the ordinary representation theory of a finite group.** The "ordinary" in the title refers to the fact that the base field is algebraically closed of characteristic zero. Let F be such a field, e.g., that of complex numbers. Let G be a finite group. The group ring FG of G with coefficients in F evidently has F-dimension $|G|$ (the cardinality of G). Moreover, it is semisimple by Maschke's theorem (Theorem 4). Wedderburn's theorem therefore applies: if $V_1, \ldots, V_k$ be simple modules such that no two are isomorphic and together represent all simple isomorphism classes, then $FG \simeq \mathrm{End}_F V_1 \times \cdots \times \mathrm{End}_F V_k$.

- Equating the dimensions of the centres on both sides, we see that there are as many simple isomorphism classes as conjugacy classes in the group.
- Equating F-dimensions on both sides, we get $|G| = (\dim V_1)^2 + \cdots + (\dim V_k)^2$.

4. The commutant of a semisimple algebra

We begin with a simple but crucial observation. Let F be a field and V, W finite dimensional vector spaces over F. Consider the subalgebras $\mathrm{End}_F V$ and $\mathrm{End}_F W$ of $\mathrm{End}_F(V \otimes W)$. As can be easily checked, they are *commutants* of each other:

$$\begin{array}{l}\mathrm{End}_F W \text{ is precisely the subset of those elements of } \mathrm{End}_F(V\otimes W)\\ \text{that commute with } \mathrm{End}_F V \text{ (and vice versa).}\end{array} \tag{3}$$

4.1. **Commutant and bicommutant, after Schur.** Let now F be algebraically closed, V a finite dimensional F-vector space, and A a semisimple subalgebra of $\mathrm{End}_F V$. Then V is semisimple as an A-module (every A-module is semisimple). Following Schur, we pose:

$$\text{Can we identify the commutant } C \ (:= \mathrm{End}_A V) \text{ of } A \text{ inside } \mathrm{End}_F V? \tag{4}$$

Towards an answer to the above, let $\ell_1, \ldots, \ell_k$ be minimal left ideals of A so chosen that no two are isomorphic as A-modules and together they represent all simple isomorphism classes of A-modules. Rewrite the isotypical decomposition $\ell_1^{\oplus r_1} \oplus \cdots \oplus \ell_k^{\oplus r_k}$ of V as

$$V = \ell_1 \otimes \rho_1 \oplus \cdots \oplus \ell_k \otimes \rho_k \tag{5}$$

where $\rho_1, \ldots, \rho_k$ are F-vector spaces of respective dimensions $r_1, \ldots, r_k$, and A acts on $\ell_i \otimes \rho_i$ by $a(x \otimes y) = ax \otimes y$.

Let S denote the subalgebra $\mathrm{End}_F(\ell_1 \otimes \rho_1) \times \cdots \times \mathrm{End}_F(\ell_k \otimes \rho_k)$ of $\mathrm{End}_F V$. The image of A in $\mathrm{End}_F V$ (under the map that defines V as an A-module) is contained in S: it is in fact $A_1 \times \cdots \times A_k$, where A_i denotes the image of $\mathrm{End}_F \ell_i$ in $\mathrm{End}_F(\ell_i \otimes \rho_i)$. Since $\mathrm{End}_A V$ preserves the isotypical A-components of V, it is a subalgebra of S. From observation (3) it now follows that it is $C_1 \times \cdots \times C_k$ where C_i denotes the image of $\mathrm{End}_F \rho_i$ in $\mathrm{End}_F(\ell_i \otimes \rho_i)$ (which, let us note, is isomorphic to $\mathrm{End}_F \rho_i$).

Thus the commutant C is isomorphic to $\mathrm{End}_F\rho_1 \times \cdots \times \mathrm{End}_F\rho_k$. It is therefore semisimple (§3.1). Moreover there is a bijective correspondence $\ell_i \leftrightarrow \rho_i$ between simple isomorphism classes of A and those of C. Since A and C commute, we may consider V as an $A \otimes C$-module with $(a \otimes c)v = a(cv) = c(av)$. The decomposition (5) of V is also its isotypic decomposition as an $A \otimes C$-module: each $\ell_i \otimes \rho_i$ is simple (Corollary 8) and occurs exactly once. This last fact is expressed by saying that V is *multiplicity free* as an $A \otimes C$-module. Note that out of the k^2 simple modules of $A \otimes C$ (namely $\ell_i \otimes \rho_j$), only k appear in V (those of the form $\ell_i \otimes \rho_i$, which defines the bijective correspondence).

Finally, observe that A is the commutant of C: this follows from the same kind of reasoning that we used to identify C explicitly. In other words, A is its own *bicommutant*.

4.2. **Examples.** `At least two examples to be included here: schur-weyl, cauchy`

Exercise set 1

(1) Show that any ring A with $1 \neq 0$ admits simple modules.

(2) Show that any module over a semisimple ring is semisimple.

(3) (Example of a non-semisimple finite dimensional complex algebra) Let B be the set of 2×2 upper triangular matrices with complex entries. With respect to standard matrix addition and multiplication, B is a $\mathbb{C}$-algebra. Let M be the set of 2×1 matrices with complex entries. Then with respect to multiplication from the left, M is a B module. Show that M is not semisimple. (Hint: Consider the submodule N consisting of 2×1 matrices with the $(2, 1)$ entry being 0. Show that N has no complement. In fact, show that N is the only submodule of M.) Conclude that B is not semisimple.

(4) True or false?: If N is a semisimple submodule of a module M such that M/N is also semisimple, then M is semisimple.

(5) Let F be a field, G a group, and FG the group ring. Determine the centre of FG. Assuming G is finite, conclude that the F-dimension of the centre is the number of conjugacy classes of G.

(6) (Converse of Maschke) Let F be a field, G a group, and FG the group ring. We can turn F into an FG-module by letting each g in G act as identity and extending linearly: $(\sum \lambda_g g) \cdot \mu = (\sum \lambda_g)\mu$. This module is called the *trivial* FG-module. Let $H := \{\sum_{g \in G} \lambda_g g \in FG \mid \sum_{g \in G} \lambda_g = 0\}$. Then H is a codimension 1 subspace of FG and is an FG-submdoule. Moreover, FG/H is trivial.

Now assume that G is finite. Show that the span of $\sum_{g \in G} g$ is the only 1-dimensional trivial FG-submodule of FG. Conclude that H does not have a complemtary submodule if the characteristic of F is positive and divides $|G|$.

(7) Let A be a ring, M a semisimple A-module, and $C := \operatorname{End}_A M$. For K a simple A-module, let D denote the division ring $\operatorname{End}_A K$, and let S denote $\operatorname{Hom}_A(K, M)$ with its natural C-module structure. Suppose that S is non-zero. Then:

  (a) S is simple.

  (b) $\operatorname{End}_C S$ is naturally isomorphic to D^{opp}, where D^{opp} denotes the opposite of D.

  (c) $\operatorname{End}_C(S, M)$ is naturally isomorphic to K simultaneously as A- and D-modules.

  (d) Via the above isomorphisms, the evaluation map $\operatorname{Hom}_C(S, M) \otimes_{\operatorname{End}_C S} S \to M$ becomes identified with the other evaluation map $\operatorname{Hom}_A(K, M) \otimes_D K \to M$.

  (e) M is semisimple as a C-module.

  (f) There is a bijective correspondence between the simple A-modules that occur (with non-trivial multiplicity) in M and the simple C-modules that occur in M. This is given by $K \mapsto S := \operatorname{Hom}_A(K, M)$. Moreover, the multiplicity of K in M equals the D-dimension of its (right) module S, and the multiplicity of S in M equals the D-dimension of K. The K-isotypic component of M as an A-module is the same as its S-isotypic component as a C-module.

Exercise set 2

(1) A finite dimensional algebra over a field admits only finitely many isomorphism classes of simple modules. Solution: For any ring A, any simple module is a quotient of ${}_AA$: indeed, for any $m \neq 0$ in M, the homomorphism ${}_AA \to M$ by $a \mapsto am$ is onto. Now let A be an algebra of finite dimension over a field F. Fix a sequence ${}_AA = N_0 \supsetneq N_1 \supsetneq \ldots \supsetneq N_{k-1} \supsetneq N_k = 0$ of submodules of ${}_AA$ with all quotients N_j/N_{j+1} being simple. Such a sequence exists: indeed the length k of such a strictly decreasing sequence is bounded by $\dim_F A$. Given a simple A-module M, let $\varphi : A \twoheadrightarrow M$ be an A-module epimorphism, and let r be the least integer such that $\varphi|_{N_r} = 0$. Then $r \geq 1$ and $M \simeq N_{r-1}/N_r$. Thus any simple module is isomorphic to one of the quotients N_j/N_{j+1}.

(2) Show that a ring that admits a finitely generated faithful[3] semisimple module is itself semisimple. Find a ring that is not semisimple but admits a faithful (necessarily non-finitely generated) module. (HInt: A commutative ring admits a faithful semisimple module iff its Jacobson radical is trivial.)

(3) (A version of Nakayama's lemma) Let M be a finitely generated A-module. Given a submodule N, there exists a maximal (proper) submodule of M containing N. (Hint: Zorn.) If M is finitely generated and non-zero, then there exists a primitive ideal $\mathfrak{a}$ such that $\mathfrak{a}M \subsetneq M$.[4] (Hint: Let N be a maximal proper submodule of M and take $\mathfrak{a}$ to be the annihilator of M/N.) Deduce the following: if A is a commutative local ring with maximal ideal $\mathfrak{m}$ and M a non-zero finitely generated A-module, then $\mathfrak{m}M \subsetneq M$.

(4) (Equivalence in general of the conditions in the definition of a semisimple module) Let M be a module such that any submodule of it admits a complement. Then M is a direct sum of (some of its) simple submodules. Solution: Fix a maximal collection $\mathfrak{C}$ of simple submodules of M whose sum is their direct sum: such a collection exists by Zorn. Let N be the sum of submodules in $\mathfrak{C}$, and suppose that $N \subsetneq M$. Choose $y \in M \setminus N$. Choose, by Zorn, a maximal proper submodule P containing N of $N + Ay$. Let S be a complement to P in $N + Ay$ (it exists because the hypothesis on M passes to submodules). Being isomorphic to $(N + Ay)/P$, it is simple. And its existence violates the maximality of $\mathfrak{C}$.

(5) Let F be the finite field $\mathbb{Z}/p\mathbb{Z}$ and $M_3(F)$ the ring of 3×3 matrices with coefficients in F. What are the possible dimensions of left ideals in $M_3(F)$? How many left ideals are there of each such dimension?

(6) Let F be a field, V a finite dimensional F-vector space, and T a linear transformation of V. Then V becomes naturally a module over the polynomial ring $F[t]$ with t acting as T. As we know (e.g., from the RT-1 module of this school) there exists a unique sequence $g_1, \ldots, g_n$ of monic polynomials in $F[t]$ such that $V \simeq F[t]/(g_1) \oplus \cdots \oplus F[t]/(g_n)$ as $F[t]$-modules. In terms of the sequence g_i, identify when V is (a) simple, (b) semisimple. Deduce that (a) V is simple iff its characteristic polynomial is irreducible (in which case the minimal and characteristic polynomial coincide), (b) V is semisimple iff its minimal polynomial is a product of distinct irreducible factors.

[3] A module is faithful if no non-zero element of the ring kills the module.

[4] A ring is *primitive* if it admits a faithful simple module. A two sided ideal is *primitive* if the quotient by it is a primitive ring.

Exercise set 3

Let F be a field and $M_n(F)$ the F-algebra of $n \times n$ matrices over F. The symbols A and M stand for a ring and an A-module respectively.

(1) (This is used in the proof of Burnside's lemma.) If M_1 and M_2 are matrices over F of sizes $n \times r$ and $r \times n$ such that $r \leq n$ and $M_1 M_2$ is an invertible $n \times n$ matrix, then $r = n$.

(2) Let K_α, where α is over an index set, be simple A-modules no two of which are isomorphic. Let $\mathrm{Iso}(K_\alpha, M)$ be the sum of all simple submodules of M that are isomorphic to K_α. Show that the sum, over α, of $\mathrm{Iso}(K_\alpha, M)$ is direct. (The submodule $\mathrm{Iso}(K_\alpha, M)$ is called the *α-isotypic component* of M—see §1.4.)

(3) For K a simple A-module, the map $\mathrm{Hom}_A(K, M) \otimes_{\mathrm{End}_A K} K \to M$ given by "evaluation" (see §1.4) is an injection.

(4) If A and B are F-algebras, then $A \otimes B$ (where the tensor is over F) is naturally an F-algebra by: $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$. If G and H are groups, then $F(G \times H)$ is naturally isomorphic to $FG \otimes FH$.

(5) Let A be an F-algebra and B a subalgebra. The *commutant C of B in A* is the subalgebra of A consisting of all elements of A that commute with all elements of B. As we have seen, if B is a semisimple subalgebra of the matrix algebra $A = M_n(F)$ (with F being algebraically closed), then C is semisimple too and B is the commutant of C. Find a (necessarily not semisimple) subalgebra B of $M_n(F)$ such that the commutant of its commutant C strictly contains B.

(6) True or false?: If a subalgebra B of $M_n(F)$ is such that its commutant C is semisimple, then B is semisimple too.